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ROBUST STABILITY OF NON LINEAR TIME VARYING SYSTEMS[1,2]

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Systems with time-varying non-linearity confined to a given sector (Luré type) and a linear part with uncertainty formulated by an interval transfer function, are considered.

Sufficient conditions satisfying the Popov criterion for stability, which are computationally tractable, are derived.

The problem of checking the Popov criterion for an infinite set of systems, is reduced to that of checking the Popov criterion for a finite number of fixed coefficient systems, each in a prescribed frequency interval.

Illustrative numerical examples are provided.

1. INTRODUCTION

A large group of "real life" engineering systems, which are non-linear and (possibly) time-varying, can be classified as Luré type systems. This class of systems will be defined formally in the next section, but it is a well known one and extensively treated in the literature for many years. Essentially, the (single input single output case) system is composed of a single non-linear and (possibly) time varying element, in cascade, or in the feedback path, of a *linear* system.

The non-linear element, although constrained by some conditions, is of a very broad nature and allows a large class of non-linearities, so that uncertainties and ignorance about the exact type of non-linearity are taken care of, and do not impair stability analysis of the system. On the other hand, with a few exceptions [2]-[5], [10], [11] the linear part of the system is assumed, in the vast majority of publications on the subject, to be exactly known and precisely modeled by its transfer function or state-space description, with no uncertainties. This is obviously not a realistic assumption, even if the model is precise with no neglected dynamics, since the physical parameters of the system are never known exactly and, in addition, they are subject to changes.

In [3], [5], [10], [11] continuous-time systems are considered, whereas in [2], [4] discrete-time systems are considered. In [10], [11] parameter uncertainties in the

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linear part of the system are assumed, and *sufficient* conditions for the existence of the Popov stability criterion [8] are derived. Note that the Popov stability criterion is itself only a sufficient condition for stability, but not a necessary one. In [3], parameters uncertainties in the linear part of the system are again assumed, and a *necessary and sufficient* algorithm is derived for the existence of the Popov stability criterion. The price is in the computational complexity of the algorithm. The results in [5] pertain to uncertainty in the frequency response of the linear part of the system, which is a non-parametric form of uncertainty.

In this paper, we consider the parametric form of uncertainty. We use some recent results [7] on the tight envelopes of the frequency response of a family of interval coefficients transfer function of a continuous-time system. These results allow us to obtain sufficient conditions satisfying the Popov criterion, which are computationally tractable. The computational tractability is the main advantage of the present approach relative to previous work. In fact, checking stability of the entire (infinite) family of systems, is reduced to checking the Popov condition for a finite number of systems, each with a fixed coefficient linear part, and each in a prescribed frequency interval. Some of these results are described in [12].

The structure of the paper is as follows: In Section 2, some preliminary derivations are presented and the problem is stated formally. The main results are presented in Section 3, illustrative numerical examples are provided in Section 4, and the paper is concluded in Section 5.

2. PRELIMINARIES AND STATEMENT OF THE PROBLEM

Consider a single-input single-output Luré type continuous-time system, as described in Figure 1 and formulated by its state space representation:

$$\dot{x} = Ax + bF(y,t), \qquad y = cx \tag{1}$$

where

$$x = x(t) \in I\!R^n\ ,\ A \in I\!R^{n\times n}\ ,\ b \in I\!R^{n\times 1}\ ,\ y \in I\!R\ ,\ c \in I\!R^{1\times n} \tag{2}$$

and F is a non-linear (possibly time-variable) continuous function from $I\!R$ to $I\!R$ satisfying the following sector conditions:

$$F(0,t) = 0\ , \quad 0 < K_1 < \frac{F(y,t)}{y} < K_2 < \infty \quad \text{for } y \neq 0. \tag{3}$$

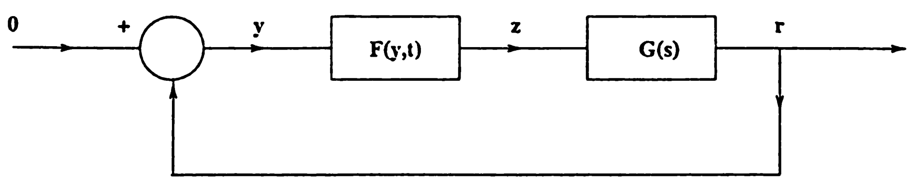


Fig. 1. Luré type continuous-time (possibly time-variable) system.

The relationship between the state space representation and the transfer function $G(s)$ of the linear part of the system, from input z to output $r = -y$, is given by

$$G(s) = c(sI - A)^{-1}b. \tag{4}$$

If the system is *not* time-variable, i.e. $F(y,t) \equiv F(y)$, then it was shown by Popov [8] that such a system is stable (the equilibrium $x = 0$ is asymptotically stable in the large) for every non-linearity as in (3), if the linear part of the system (the transfer function $G(s)$ or the matrix A) is stable and, in addition, there exists a real number q such that

$$\frac{1}{K_2 - K_1} + \operatorname{Re}\left[(1 + j\omega q)\, G(j\omega)\right] + \frac{K_2}{K_2 + K_1}\, |G(j\omega)|^2 > 0 \quad \forall\, \omega \geq 0. \tag{5}$$

The time-variable case has been considered in [9], where it was shown that the system is stable for every non-linearity as in (3), if the Popov criterion is satisfied with $q = 0$. It can be verified that imposing $q = 0$ in (5) yields the condition

$$\operatorname{Re} \frac{1 + K_2 G(j\omega)}{1 + K_1 G(j\omega)} > 0 \quad \forall\, \omega \geq 0. \tag{6}$$

Suppose now that the uncertainty is not only with regard to the non-linear part of the system, expressed in (3), but there is also parametric uncertainty with regard to the linear part of the system namely, the numerator and denominator of the rational transfer function $G(s)$ are interval polynomials. In other words, their coefficients are not known exactly, but only known to take on values in given intervals. It will be shown in the next section how to check the above stability conditions, in this uncertainty case.

3. CHECKING STABILITY IN THE CASE OF UNCERTAINTY

Let

$$G(s) = \frac{A(s)}{B(s)} \tag{7}$$

where

$$A(s) = \sum_{i=0}^{m} a_i s^i\,, \quad B(s) = \sum_{j=0}^{\ell} b_j s^j \tag{8.1}$$

$$\underline{a}_i \leq a_i \leq \overline{a}_i \quad (i = 0, \ldots, m)\,, \quad \underline{b}_j \leq b_j \leq \overline{b}_j \quad (j = 0, \ldots, \ell). \tag{8.2}$$

The first stability condition is to ensure the stability of the family of linear systems (7), (8). To this end, it is only necessary (and sufficient) to check that the following four Kharitonov [6] polynomials with *fixed* coefficients have all their zeros in the open left half complex plane:

$$\begin{aligned}
B_1(s) &= \overline{b}_0 + \overline{b}_1 s + \underline{b}_2 s^2 + \underline{b}_3 s^3 + \overline{b}_4 s^4 + \cdots \\
B_2(s) &= \overline{b}_0 + \underline{b}_1 s + \underline{b}_2 s^2 + \overline{b}_3 s^3 + \overline{b}_4 s^4 + \cdots \\
B_3(s) &= \underline{b}_0 + \overline{b}_1 s + \overline{b}_2 s^2 + \underline{b}_3 s^3 + \underline{b}_4 s^4 + \cdots \\
B_4(s) &= \underline{b}_0 + \underline{b}_1 s + \overline{b}_2 s^2 + \overline{b}_3 s^3 + \underline{b}_4 s^4 + \cdots
\end{aligned} \tag{9}$$

Turn now to the second stability condition, namely (6). It can be verified that its geometrical interpretation is that there is no intersection between the locus of $G(j\omega)$ and a circle whose center is at $(d, 0)$ and whose radius is r, where

$$d = -\frac{1}{2}\left(\frac{1}{K_1} + \frac{1}{K_2}\right), \quad r = \frac{1}{2}\left(\frac{1}{K_1} - \frac{1}{K_2}\right). \tag{10}$$

Thus, condition (6) can be re-written as

$$|-d + G(j\omega)| > r \quad \forall\, \omega \geq 0. \tag{11}$$

Substituting (7) in (11) yields

$$\frac{|A(j\omega) - dB(j\omega)|}{|B(j\omega)|} > r \quad \forall\, \omega \geq 0 \tag{12}$$

where the coefficients of $A(j\omega)$ and $B(j\omega)$ take on values in the intervals (8.2).

Let

$$c_i = a_i - db_i\,, \qquad i = 0, 1, \ldots, \max(\ell, m) \tag{13}$$

where it is understood that $a_i = 0$ for $i > m$ or $b_i = 0$ for $i > \ell$.
Then, a sufficient condition to ensure (12) is that at each frequency $\omega_0 \geq 0$, the ratio between

$$\text{Min}\left|\sum_{i=0}^{\max(\ell,m)} c_i(j\omega_0)^i\right| \quad \text{over} \quad \underline{a}_i - d\underline{b}_i \leq c_i \leq \overline{a}_i - d\overline{b}_i \tag{14}$$

and

$$\text{Max}\left|\sum_{i=0}^{\ell} b_i(j\omega_0)^i\right| \quad \text{over} \quad \underline{b}_i \leq b_i \leq \overline{b}_i \tag{15}$$

is greater than r.

Remark 1. This condition is sufficient but not necessary, since b_i and c_i were assumed to be independent interval coefficients, even though there is a dependency of the value of c_i on the value of b_i.

Remark 2. The intervals of c_i in (14) were determined taking into account the fact that $d < 0$.

The results in [7] are particularly applicable to carry out (14) and (15). It is shown in [7] that (14) must coincide, at each frequency $\omega_0 \geq 0$, with one of the following nine possibilities:

$$\{|C_1(j\omega)|,\ |C_2(j\omega)|,\ |C_3(j\omega)|,\ |C_4(j\omega)|,\ |\text{Re}\,[C_1(j\omega)]|,\ |\text{Re}\,[C_4(j\omega)]|,\ |\text{Im}\,[C_2(j\omega)]|,\ |\text{Im}\,[C_3(j\omega)]|,\ 0\} \tag{16}$$

where $C_i(s)$, $i = 1, \ldots, 4$ are the four Kharitonov polynomials associated with the family

$$C(s) = \sum_{i=0}^{\max(\ell,m)} c_i s^i. \tag{17}$$

Moreover, the frequencies where the minimum in (14) "jumps" from one expression in (16) to another expression in (16) are given by the real roots with odd multiplicity, of the following four equations:

$$\mathrm{Re}\,[C_1(j\omega)] = 0 \tag{18.1}$$
$$\mathrm{Re}\,[C_4(j\omega)] = 0 \tag{18.2}$$
$$\frac{1}{\omega}\,\mathrm{Im}\,[C_2(j\omega)] = 0 \tag{18.3}$$
$$\frac{1}{\omega}\,\mathrm{Im}\,[C_3(j\omega)] = 0. \tag{18.4}$$

It can be readily verified that the various expressions in (16) which coincide with (14) can be chosen according to the following Table.

Table 1. "Sign rule" to choose the pertinent expression for (14).

$\mathrm{Re}\,[C_1(j\omega)]$	$\mathrm{Re}\,[C_4(j\omega)]$	$\mathrm{Im}\,[C_3(j\omega)]$	$\mathrm{Im}\,[C_2(j\omega)]$	(14)
+	+ or 0	+	+ or 0	$\lvert C_4(j\omega)\rvert$
+	−	+	+ or 0	$\lvert \mathrm{Im}\,[C_2(j\omega)]\rvert$
− or 0	−	+	+ or 0	$\lvert C_2(j\omega)\rvert$
− or 0	−	+	−	$\lvert \mathrm{Re}\,[C_1(j\omega)]\rvert$
− or 0	−	− or 0	−	$\lvert C_1(j\omega)\rvert$
+	−	− or 0	−	$\lvert \mathrm{Im}\,[C_3(j\omega)]\rvert$
+	+ or 0	− or 0	−	$\lvert C_3(j\omega)\rvert$
+	+ or 0	+	−	$\lvert \mathrm{Re}\,[C_4(j\omega)]\rvert$
+	−	+	−	0

This is due to the special structure of the polynomials $C_i(s)$ $(i = 1, \ldots, 4)$ e. g.

$$\mathrm{Re}\,[C_1(j\omega)] \geq \mathrm{Re}\,[C_4(j\omega)]$$
$$\mathrm{Im}\,[C_3(j\omega)] \geq \mathrm{Im}\,[C_2(j\omega)]$$

and the special shape of the value set of the family $C(j\omega)$. (See Figure 2).

Also, (15) must coincide, at each frequency $\omega_0 \geq 0$, with one of the following four possibilities

$$\{|B_1(j\omega)|,\ |B_2(j\omega)|,\ |B_3(j\omega)|,\ |B_4(j\omega)|\} \tag{19}$$

where $B_i(s)$, $i = 1, \ldots, 4$, are defined in (9). Moreover, the frequencies where the maximum in (15) "jumps" from one expression in (19) to another expression in (19) are given by the real roots with odd multiplicity, of the following two equations:

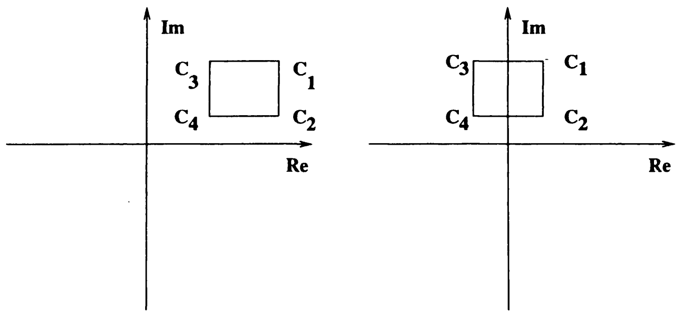
$$b_0^0 - b_2^0\omega^2 + b_4^0\omega^4 - b_6^0\omega^6 + \cdots = 0 \tag{20.1}$$
$$b_1^0 - b_3^0\omega^2 + b_5^0\omega^4 - b_7^0\omega^6 + \cdots = 0 \tag{20.2}$$

where

$$b_i^0 = \underline{b}_i + \overline{b}_i\,, \quad i = 0, \dots, \ell. \tag{21}$$

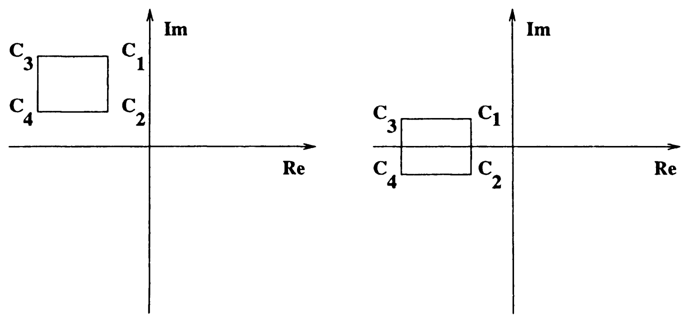
Furthermore, since the polynomials $B_i(s)$, $i = 1, \dots, 4$ are required to be Hurwitz polynomials by the first stability condition, so is the polynomial

$$B^0(s) = \sum_{i=0}^{\ell} b_i^0 s^i \tag{22}$$



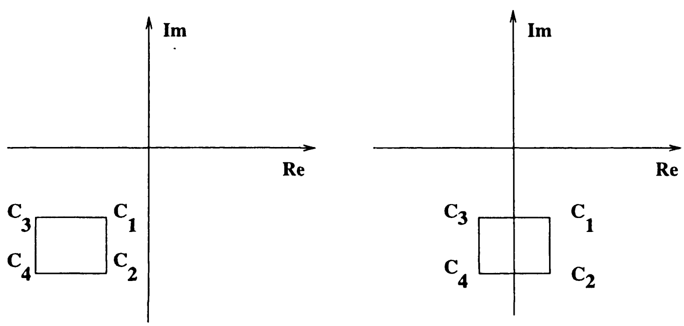
(a) Min | C (jω) | = | C_4 (jω) |

(b) Min | C(jω) | = Im [C_2(jω)]



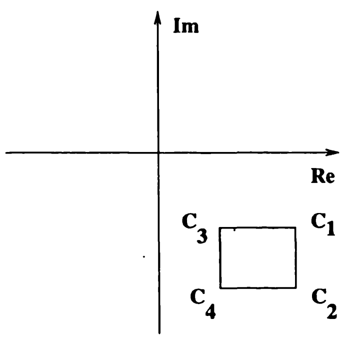
(c) Min | C(jω) | = | C_2(jω) |

(d) Min | C(jω) | = | Re [C_1 (jω)] |

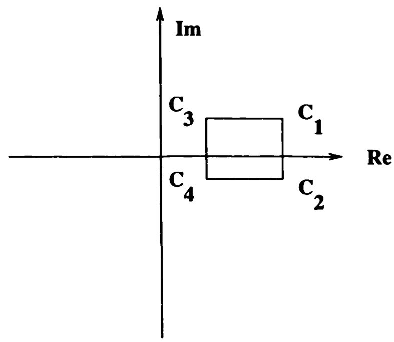


(e) Min | C(jω) | = | C_1 (jω) |

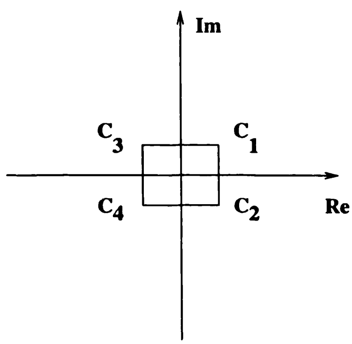
(f) Min | C(j ω) | = | Im [C_3 (jω)] |



(g) Min | C (j ω) | = | C_3 (j ω) |



(h) Min | C (j ω) | = Re [C_4 (j ω)]



(i) Min | C (jω) | = 0

Fig. 2. The minimum amplitude of an interval polynomial.

However, for this special case, it is shown in [7] that (19) can be more explicit: In the interval between $\omega = 0$ to the next "change frequency" (the smallest positive ω which is an odd multiplicity root of (20)), (15) coincides with $|B_1(j\omega)|$. At each consecutive frequency interval, created by the "change frequencies", the order of expressions coinciding with (15) is given by the cycle:

$$|B_1(j\omega)| \Rightarrow |B_3(j\omega)| \Rightarrow |B_4(j\omega)| \Rightarrow |B_2(j\omega)| \Rightarrow |B_1(j\omega)| \Rightarrow \dots \tag{23}$$

To conclude this section, it is clear from the above discussion that to ensure stability of a system as described in Figure 1, with uncertainty in the linear part as formulated in (8), it is sufficient to:

1. Check $B_i(s)$, $i = 1, \dots, 4$, in (9), to be Hurwitz polynomials.
2. Solve Eqns. (18) and (20) to find their real positive roots with odd multiplicity. It can be shown that the maximal number of such roots is $(n-1)$ for Eqns. (20) and $2(n-1)$ for Eqns. (18).
3. Divide the positive frequency axis into a finite number of intervals created by the roots found in Step 2, and choose an arbitrary frequency ω_i in the interior of each of these intervals.
4. Determine which of the expressions in (19) coincides with (15) at each ω_i (and hence, at each interval associated with ω_i) by the sequence (23). Determine which of the expressions in (16) coincides with (14) at each ω_i (and hence, at each interval associated with ω_i) by Table 1.
   Note that the intervals associated with ω_i may be different for (14) than for (15).
5. For each interval created in Step 3, check if the *fixed coefficient* expressions determined in Step 4 satisfy,

$$\frac{(14)}{(15)} > r. \tag{24}$$

4. EXAMPLES

Consider the following nominal plant studied in [1]:

$$G(s) = \frac{a_0 + a_1 s}{b_0 + b_1 s + b_2 s^2} = \frac{1 - s}{1 + 2s + s^2} \tag{25}$$

and let

$$K_1 = 1/2, \qquad K_2 = 1, \tag{26}$$

so that the circle defined in (10) is given by

$$d = -3/2, \qquad r = 1/2. \tag{27}$$

Example 1.

Suppose the coefficients of the nominal plant are subject to 40 % tolerance, namely:

$$0.6 \le a_0 \le 1.4, \qquad -1.4 \le a_1 \le -0.6 \tag{28.1}$$

$$0.6 \le b_0 \le 1.4, \quad 1.2 \le b_1 \le 2.8, \quad 0.6 \le b_2 \le 1.4. \tag{28.2}$$

Using (28) and (27) we have for the coefficients c_i defined in (13), the following intervals (needed in (14)):

$$1.5 \le c_0 \le 3.5\,, \quad 0.4 \le c_1 \le 3.6\,, \quad 0.9 \le c_2 \le 2.1 \tag{29}$$

so that the four Kharitonov polynomials associated with the family (17) are

$$C_1(s) = 3.5 + 3.6s + 0.9s^2 \quad (30.1)$$
$$C_2(s) = 3.5 + 0.4s + 0.9s^2 \quad (30.2)$$
$$C_3(s) = 1.5 + 3.6s + 2.1s^2 \quad (30.3)$$
$$C_4(s) = 1.5 + 0.4s + 2.1s^2. \quad (30.4)$$

Refer now to the steps numbered in the summary of the test algorithm at the end of Section 3.

1. $B_i(s),\ i = 1, \ldots, 4$ are obviously Hurwitz polynomials, since their coefficients $\underline{b}_i$ and $\bar{b}_i$ are all positive and the degree of the polynomials is 2.

2. Solving Eqns. (18) we obtain:

$$\mathrm{Re}[C_1(j\omega)] = 3.5 - 0.9\omega^2 = 0 \Rightarrow \omega = 1.972$$
$$\mathrm{Re}[C_4(j\omega)] = 1.5 - 2.1\omega^2 = 0 \Rightarrow \omega = 0.845$$
$$\mathrm{Im}[C_2(j\omega)] = 0.4\omega$$
$$\mathrm{Im}[C_3(j\omega)] = 3.6\omega.$$

Therefore, the "jump" frequencies of the Min expression in (14) are

$$\omega = 0.845 \quad \text{and} \quad \omega = 1.972\,. \quad (31)$$

Solving Eqns. (20) we obtain

$$b_0^0 = b_2^0 = 2, \qquad b_1^0 = 4$$

and

$$2 - 2\omega^2 = 0 \Rightarrow \omega = 1.$$

Therefore, there is only one "jump" frequency of the Max expression in (15):

$$\omega = 1. \quad (32)$$

3. Let the arbitrary frequencies ω_i in the interior of each interval be chosen as (see Figure 3):

$$\omega_1 = 0.5\,, \quad \omega_2 = 1.3\,, \quad \omega_3 = 3. \quad (33)$$

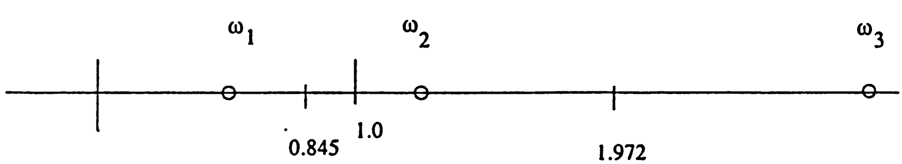


Fig. 3. Divison of the frequency axis into a finite number of intervals (Example 1).

4. In the interval

$$0 \leq \omega \leq 1 \tag{34.1}$$

expression (15) coincides with

$$\begin{aligned} |B_1(j\omega)| &= \left|\bar{b}_0 + \bar{b}_1 j\omega - \underline{b}_2\omega^2\right| \\ &= \left[\left(1.4 - 0.6\omega^2\right)^2 + (2.8\omega)^2\right]^{1/2} \end{aligned} \tag{34.2}$$

In the interval

$$1 \leq \omega < \infty \tag{35.1}$$

expression (15) coincides with

$$\begin{aligned} |B_3(j\omega)| &= \left|\underline{b}_0 + \bar{b}_1 j\omega - \bar{b}_2\omega^2\right| \\ &= \left[\left(0.6 - 1.4\omega^2\right)^2 + (2.8\omega)^2\right]^{1/2}. \end{aligned} \tag{35.2}$$

For $\omega_1 = 0.5$ we readily obtain

$$\mathrm{Re}[C_1(j0.5)] = 3.275 > 0, \quad \mathrm{Re}[C_4(j0.5)] = 0.975 > 0,$$

$$\mathrm{Im}[C_2(j0.5)] = 0.2 > 0, \quad \mathrm{Im}[C_3(j0.5)] = 1.8 > 0.$$

Hence, by Table 1, in the interval

$$0 \leq \omega \leq 0.845 \tag{36.1}$$

expression (14) coincides with

$$|C_4(j\omega)| = \left[\left(1.5 - 2.1\omega^2\right)^2 + (0.4\omega)^2\right]^{1/2}. \tag{36.2}$$

For $\omega_2 = 1.3$ we readily obtain

$$\mathrm{Re}[C_1(j1.3)] = 1.979 > 0, \quad \mathrm{Re}[C_4(j1.3)] = -2.049 < 0,$$

$$\mathrm{Im}[C_2(j1.3)] = 0.52 > 0, \quad \mathrm{Im}[C_3(j1.3)] = 4.68 > 0.$$

Hence, by Table 1, in the interval

$$0.845 \leq \omega \leq 1.972 \tag{37.1}$$

expression (14) coincides with

$$|\mathrm{Im}\,[C_2(j\omega)]| = 0.4\omega. \tag{37.2}$$

For $\omega_3 = 3$ we readily obtain

$$\mathrm{Re}[C_1(j3)] = -4.6 < 0, \quad \mathrm{Re}[C_4(j3)] = -17.4 < 0,$$

$$\mathrm{Im}[C_2(j3)] = 1.2 > 0, \quad \mathrm{Im}[C_3(j3)] = 10.8 > 0.$$

Hence, by Table 1, in the interval

$$1.972 \leq \omega < \infty \tag{38.1}$$

expression (14) coincides with

$$|C_2(j\omega)| = \left[\left(3.5 - 0.9\omega^2\right)^2 + (0.4\omega)^2\right]^{1/2}. \tag{38.2}$$

5. Using the results in Step 4, stability is ensured by the following four conditions:

$$\frac{(36.2)}{(34.2)} > r = \frac{1}{2} \quad \text{for} \quad 0 \leq \omega \leq 0.845 \tag{39.1}$$

$$\frac{(37.2)}{(34.2)} > r = \frac{1}{2} \quad \text{for} \quad 0.845 \leq \omega \leq 1 \tag{39.2}$$

$$\frac{(37.2)}{(35.2)} > r = \frac{1}{2} \quad \text{for} \quad 1 \leq \omega \leq 1.972 \tag{39.3}$$

$$\frac{(38.2)}{(35.2)} > r = \frac{1}{2} \quad \text{for} \quad 1.972 \leq \omega < \infty. \tag{39.4}$$

It is readily verified that, say (39.1), is *not* satisfied. Hence, we cannot determine that the system is robustly stable with the uncertainty of 40 % tolerance. Note, however, that the nonlinear nominal system (25) – (26) is stable. Therefore, it is reasonable to assume that the system is stable for a smaller tolerance. To this end, consider

Example 2.

Suppose the coefficients of the nominal plant (25) are subject to 10 % tolerance, namely:

$$0.9 \leq a_0 \leq 1.1 \qquad -1.1 \leq a_1 \leq -0.9 \tag{40.1}$$

$$0.9 \leq b_0 \leq 1.1 \quad 1.8 \leq b_1 \leq 2.2 \quad 0.9 \leq b_2 \leq 1.1 \tag{40.2}$$

and

$$2.25 \leq c_0 \leq 2.75 \quad 1.6 \leq c_1 \leq 2.4 \quad 1.35 \leq c_2 \leq 1.65 \tag{41}$$

so that the four Kharitonov polynomials associated with the family (17) are:

$$C_1(s) = 2.75 + 2.4s + 1.35s^2 \tag{42.1}$$

$$C_2(s) = 2.75 + 1.6s + 1.35s^2 \tag{42.2}$$

$$C_3(s) = 2.25 + 2.4s + 1.65s^2 \tag{42.3}$$

$$C_4(s) = 2.25 + 1.6s + 1.65s^2. \tag{42.4}$$

The test algorithm is thus:

1. $B_i(s)$, $i = 1, \ldots, 4$ are obviously Hurwitz polynomials, as in Example 1.

2. Solving Eqns. (18) we obtain:

$$\text{Re}[C_1(j\omega)] = 0 \Rightarrow \omega = 1.427 \tag{43.1}$$

$$\text{Re}[C_4(j\omega)] = 0 \Rightarrow \omega = 1.168. \tag{43.2}$$

Solving Eqns. (20) we obtain

$$\omega = 1. \tag{43.3}$$

3. Let the arbitrary frequencies ω_i in the interior of each interval be chosen as (see Figure 4):

$$\omega_1 = 0.5\,, \quad \omega_2 = 1.3\,, \quad \omega_3 = 2. \tag{44}$$

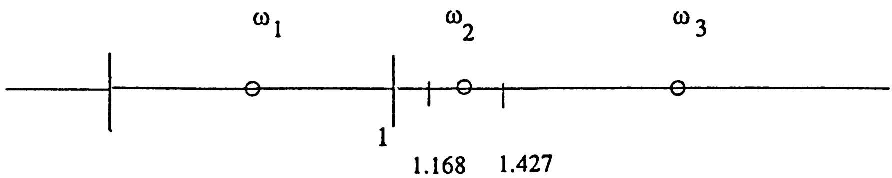


Fig. 4. Divison of the frequency axis into a finite number of intervals (Example 2).

4. In the interval

$$0 \le \omega \le 1 \tag{45.1}$$

expression (15) coincides with

$$|B_1(j\omega)| = \left[\left(1.1 - 0.9\omega^2\right)^2 + (2.2\omega)^2\right]^{1/2}. \tag{45.2}$$

In the interval

$$1 \le \omega < \infty \tag{46.1}$$

expression (15) coincides with

$$|B_3(j\omega)| = \left[\left(0.9 - 1.1\omega^2\right)^2 + (2.2\omega)^2\right]^{1/2}. \tag{46.2}$$

For $\omega_1 = 0.5$ we readily obtain

$$\operatorname{Re} C_1 > 0, \quad \operatorname{Re} C_4 > 0, \quad \operatorname{Im} C_2 > 0, \quad \operatorname{Im} C_3 > 0.$$

Hence, by Table 1, in the interval

$$0 \le \omega \le 1.168 \tag{47.1}$$

expression (14) coincides with

$$|C_4(j\omega)| = \left[\left(2.25 - 1.65\omega^2\right)^2 + (1.6\omega)^2\right]^{1/2}. \tag{47.2}$$

For $\omega_2 = 1.3$ we readily obtain

$$\operatorname{Re} C_1 > 0, \quad \operatorname{Re} C_4 < 0, \quad \operatorname{Im} C_2 > 0, \quad \operatorname{Im} C_3 > 0.$$

Hence, by Table 1, in the interval

$$1.168 \le \omega \le 1.427 \tag{48.1}$$

expression (14) coincides with

$$\operatorname{Im}\left[C_2(j\omega)\right] = 1.6\omega. \tag{48.2}$$

For $\omega_3 = 2$ we readily obtain

$$\operatorname{Re} C_1 < 0, \quad \operatorname{Re} C_4 < 0,$$

$$\operatorname{Im} C_2 > 0, \quad \operatorname{Im} C_3 > 0.$$

Hence, by Table 1, in the interval

$$1.427 \leq \omega < \infty \tag{49.1}$$

expression (14) coincides with

$$|C_2(j\omega)| = \left[\left(2.75 - 1.35\omega^2\right)^2 + (1.6\omega)^2\right]^{1/2}. \tag{49.2}$$

5. Using the results in Step 4, stability is ensured by the following four conditions:

$$\frac{(47.2)}{(45.2)} > r = \frac{1}{2} \quad \text{for} \quad 0 \leq \omega \leq 1 \tag{50.1}$$

$$\frac{(47.2)}{(46.2)} > r = \frac{1}{2} \quad \text{for} \quad 1 \leq \omega \leq 1.168 \tag{50.2}$$

$$\frac{(48.2)}{(46.2)} > r = \frac{1}{2} \quad \text{for} \quad 1.168 \leq \omega \leq 1.427 \tag{50.3}$$

$$\frac{(49.2)}{(46.2)} > r = \frac{1}{2} \quad \text{for} \quad 1.427 \leq \omega < \infty. \tag{50.4}$$

It is readily verified that all four conditions are satisfied, hence we conclude that the nonlinear system (25) – (26) *is* robustly stable with the uncertainty of 10 % tolerance.

5. CONCLUSION

A non-linear time-varying system is considered, where both the non-linear part and the linear part are only partially known. There is a lot of uncertainty about the behaviour of the system which, presumably, makes it very difficult to analyze the system, even just for stability. Nevertheless, sufficient conditions, for the parametric type uncertainty of the linear part and Luré type uncertainty of the nonlinear part, which ensure stability of the system and which are computationally tractable, are presented here. These conditions are based on some recently derived results on the frequency response of continuous-time systems with uncertainties formulated by interval transfer functions. Using these results, we are able to reduce the necessity to check the Popov condition for an infinite set of systems, to checking the Popov condition for a finite number of fixed coefficients systems, each in a prescribed (calculated) frequency interval.

A final remark concerns the sufficiency of the conditions. Since necessary and sufficient conditions for stability of a Luré type non-linear system do not exist even for the standard case of a completely specified and exact linear part and time-invariant non linear part, it would be too ambitious and non-realistic to expect such for the case with uncertainty.

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