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FINITE-TO-ONE FUZZY MAPS AND FUZZY PERFECT MAPS

FRANCISCO GALLEGO LUPIÁÑEZ

In this paper we define, for fuzzy topology, notions corresponding to finite-to-one and k -to-one maps. We study the relationship between these new fuzzy maps and various kinds of fuzzy perfect maps. Also, we show the invariance and the inverse invariance under the various kinds of fuzzy perfect maps (and the finite-to-one fuzzy maps), of different properties of fuzzy topological spaces.

1. INTRODUCTION AND DEFINITIONS

In General Topology, the open maps has been very studied. The efforts has been particularly concentrated on determining which classes of spaces are invariant or inverse invariant under various kinds of such maps: open compact maps, open finite-to-one maps, open k -to-one maps, ... [1, 2, 3, 6, 9, 10, 11, 17, 18, 19, 21, 26].

Definition 1. [19] Let X and Y be topological spaces and $f : X \rightarrow Y$ be a map. The map is said finite-to-one if $f^{-1}(y)$ is a finite subset of X for each $y \in Y$, and is k -to-one if $f^{-1}(y)$ consists of exactly k points in X for each $y \in Y$.

In this paper we define for fuzzy topology notions corresponding to these maps. We study the relationship between these new fuzzy maps and various kinds of fuzzy perfect maps. Also, we show the invariance and the inverse invariance under the various kinds of fuzzy perfect maps (and the finite-to-one fuzzy maps) of different properties of fuzzy topological spaces in the Chang's sense.

All the maps are assumed to be continuous and onto.

First, we summarize the definitions which we will use along this paper.

Definition 2. [13] Let (X, τ) be a topological space and $\omega(\tau)$ be the set of all semicontinuous functions from (X, τ) to the unit interval equipped with the usual topology, then $(X, \omega(\tau))$ is called the weakly induced fuzzy topological space by (X, τ) .

Definition 3. [13] A fuzzy extension of a topological property is said to be good, when it is possessed by $(X, \omega(\tau))$ if, and only if, the original property is possessed by (X, τ) .

Definition 4. [20] If μ is a fuzzy set in X , the set $\{x \in X | \mu(x) > 0\}$ is called the support of μ and is denoted by $\text{supp } \mu$.

Definition 5. [20] Let μ_1, μ_2 be two fuzzy sets in X , μ_1 is said to be quasi-coincident with μ_2 , denoted by $\mu_1 q \mu_2$, if there exists $x \in X$ such that $\mu_1(x) > \mu_2'(x)$ where μ_2' is the fuzzy complement of μ_2 .

Definition 6. [20] A fuzzy set in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If this value at x is α ($0 < \alpha \leq 1$) we denote this fuzzy point by x_α .

Definition 7. [4] Let f be a map from X to Y . Let μ be a fuzzy set in Y , then the inverse of μ , denoted as $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all x in X . Conversely, if μ is a fuzzy set in X , the image of μ , written as $f(\mu)$ is a fuzzy set in Y given by

$$f(\mu)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \{\mu(z)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Definition 8. [4] A map f from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, s) is F -continuous if the inverse of each open fuzzy set is open fuzzy in (X, τ) .

Definition 9. [27] A map f from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, s) is fuzzy closed (resp. fuzzy open) if $f(\mu)$ is fuzzy closed (resp. fuzzy open) in (Y, s) , for each fuzzy closed (resp. fuzzy open) set μ in X .

Definition 10. [4] A family of fuzzy sets $\{\mu_j | j \in J\}$ is a cover of a fuzzy set μ if $\mu \leq \bigvee \{\mu_j | j \in J\}$. A subcover of $\{\mu_j | j \in J\}$ is a subfamily of it which is also a cover. A fuzzy topological space is compact in the Chang's sense, if each cover has a finite subcover.

Definition 11. [12] A fuzzy set μ in (X, T) is fuzzy compact in the Lowen's sense, if for all family of fuzzy open sets $\{\mu_j | j \in J\}$ such that $\mu \leq \bigvee \{\mu_j | j \in J\}$ and for all $\varepsilon > 0$ there exists a finite subfamily $\{\mu_j | j \in J_0\}$ such that $\mu - \varepsilon \leq \bigvee \{\mu_j | j \in J_0\}$.

Definition 12. [8] A collection \mathcal{U} of fuzzy sets in (X, T) is called q -cover of a fuzzy set μ if for each $x \in \text{supp } \mu$, $x_{\mu(x)} q \bigvee_{U \in \mathcal{U}} U$. If each member of \mathcal{U} is fuzzy open, then \mathcal{U} is called an open q -cover. A fuzzy set μ in a fuzzy topological space (X, T) is called q -compact if for every open q -cover \mathcal{U} , of μ , there exists a finite subcollection \mathcal{U}_0 of \mathcal{U} such that $\sup\{1 - \bigvee_{U \in \mathcal{U}_0} U(z) | z \in \text{supp } \mu\} < \mu(x)$ for every $x \in \text{supp } \mu$.

Definition 13. [8] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect, in the Ghosh's sense, if f is onto, fuzzy closed, F -continuous and $f^{-1}(y_\alpha)$ is q -compact, for each fuzzy point y_α in Y .

Definition 14. [23] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Srivastava and Lal's sense, if f is onto, fuzzy closed, F -continuous and $f^{-1}(y_\alpha)$ is fuzzy compact in the Lowen's sense, for each fuzzy point y_α in Y .

Definition 15. [5] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Christoph's sense, if f is onto, fuzzy closed, F -continuous and $f^{-1}(y_\alpha)$ is compact in the Chang's sense, for each fuzzy point y_α in Y .

Let (X, τ) and (Y, s) be two topological spaces and $f : (X, \tau) \rightarrow (Y, s)$ an onto map. We denote $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ the map given by $\tilde{f}(x_\alpha)(y) = \sup_{t \in f^{-1}(y)} \{x_\alpha(t)\}$ for each fuzzy point x_α in X .

Definition 16. Let $(X, T), (Y, S)$ be two fuzzy topological spaces and $\tilde{f} : (X, T) \rightarrow (Y, S)$ be a fuzzy map. The map will be called finite-to-one fuzzy if $\text{supp } \tilde{f}^{-1}(y_\alpha)$ is a finite subset of X for each fuzzy point y_α in Y , and will be called k -to-one fuzzy if $\text{supp } \tilde{f}^{-1}(y_\alpha)$ consists of exactly k points in X for each fuzzy point y_α in Y .

Lemma 1.1. If the map $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ is open fuzzy then the map $f : (X, \tau) \rightarrow (Y, s)$ is open.

Proof. If A is open in (X, τ) , we have $A = \chi_A^{-1}((a, 1])$ for all $a \in [0, 1)$, $\chi_A \in \omega(\tau)$, and then $\tilde{f}(\chi_A) = \chi_{f(A)} \in \omega(s)$ and $f(A)$ is open in (Y, s) . □

Proposition 1.2. If the map $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ is finite-to-one fuzzy (resp. k -to-one fuzzy), then the map $f : (X, \tau) \rightarrow (Y, s)$ is finite-to-one (resp. k -to-one).

Proof. If there exists $y \in Y$ such that $f^{-1}(y)$ is infinite (resp. $\text{card } f^{-1}(y) \neq k$) let

$$y_1(y') = \begin{cases} 1 & \text{if } y' = y \\ 0 & \text{if } y' \neq y \end{cases}$$

Then, for each $x \in X$,

$$\tilde{f}^{-1}(y_1)(x) = y_1(f(x)) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{if } f(x) \neq y \end{cases} \quad \square$$

Thus, $\text{supp } \tilde{f}^{-1}(y_1)$ and $f^{-1}(y)$ have equal cardinals, and $\text{supp } \tilde{f}^{-1}(y_1)$ is infinite (resp. its cardinal is $\neq k$).

2. RELATIONSHIP BETWEEN THESE FUZZY MAPS AND FUZZY PERFECT MAPS

It is known that every open k -to-one continuous and onto map is a perfect map [1, Lemmas 1 and 2]. Then it is natural to study the relationship between k -to-one fuzzy maps and the various kinds of fuzzy perfect maps.

Remark 2.1. In general, \tilde{f} open- k -to-one fuzzy map $\not\approx \tilde{f}$ fuzzy perfect map in the Christoph's sense.

Indeed, let $1_x : (X, T) \rightarrow (X, T)$ be the identity fuzzy map. Clearly, 1_x is 1-to-one fuzzy map. But $1_x^{-1}(x_1)$ for x_1 crisp fuzzy point in X , is not fuzzy compact in the Chang's sense, because if $x_1 \leq \bigvee_{j \in (0,1)} \mu_j$, then $1 = \sup\{\mu_j(x) | j \in (0, 1)\}$, and if μ_j is an open fuzzy set in X such that $\mu_j(x) = j$, for all $j \in (0, 1)$, there is not a finite subfamily $J_0 \subset (0, 1)$ such that $1 = \sup\{\mu_j(x) | j \in J_0\}$. Then, 1_x is not fuzzy perfect map in the Christoph's sense.

Proposition 2.2. If $\tilde{f} : (X, T) \rightarrow (Y, S)$ is a finite-to-one fuzzy map, then $\tilde{f}^{-1}(y_\alpha)$ is fuzzy compact in the Lowen's sense for each fuzzy point y_α in Y .

Proof. Let $\text{supp } \tilde{f}^{-1}(y_\alpha) = \{x_r | r = 1, \dots, k\}$ for some fuzzy point y_α in Y , and $\{\mu_j | j \in J\}$ be a family of open fuzzy sets in X such that $\tilde{f}^{-1}(y_\alpha) \leq \bigvee_{j \in J} \mu_j$. Then $\alpha \leq \sup\{\mu_j(x_r) | j \in J\}$ for every $r = 1, \dots, k$. Thus, for each $\varepsilon > 0$ and each $r = 1, \dots, k$ there exists $j_r \in J$ such that $\alpha - \varepsilon < \mu_{j_r}(x_r)$. We denote $J_0 = \{j_r | r = 1, \dots, k\}$, and we have that $\alpha - \varepsilon < \sup\{\mu_j(x_r) | j \in J_0\}$ for every $r = 1, \dots, k$, and $\tilde{f}^{-1}(y_\alpha) - \varepsilon \leq \bigvee_{j \in J_0} \mu_j$. Hence $\tilde{f}^{-1}(y_\alpha)$ is fuzzy compact in the Lowen's sense. \square

Corollary 2.3. If \tilde{f} is an onto, F -continuous, finite-to-one closed fuzzy map, then \tilde{f} is fuzzy perfect in the Srivastava and Lal's sense.

Proposition 2.4. If $\tilde{f} : (X, T) \rightarrow (Y, S)$ is a finite-to-one fuzzy map, then $\tilde{f}^{-1}(y_\alpha)$ is q -compact for each fuzzy point y_α in Y .

Proof. Let $\text{supp } \tilde{f}^{-1}(y_\alpha) = \{x_r | r = 1, \dots, k\}$ for some fuzzy point y_α in Y , and \mathcal{U} be an open q -cover of $\tilde{f}^{-1}(y_\alpha)$. Then for each $x \in \text{supp } \tilde{f}^{-1}(y_\alpha)$, $x_\alpha q \bigvee_{U \in \mathcal{U}} U$. Thus, we have that $\alpha > 1 - \bigvee_{U \in \mathcal{U}} U(x_r)$ for each $r = 1, \dots, k$, and there exists an $U_r \in \mathcal{U}$ such that $U_r(x_r) > 1 - \alpha$ for each $r = 1, \dots, k$. Let $\mathcal{U}_0 = \{U_r | r = 1, \dots, k\}$. We have that \mathcal{U}_0 is a finite subcollection of \mathcal{U} and $(1 - \bigvee_{U \in \mathcal{U}_0} U)(x_r) = 1 - \sup\{U(x_r) | U \in \mathcal{U}_0\} < \alpha$. Thus, $\sup\{(1 - \bigvee_{U \in \mathcal{U}_0} U)(x_r) | r = 1, \dots, k\} < \alpha = f^{-1}(y_\alpha)(x)$ for every $x \in \text{supp } \tilde{f}^{-1}(y_\alpha)$. Hence $\tilde{f}^{-1}(y_\alpha)$ is q -compact. \square

Corollary 2.5. If \tilde{f} is an onto, F -continuous, finite-to-one closed fuzzy map, then \tilde{f} is fuzzy perfect in the Ghosh's sense.

3. INVARIANCE THEOREMS

Now, we show the invariance's theorems for the various kinds of fuzzy perfect maps:

Proposition 3.1. If the map $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ verifies that $\tilde{f}^{-1}(y_\alpha)$ is q -compact for every fuzzy point y_α in Y , then $f^{-1}(y)$ is compact for every $y \in Y$.

Proof. For every open cover \mathcal{U} of $f^{-1}(y)$, we have that $\mathcal{U}^* = \{\chi_U \mid U \in \mathcal{U}\}$ is open q -cover of $\tilde{f}^{-1}(y_\alpha)$. Indeed: for each $x \in \text{supp } \tilde{f}^{-1}(y_\alpha)$ is $\tilde{f}^{-1}(y_\alpha)(x) = y_\alpha(f(x)) = \alpha$, then $f(x) = y$, $x_{\tilde{f}^{-1}(y_\alpha)(x)} + \chi_{\bigcup_{U \in \mathcal{U}} U}(x) = \alpha + 1 > 1$, thus $x_{\tilde{f}^{-1}(y_\alpha)(x)}^q \chi_{\bigcup_{U \in \mathcal{U}} U}$.

By the hypothesis, there exists a finite subcollection \mathcal{U}_0^* of \mathcal{U}^* such that $\text{sup}\{(1 - \bigvee_{\chi_U \in \mathcal{U}_0^*} \chi_U)(x) \mid x \in \text{supp } \tilde{f}^{-1}(y_\alpha)\} < \tilde{f}^{-1}(y_\alpha)(x)$, for every $x \in \text{supp } \tilde{f}^{-1}(y_\alpha)$ (or, equivalently $x \in f^{-1}(y)$). Then, $(\bigvee_{\chi_U \in \mathcal{U}_0^*} \chi_U)(x) + \tilde{f}^{-1}(y_\alpha)(x) > 1$ for every $x \in f^{-1}(y)$ [8, Corollary 2.1], $\chi_{\bigcup_{U \in \mathcal{U}_0} U}(x) + \alpha > 1$ for every $x \in f^{-1}(y)$ (where $\mathcal{U}_0 \subset \mathcal{U}$ is a finite subfamily), $x \in \bigcup_{U \in \mathcal{U}_0} U$ for all $x \in f^{-1}(y)$ and, finally \mathcal{U}_0 covers $f^{-1}(y)$. Thus $f^{-1}(y)$ is compact. \square

Corollary 3.2. If the map $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ is fuzzy perfect in the Ghosh's sense, then the map $f : (X, \tau) \rightarrow (Y, s)$ is perfect.

Proof. It follows from the proposition and Lemmas 2.1 and 2.2 in [15]. \square

Proposition 3.3. If the map $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ is fuzzy perfect in the Srivastava and Lal's sense (or in the Christoph's sense), then the map $f : (X, \tau) \rightarrow (Y, s)$ is perfect.

Proof. We have proved this in [15]. \square

Theorem 3.4. Let $(X, \omega(\tau)), (Y, \omega(s))$ be two weakly induced fuzzy topological spaces and $\tilde{f} : (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense), if:

- (a) $(X, \omega(\tau))$ verifies the fuzzy version of a property \mathcal{P} of topological spaces,
- (b) the property \mathcal{P} is invariant under perfect maps,
- (c) the fuzzy version of \mathcal{P} is a good extension of \mathcal{P} .

Then $(Y, \omega(s))$ verifies the fuzzy version of \mathcal{P} .

Proof. The map $f : (X, \tau) \rightarrow (Y, s)$ is perfect by Corollary 3.2 and Proposition 3.3. Then the result is clear. \square

Corollaries 3.5.

1. Hausdorff fuzzy topological spaces [24] are invariant by fuzzy perfect maps in the various senses.
2. All the good extensions of paracompactness are invariant by fuzzy perfect maps in the various senses.

Theorem 3.6. Let $(X, \omega(\tau)), (Y, \omega(s))$ be two weakly induced fuzzy topological spaces and $\tilde{f}: (X, \omega(\tau)) \rightarrow (Y, \omega(s))$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense), if:

- (a) $(Y, \omega(s))$ verifies the fuzzy version of a property \mathcal{P} of topological spaces.
- (b) the property \mathcal{P} is inverse invariant under perfect maps,
- (c) the fuzzy version of \mathcal{P} is a good extension of \mathcal{P} .

Then $(X, \omega(\tau))$ verifies the fuzzy version of \mathcal{P} .

Proof. The map $f: (X, \tau) \rightarrow (Y, s)$ is perfect by Corollary 3.2 and Proposition 3.3. Then the result is clear. \square

Corollaries 3.7.

1. Hausdorff fuzzy topological spaces are inverse invariant by fuzzy perfect maps in the various senses.
2. All the good extensions of paracompactness are inverse invariant by the various kinds of fuzzy perfect maps.

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