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ENTROPY OF T–SUMS AND T–PRODUCTS OF L–R FUZZY NUMBERS

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In the paper the entropy of L–R fuzzy numbers is studied. It is shown that for a given norm function, the computation of the entropy of L–R fuzzy numbers reduces to using a simple formula which depends only on the spreads and shape functions of incoming numbers. In detail the entropy of T_M–sums and T_M–products of L–R fuzzy numbers is investigated. It is shown that the resulting entropy can be computed only by means of the entropy of incoming fuzzy numbers or by means of their parameters without the computation of membership functions of corresponding sums or products. Moreover, the results for some other t-norm–based sums and products are derived. Several examples are included.

1. INTRODUCTION

Since early beginnings of fuzzy set theory the problem of measuring the degree of fuzziness of fuzzy sets was discussed by many authors, e. g., by De Luca and Termini [8], Kaufmann [13], Knopfmacher [17], Loo [21], Trilas and Riera [34], Yager [37], Ebanks [11], Sander [32], Pal and Bezdek [30], Benvenuti, Vivona and Divari [3], Vivona [36], among others, compare also [1, 2, 4, 5, 35].

In general, a measure of fuzziness H is a mapping which assigns to each fuzzy subset F of a considered universal set X a non–negative number $H(F)$ that quantifies the degree of fuzziness present in F. All measures of fuzziness should satisfy at least two very natural properties, namely

(i) fuzziness of crisp sets should be equal to zero,

(ii) if a fuzzy set F_1 is sharper than F_2, which expressed by membership degrees means that

$$F_1(x) \le F_2(x) \quad \text{if} \quad F_2(x) < 0.5$$

and

$$F_1(x) \ge F_2(x) \quad \text{if} \quad F_2(x) > 0.5$$

for all $x \in X$, then $H(F_1)$ should not be greater than $H(F_2)$.

Usually also conditions concerning the maximal fuzziness and equality of degrees of fuzziness of a fuzzy set F and its complement F^c for each F are required.

Various types of measures of fuzziness were proposed and investigated in the literature: entropy like measures, distance like measures, or general measures of fuzziness, see [31]. The concept of "entropy" in fuzzy set theory has been already mentioned by Zadeh [38]. De Luca and Termini using the functional formally similar to the Shannon entropy and its generalization, defined the "entropy" of a fuzzy set F (on a finite universal set) by

$$H(F) = -K \sum_{i=1}^{n} \left(F(x_i) \log F(x_i) + (1 - F(x_i)) \log (1 - F(x_i))\right) ,$$

where K is a positive constant and $F(x_i)$ is a membership degree of the element x_i in F.
$H(F)$ can be regarded as an "entropy" in the sense that it measures the uncertainty about presence or absence a certain property described by F. A deeper explanation can be found in [8, 11, 31, 32]. An exhaustive overview of measures of fuzziness is given in [31], including the relevant philosophical and axiomatical backgrounds.

In this paper we will deal with special types of fuzziness measures (fuzzy entropy measures) defined on the set of all fuzzy quantities. A special attention will be paid to the fuzzy entropy of L–R fuzzy numbers. Finally, the entropy of T–sums and T–products of L–R fuzzy numbers will be studied.

2. BASIC NOTATIONS AND DEFINITIONS

Denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy quantities, i. e., fuzzy subsets of the real line which can be identified with Borel measurable functions $F : \mathbb{R} \to [0, 1]$.
Let $\mathcal{C}(\mathbb{R})$ denotes the set of all crisp Borel subsets of $\mathbb{R}$. Any $C \in \mathcal{C}(\mathbb{R})$ can be regarded as a special fuzzy set for which its membership degree assumes only the values 0 and 1, thus $\mathcal{C}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$.
On the set $\mathcal{F}(\mathbb{R})$ we will consider the following (partial) order [6]:
Let $F_1, F_2 \in \mathcal{F}(\mathbb{R})$. We say that F_1 is sharper than F_2, with the notation $F_1 \succeq F_2$, if

$$|F_1(x) - 0.5| \geq |F_2(x) - 0.5| \quad \text{for all} \;\; x \in \mathbb{R}. \tag{1}$$

The relation $\succeq$ is larger with respect the relation mentioned in Introduction. Fuzzy sets comparable by the relation $\succeq$ need not be comparable by that one. We remark that for any fuzzy subset F it is $F \succeq F^c$ and $F^c \succeq F$ together, i. e., F and F^c are equivalent with respect to the relation $\succeq$. Note that only the standard complement $F^c(x) = 1 - F(x)$ is considered.

Definition 1. A mapping $H : \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$ is called an entropy measure if it satisfies the properties:

(i) $H(F) = 0 \quad \text{if } F \in \mathcal{C}(\mathbb{R})$,

(ii) $H(F_1) \leq H(F_2)$ whenever $F_1, F_2 \in \mathcal{F}(\mathbb{R})$ such that $F_1 \succeq F_2$.

In what follows we will define entropy measures by means of so-called norm functions [17].

Definition 2. A continuous function $h : [0,1] \to [0,1]$ with the properties:

(i) $h(0) = h(1) = 0$, and $h(0.5) = 1$,

(ii) h is non-decreasing on the interval $[0, 0.5]$,

(iii) $h(x) = h(1-x)$ for each $x \in [0,1]$,

will be called a norm function.

Example 1. The following functions are norm functions:

(i) $h_1(x) = \min\{2x, 2-2x\}, \quad x \in [0,1]$,

(ii) $h_k(x) = 1 - |2x-1|^k, \; x \in [0,1], \; k \in]0, \infty[$,

(iii) $h_s(x) = -x \log x - (1-x) \log(1-x), \; x \in [0,1]$, where $0. \log 0 = 0$ by convention,

(iv) $h_l(x) = 4x(1-x), \; x \in [0,1]$.

Note that the function h_1 is the classical "tent" function, functions h_k are its generalizations (for $k = 1$ (ii) gives the function h_1 from (i)), h_s is called the Shannon function, because it has been derived from the Shannon entropy and h_l is the logistic function [35].

The global entropy $H(F)$ of a fuzzy quantity $F \in \mathcal{F}(\mathbb{R})$ can be defined by means of a norm function h and the Lebesgue integral with respect to the Lebesgue measure as follows:

$$H(F) = \int_{\mathbb{R}} h(F(x))\, \mathrm{d}x\,, \quad F \in \mathcal{F}(\mathbb{R})\,. \tag{2}$$

Using the properties of norm functions and the monotonicity of the integral, it can be directly shown that the mapping $H : \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$ defined by (2), is an entropy measure in the sense of Definition 1.

3. ENTROPY OF L–R FUZZY NUMBERS

In what follows we will deal only with the entropy of L–R fuzzy numbers. L–R fuzzy numbers were introduced by Dubois and Prade [9]. Their arithmetic was discussed in several papers, see, e. g., [7, 9, 10, 12, 22, 23, 26, 28]. An L–R fuzzy number is a convex normal fuzzy subset of the real line $\mathbb{R}$ with continuous membership function and bounded support. The formal definition follows.

Definition 3. A fuzzy set $F \in \mathcal{F}(\mathbb{R})$ is called an L–R fuzzy number with the notation $F = (a, b, \alpha, \beta)_{LR}$, if its membership function is given by

$$F(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ L\left(\frac{a-x}{\alpha}\right) & \text{if } x \in [a-\alpha, a[\\ R\left(\frac{x-b}{\beta}\right) & \text{if } x \in]b, b+\beta] \\ 0 & \text{otherwise,} \end{cases}$$

where $a, b \in \mathbb{R}$, $\alpha, \beta > 0$ and $L, R : [0,1] \to [0,1]$ are continuous non–increasing functions such that $L(x) = R(y) = 1$ iff $x = y = 0$ and $L(x) = R(y) = 0$ iff $x = y = 1$.

The functions L, R are called shape functions and the constants α, β are spreads. The set of all L–R fuzzy numbers will be denoted by $\mathcal{S}$.
Recall that for $L(u) = R(u) = 1 - u$, $u \in [0,1]$, we obtain linear fuzzy numbers, which are called triangular if $a = b$, and trapezoidal if $a < b$. The notation of a linear fuzzy number will be only $F = (a, b, \alpha, \beta)$.

For L–R fuzzy numbers the entropy defined by (2) can be simplified and by a direct computation it can be shown that the entropy $H(F)$ of an L–R fuzzy number F depends only on h, L, R and the spreads α, β.

Proposition 1. If $F = (a, b, \alpha, \beta)_{LR}$, then $H(F) = \alpha c_L + \beta c_R$, where

$$c_L = \int_0^1 (h \circ L)(u)\, \mathrm{d}u\,, \qquad c_R = \int_0^1 (h \circ R)(u)\, \mathrm{d}u\,.$$

Proof. Since $\operatorname{supp}(F) =]a-\alpha, b+\beta[$ and $h(0) = 0$, instead of (2) we can immediately write

$$H(F) = \int_{a-\alpha}^{b+\beta} h(F(x))\, \mathrm{d}x\,,$$

and next

$$H(F) = \int_{a-\alpha}^{a} h\left(L\left(\frac{a-x}{\alpha}\right)\right) \mathrm{d}x + \int_a^b h(1)\, \mathrm{d}x + \int_b^{b+\beta} \left(R\left(\frac{x-b}{\beta}\right)\right) \mathrm{d}x.$$

Using $h(1) = 0$ and substitutions $\frac{a-x}{\alpha} = u$ and $\frac{x-b}{\beta} = v$, respectively, we obtain the claim. □

Before going further let us recall that the multiplication by a real constant c in the set $\mathcal{S}$ is defined by

$$c(a, b, \alpha, \beta)_{LR} = \begin{cases} (ca, cb, c\alpha, c\beta)_{LR} & c \geq 0 \\ (cb, ca, -c\beta, -c\alpha)_{RL} & c < 0. \end{cases} \tag{3}$$

Note that if $c = 0$, $c(a,b,\alpha,\beta)_{LR} = \mathbf{1}_{\{0\}}$ is not more an L–R fuzzy number in the sense of Definition 3. However, admitting $\alpha = 0$ or $\beta = 0$ in Definition 3 will not influence any of the discussed properties, only the continuity of corresponding fuzzy numbers may be violated. So, e.g., $F = (a,b,0,0)_{LR}$ is in fact a crisp subset of $\mathbb{R}$ and hence $H(F) = 0$.

Proposition 2. The entropy measure defined by (2) is positively homogeneous on $\mathcal{S}$, i.e.,

$$H(cF) = |c| H(F)\,, \quad \text{for all } c \in \mathbb{R},\ F \in \mathcal{S}\,.$$

Proof. The claim is an easy consequence of (3) and Proposition 1. For $c \geq 0$ the claim is evident. We show it only for $c < 0$.
Consider $F = (a,b,\alpha,\beta)_{LR}$. Since $cF = (cb, ca, -c\beta, -c\alpha)_{RL}$, for its entropy we obtain

$$H(cF) = -c\beta c_R - c\alpha c_L = -c(\alpha c_L + \beta c_R) = |c| H(F)\,. \qquad \square$$

In the following Examples 2 – 5 we compute the entropy of a linear fuzzy number $F = (a,b,\alpha,\beta)$ by means of norm functions introduced in Example 1 (i) – (iv), and Proposition 1. Some of these examples can also be found in [35].

Example 2. For the tent function h_1 we have

$$c_L = c_R = \int_0^1 h_1(1-u)\,\mathrm{d}u = \int_0^{0.5} 2u\,\mathrm{d}u + \int_{0.5}^1 (2-2u)\,\mathrm{d}u = 0.5\,.$$

Therefore

$$H_1\left((a,b,\alpha,\beta)\right) = \frac{\alpha+\beta}{2}\,.$$

Example 3. For norm functions h_k from Example 1 (ii), we have

$$c_L = c_R = \int_0^1 h_k(1-u)\,\mathrm{d}u = \int_0^{0.5} \left(1-(1-2u)^k\right)\,\mathrm{d}u + \int_{0.5}^1 \left(1-(2u-1)^k\right)\,\mathrm{d}u = \frac{k}{k+1}\,,$$

which implies

$$H_k\left((a,b,\alpha,\beta)\right) = \frac{k}{k+1}(\alpha+\beta)\,.$$

Example 4. For the Shannon function h_s we obtain

$$\begin{aligned} c_L = c_R &= \int_0^1 \left(-(1-u)\log_2(1-u) - u\log_2 u\right)\,\mathrm{d}u \\ &= -2\int_0^1 u\log_2 u\,\mathrm{d}u = \frac{1}{2\log 2} = \log_4 e\,. \end{aligned}$$

Thus

$$H_s\left((a,b,\alpha,\beta)\right) = (\alpha+\beta)\log_4 e\,.$$

Example 5. For the logistic function h_l

$$c_L = c_R = \int_0^1 4u(1-u)\,\mathrm{d}u = \frac{2}{3},$$

and so

$$H_l((a,b,\alpha,\beta)) = \frac{2}{3}(\alpha+\beta).$$

It is evident that for an arbitrary norm function h and a fuzzy quantity $F \in \mathcal{F}(\mathbb{R})$

$$0 \le H(F) \le m(\operatorname{supp}(F)),$$

where m is the standard Lebesgue measure. For the class $\mathcal{S}$ of L–R fuzzy numbers we can prove the following result.

Proposition 3. Let h be a given norm function and $\mathcal{S}_{c,d}$ be the set of all L–R fuzzy numbers with support $]c,d[$. Then for the entropy H induced by h via (2) we have $\operatorname{Ran}\left(H_{|\mathcal{S}_{c,d}}\right) =]0,d-c[$.

Proof. Directly from (2) it is clear that for each $F \in \mathcal{S}_{c,d}$ and for each norm function h

$$0 < H(F) < d-c$$

holds.
Now we show that for an arbitrary element $k \in]0,d-c[$ there exists $F^* \in \mathcal{S}_{c,d}$ such that $H(F^*) = k$. To this purpose choose any $\alpha \in]0,\frac{d-c}{2}]$. By Proposition 1 the entropy of any L–R fuzzy number of the form $(c+\alpha,d-\alpha,\alpha,\alpha)_{LR}$ is $H((c+\alpha,d-\alpha,\alpha,\alpha)_{LR}) = \alpha(c_L+c_R)$.
Now, let $F_n = (c+\alpha_n,d-\alpha_n,\alpha_n,\alpha_n)_{K_nK_n}$, $n \in \mathbb{N}$, where

$$K_n(x) = \frac{1+(1-2x)^{2n-1}}{2} \quad \text{and} \quad \alpha_n \in \left]0,\frac{d-c}{2}\right].$$

Since

$$\lim_{n\to\infty} \frac{1+(1-2u)^{2n-1}}{2} = \frac{1}{2},$$

it holds

$$\lim_{n\to\infty} c_{K_n} = \lim_{n\to\infty} \int_0^1 h\left(\frac{1+(1-2u)^{2n-1}}{2}\right) \mathrm{d}u = 1,$$

and we can conclude that $\sup_{n\in\mathbb{N}} c_{K_n} = 1$.
For a given $k \in]0,d-c[$ we have $\frac{k}{d-c} < 1$, and thus there exists $p \in \mathbb{N}$ such that $c_{K_p} > \frac{k}{d-c}$.
Put $\alpha_p = \frac{k}{2c_{K_p}}$. Then $\alpha_p < \frac{d-c}{2}$ and for $F^* = F_p = (c+\alpha_p,d-\alpha_p,\alpha_p,\alpha_p)_{K_pK_p}$ we obtain

$$H(F^*) = \frac{k}{2c_{K_p}} 2c_{K_p} = k.$$

So we have shown that on the interval $[c,d]$ we are able for any fixed norm function h and arbitrarily chosen number $k \in]0,d-c[$, to construct an L–R fuzzy number whose entropy is just equal to k. □

Remark 1. Similarly, it is possible to show that for any fixed L–R fuzzy number $F = (a, b, \alpha, \beta)_{LR}$ with strictly monotone shapes L, R, for any $k \in]0, \alpha + \beta[$ we can find a norm function h such that the corresponding entropy of F induced by h is $H(F) = k$.

4. ENTROPY OF T-SUMS OF L–R FUZZY NUMBERS

Using the generalized extension principle of Zadeh we can introduce the sum of fuzzy quantities $F_1, \ldots, F_n \in \mathcal{F}(\mathbb{R})$ based on a t-norm T (T-sum for short) as a fuzzy quantity with the membership function

$$F(x) = (F_1 \oplus_T \cdots \oplus_T F_n)(x) = \sup_{\sum x_i = x} T\left(F_1(x_1), \ldots, F_n(x_n)\right) \ , \ x \in \mathbb{R}.$$

If $F_1, \ldots, F_n \in \mathcal{S}$ and T is continuous on $[0,1[^2$, also $F \in \mathcal{S}$.
For the definition of a t–norm and more properties we refer the reader e. g. to [14, 16, 33].

In what follows we will investigate the entropy of T-sums of L–R fuzzy numbers. Our aim is to discuss the possibility of computing the resulting entropy $H(F)$ only by means of entropy $H(F_i)$ of incoming summands or their parameters.

Let us briefly recall two basic cases. Consider L–R fuzzy numbers $F_1, \ldots, F_n$ with the same shape functions L, R, i. e., let $F_i = (a_i, b_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$. By the well–known results [9, 10]

$$(F_1 \oplus_{T_M} \cdots \oplus_{T_M} F_n) = \left(\sum a_i, \sum b_i, \sum \alpha_i, \sum \beta_i\right)_{LR},$$

and

$$(F_1 \oplus_{T_D} \cdots \oplus_{T_D} F_n) = \left(\sum a_i, \sum b_i, \max \alpha_i, \max \beta_i\right)_{LR},$$

where T_M is the minimum t-norm defined by $T_M(x, y) = \min(x, y)$, $(x, y) \in [0,1]^2$, and T_D is the drastic product given by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

As a direct consequence of Proposition 1 we obtain the following results.

Corollary 1. Let $F_i = (a_i, b_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$. Then

(i) $H(F_1 \oplus_{T_M} \cdots \oplus_{T_M} F_n) = \sum\limits_{i=1}^{n} H(F_i)$,

(ii) $H(F_1 \oplus_{T_D} \cdots \oplus_{T_D} F_n) = \left(\max\limits_{1 \le i \le n} \alpha_i\right) c_L + \left(\max\limits_{1 \le i \le n} \beta_i\right) c_R$.

The first result means that in the considered case the entropy is T_M–additive. This result was recently proved for linear fuzzy numbers in [35]. Note that if $\alpha_i = \beta_i$ for $i = 1, \dots, n$, then $H(F_1 \oplus_{T_D} \cdots \oplus_{T_D} F_n) = \max_{1 \le i \le n} H(F_i)$.

For the t–norm T_M we can prove much stronger result which is based on the properties of α–cuts.
Recall that α–cuts of a fuzzy set $F \in \mathbb{R}$ are crisp sets

$$F^{(\alpha)} = \{x \in \mathbb{R} \mid F(x) \ge \alpha\}, \quad \alpha \in]0,1],$$

and strong α–cuts are crisp sets

$$F^{[\alpha]} = \{x \in \mathbb{R} \mid F(x) > \alpha\}, \quad \alpha \in [0,1[.$$

Now, consider a norm function h. It can be regarded as a convex normal fuzzy set with continuous membership function on $[0,1]$. For each α–cut $h^{(a)}$ put

$$l_\alpha = \inf h^{(\alpha)} \quad \text{and} \quad r_\alpha = \sup h^{(\alpha)}.$$

Because of properties of h, $h^{(\alpha)} = [l_\alpha, r_\alpha]$.
Further, let $F \in \mathcal{F}(\mathbb{R})$. Then $h \circ F \in \mathcal{F}(\mathbb{R})$ and

$$\begin{aligned} (h \circ F)^{(\alpha)} &= \{x \in \mathbb{R} \mid h(F(x)) \ge \alpha\} = \{x \in \mathbb{R} \mid F(x) \in h^{(\alpha)}\} \\ &= \{x \in \mathbb{R} \mid F(x) \in [l_\alpha, r_\alpha]\} = F^{-1}([l_\alpha, r_\alpha]) = F^{(l_\alpha)} \setminus F^{[r_\alpha]}. \end{aligned} \tag{4}$$

Lemma 1. Let $F_i \in \mathcal{S}$ (with any shapes L_i, R_i), $i = 1, \dots, n$, and $F = F_1 \oplus_{T_M} \cdots \oplus_{T_M} F_n$. Then with the previous notation we have

$$m\left((h \circ F)^{(\alpha)}\right) = \sum_{i=1}^{n} m\left(F^{(l_\alpha)} \setminus F^{[r_\alpha]}\right),$$

where m is the Lebesgue measure on $\mathbb{R}$.

Proof. It is known [29] that

$$F^{(\alpha)} = \sum F_i^{(\alpha)} \quad \text{and} \quad F^{[\alpha]} = \sum F_i^{[\alpha]}. \tag{5}$$

Note that right–hand sides expressions mean the sum of intervals defined as usually. By (4) and (5) we can write

$$(h \circ F)^{(\alpha)} = F^{(l_\alpha)} \setminus F^{[r_\alpha]} = \left(\sum F_i^{(l_\alpha)}\right) \setminus \left(\sum F_i^{[r_\alpha]}\right). \tag{6}$$

For a fixed $\alpha \in]0,1]$ denote

$$F_i^{(l_\alpha)} = [a_i, b_i] \quad \text{and} \quad F_i^{[r_\alpha]} =]c_i, d_i[, \quad i = 1, \dots, n.$$

Since $l_\alpha \le r_\alpha$, $(c_i, d_i) \subset [a_i, b_i]$ holds, and thus

$$F_i^{(l_\alpha)} \setminus F_i^{[r_\alpha]} = [a_i, c_i] \cup [d_i, b_i].$$

For the Lebesgue measure of the above difference we obtain

$$\begin{aligned} m\left(F_i^{(l_\alpha)} \setminus F_i^{[r_\alpha]}\right) &= c_i - a_i + b_i - d_i = (b_i - a_i) + (d_i - c_i) \\ &= m\left(F_i^{(l_\alpha)}\right) - m\left(F_i^{[r_\alpha]}\right). \end{aligned} \tag{7}$$

Next,

$$\sum F_i^{(l_\alpha)} = \sum [a_i, b_i] = \left[\sum a_i, \sum b_i\right] = [a, b] \tag{8}$$

and

$$\sum F_i^{[r_\alpha]} = \sum]c_i, d_i[= \left]\sum c_i, \sum d_i\right[=]c, d[\,. \tag{9}$$

Summarizing (6), (8), (9) we obtain

$$(h \circ F)^{(\alpha)} = [a, b] \setminus]c, d[= [a, c] \cup [d, b]\,.$$

Thus

$$\begin{aligned} m\left((h \circ F)^{(\alpha)}\right) &= c - a + b - d = (b - a) - (d - c) = \sum_{i=1}^{n} ((b_i - a_i) - (d_i - c_i)) \\ &= \sum_{i=1}^{n} \left(m\left(F_i^{(l_\alpha)}\right) - m\left(F_i^{[r_\alpha]}\right)\right) = \sum_{i=1}^{n} m\left(F_i^{(l_\alpha)} \setminus F_i^{[r_\alpha]}\right), \end{aligned}$$

which is the claim. □

Theorem 1. Let $F_i \in \mathcal{S}$ (with any shapes L_i, R_i), $i = 1, \ldots, n$, and $F = F_1 \oplus_{T_M} \cdots \oplus_{T_M} F_n$. Then

$$H(F) = \sum_{i=1}^{n} H(F_i)\,.$$

Proof. Applying (2) and the properties of the Lebesgue integral, especially its relationship with the Choquet integral, we obtain

$$\begin{aligned} H(F) &= \int_{\mathbb{R}} h(F(x))\,\mathrm{d}x = \int_0^1 m\left(\{x \in \mathbb{R} \mid h(F(x)) \geq \alpha\}\right)\,\mathrm{d}\alpha \\ &= \int_0^1 m\left((h \circ F)^{(\alpha)}\right)\,\mathrm{d}\alpha\,. \end{aligned}$$

We can continue applying Lemma 1:

$$\begin{aligned} H(F) &= \int_0^1 \left(\sum_{i=1}^{n} m\left(F_i^{(l_\alpha)} \setminus F_i^{[r_\alpha]}\right)\right)\,\mathrm{d}\alpha = \sum_{i=1}^{n} \left(\int_0^1 m\left(F_i^{(l_\alpha)} \setminus F_i^{[r_\alpha]}\right)\,\mathrm{d}\alpha\right) \\ &= \sum_{i=1}^{n} \left(\int_0^1 \left((h \circ F_i)^{(\alpha)}\right)\,\mathrm{d}\alpha\right) = \sum_{i=1}^{n} \left(\int_{\mathbb{R}} h(F_i(x))\,\mathrm{d}x\right) = \sum_{i=1}^{n} H(F_i)\,, \end{aligned} \tag{10}$$

which proves the claim. □

Remark 2. (i) When all shapes of fuzzy numbers F_i involved in the assumptions of Theorem 1 have finite negative derivatives, the claim of Theorem 1 can be proved in a more transparent way.
Indeed, by [25] the T_M–sum F of F_i is of the form $F = (a, b, \alpha, \beta)_{LR}$ with parameters

$$a = \sum_{i=1}^{n} a_i\,, \quad b = \sum_{i=1}^{n} b_i\,, \quad \alpha = \sum_{i=1}^{n} \alpha_i\,, \quad \beta = \sum_{i=1}^{n} \beta_i$$

and the shapes L, R for which the following is valid:

$$a - \alpha L^{-1}(u) = \sum_{i=1}^{n} \left(a_i - \alpha_i L_i^{-1}(u)\right)\,, \quad u \in [0,1]\,,$$

i. e.,

$$L^{-1}(u) = \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} L_i^{-1}(u)\,,$$

and analogously,

$$R^{-1}(u) = \sum_{i=1}^{n} \frac{\beta_i}{\beta} R_i^{-1}(u)\,.$$

Now we can compute c_L and c_R. It holds:

$$\begin{aligned} c_L &= \int_0^1 h(L(u))\,\mathrm{d}u_{|L(u)=t} = -\int_0^1 h(t)L^{-1}(t)\,\mathrm{d}t = -\int_0^1 \left(h(t)\sum_{i=1}^{n}\frac{\alpha_i}{\alpha} L_i^{-1}(t)\right)\mathrm{d}t \\ &= \sum_{i=1}^{n}\frac{\alpha_i}{\alpha}\left(-\int_0^1 h(t)L_i^{-1}(t)\,\mathrm{d}t\right) = \sum_{i=1}^{n}\frac{\alpha_i}{\alpha} c_{L_i}\,. \end{aligned}$$

Similarly,

$$c_R = \sum_{i=1}^{n} \frac{\beta_i}{\beta} c_{R_i}\,.$$

Direct application of Proposition 1 gives

$$H(F) = \alpha c_L + \beta c_R = \sum_{i=1}^{n}(\alpha_i c_{L_i} + \beta_i c_{R_i}) = \sum_{i=1}^{n} H(F_i)\,.$$

(ii) Note that the class of the shapes satisfying the requirements of (i) is dense in the class of all shapes and hence, due to the uniform continuity of all involved functions to be integrated, Remark 2(i) implies Theorem 1.

Consider again L–R fuzzy numbers $F_i = (a_i, b_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, i. e., with the same shapes. In [18, 19, 20, 27] the problems of preserving the shapes under t–norm–based additions were studied. These results have interesting impact to the entropy of T–sums.

It is well-known that T_M– and T_D–additions preserve all types of shapes, i. e., the corresponding sum has the same shape as the summands. The consequences of these facts for entropy were formulated in Corollary 1.
However, the shapes of T–sums are preserved also in some other cases. For instance, the T_L–sum of linear fuzzy numbers is again a linear fuzzy number given by

$$F = \left(\sum a_i, \sum b_i, \max \alpha_i, \max \beta_i\right),$$

i. e., the t–norm T_L works as the drastic product T_D. Note that T_L is the Łukasiewicz t–norm defined by $T_L(x,y) = \max(x+y-1, 0)$, $(x,y) \in [0,1]^2$.
When the shapes are preserved it is enough to determine the parameters of a T–sum, and then due to Proposition 1, the entropy of the result can be simply computed. Note that the mentioned result for T_L is only a special case of the claim (i) of the following theorem.

Theorem 2. Let $F_i = (a_i, b_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \dots, n$, and $F = F_1 \oplus_T \cdots \oplus_T F_n$ where T is a continuous Archimedean t–norm with an additive generator f.

(i) If the composite functions $f \circ L$ and $f \circ R$ are concave then

$$H(F) = \left(\max_{1 \le i \le n} \alpha_i\right) c_L + \left(\max_{1 \le i \le n} \beta_i\right) c_R .$$

(ii) If $f \circ L = lx^p$ and $f \circ R = rx^q$, $p, q > 1$, l, $r \in]0, \infty[$, then

$$H(F) = \left(\sum_{i=1}^{n} \alpha_i^{p^*}\right)^{1/p^*} c_L + \left(\sum_{i=1}^{n} \beta_i^{q^*}\right)^{1/q^*} c_R ,$$

where

$$\frac{1}{p} + \frac{1}{p^*} = 1 \qquad \text{and} \qquad \frac{1}{q} + \frac{1}{q^*} = 1 .$$

Proof. The conditions given in (i) and (ii) are by the results proved in [18, 19, 20, 27] sufficient for preserving the shapes L, R of T–sums. Moreover, the spreads are

$$\text{in (i):} \quad \alpha = \max_{1 \le i \le n} \alpha_i, \qquad \beta = \max_{1 \le i \le n} \beta_i,$$

and

$$\text{in (ii):} \quad \alpha = \left(\sum_{i=1}^{n} \alpha_i^{p^*}\right)^{1/p^*}, \qquad \beta = \left(\sum_{i=1}^{n} \beta_i^{q^*}\right)^{1/q^*}.$$

Now the claims follow directly from Proposition 1. □

Corollary 2. Let fuzzy numbers in the assumptions of Theorem 2 be symmetric with spreads γ_i, $i = 1, \dots, n$, and $f \circ L = f \circ R = kx^p$, $p > 1$, $k \in]0, \infty[$. Then

$$H(F) = \left(\sum_{i=1}^{n} (H(F_i))^{p^*} \right)^{1/p^*} = \| (H(F_1), \dots, H(F_n)) \|_{p^*} ,$$

where $p^* = \frac{p}{p-1}$ and $\| \, . \, \|_{p^*}$ is the norm in L^{p^*}-space.

Proof. The claim is an easy consequence of Theorem 2(ii) for the symmetric fuzzy numbers. □

Example 6. Consider the family of the Yager t-norms $\{T_p^Y\}_{p \in]0,\infty[}$, see [16], which are generated by additive generators $f(x) = (1-x)^p$, $x \in [0,1]$, and let $F_i = (a_i, b_i, \alpha_i, \beta_i)$, $i = 1, \dots, n$, be linear fuzzy numbers (i. e. with shapes $L(x) = R(x) = 1 - x \, , x \in [0,1]$).

(i) For $p \leq 1$ the assumptions of Theorem 2(i) are satisfied. Using the norm function h_1, see Examples 1(i) and 2, for the entropy of T_p^Y-based sum F of F_i we obtain

$$H(F) = \frac{1}{2}(\alpha + \beta) \, ,$$

where

$$\alpha = \max_i \alpha_i = \| (\alpha_1, \dots, \alpha_n) \|_\infty \, , \quad \beta = \max_i \beta_i = \| (\beta_1, \dots, \beta_n) \|_\infty \, .$$

(ii) For $p > 1$, the assumptions of Theorem 2(ii) are satisfied, therefore for the norm function h_1 we can also derive

$$H(F) = \frac{1}{2}(\alpha + \beta) \, ,$$

but now

$$\alpha = \left(\sum_{i=1}^{n} \alpha_i^{p^*} \right)^{1/p^*} = \| (\alpha_1, \dots, \alpha_n) \|_{p^*} \, ,$$

where $p^* = \frac{p}{p-1}$ and β can be determined analogously.
For instance, if we consider triangular fuzzy numbers $F_i = (0, 0, 1, 1)$, $i = 1, \dots, n$, then

$$H(F_1 \oplus_{T_p^Y} \cdots \oplus_{T_p^Y} F_n) = 1 \qquad \text{for } p \leq 1 \, ,$$

$$H(F_1 \oplus_{T_p^Y} \cdots \oplus_{T_p^Y} F_n) = n^{1/p^*} \quad \text{for } p > 1 \, .$$

5. ENTROPY OF T-PRODUCTS OF L–R FUZZY NUMBERS

Using the generalized extension principle, a t-norm based product of fuzzy quantities $F_1, \dots, F_n \in \mathcal{F}(\mathbb{R})$ can be introduced as a fuzzy quantity whose membership function is

$$F(x) = (F_1 \otimes_T \cdots \otimes_T F_n)(x) = \sup_{\prod x_i = x} T(F_1(x_1), \dots, F_n(x_n)), \quad x \in \mathbb{R}.$$

For a general t-norm T the exact output formula for the T-product has not been characterized yet. Therefore we restrict our considerations to non-negative L–R fuzzy numbers, i. e., with supports in $\mathbb{R}^+$ and their products based only on t-norms T_M and T_D.

For $T = T_D$ the situation is simple, since for non-negative L–R fuzzy numbers $F_i = (a_i, b_i, \alpha_i, \beta_i)_{LR}$ with $a_i - \alpha_i \geq 0$, $i = 1, \dots, n$, and with the same shapes, the output formula is [10]

$$F = (F_1 \otimes_{T_D} \cdots \otimes_{T_D} F_n)(x) = \left(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i, \alpha, \beta \right)_{LR},$$

where

$$\alpha = \prod_{i=1}^{n} a_i \max_{1 \leq i \leq n} \left(\frac{\alpha_i}{a_i} \right), \qquad \beta = \prod_{i=1}^{n} b_i \max_{1 \leq i \leq n} \left(\frac{\beta_i}{b_i} \right).$$

As we can see, the shapes are preserved, and the entropy can be computed by means of Proposition 1

$$H(F) = \alpha\, c_L + \beta\, c_R .$$

Next, let $T = T_M$. Following [28, 24] the T_M-product of two non-negative L-R fuzzy numbers $F_i = (a_i, b_i, \alpha_i, \beta_i)_{L_i R_i}$ with $a_i - \alpha_i \geq 0$, and strictly monotone shapes L_i, R_i for $i = 1, 2$, can be expressed in the form

$$F = F_1 \otimes_{T_M} F_2 = (a, b, \alpha, \beta)_{LR}, \quad \text{where}$$

$$a = a_1 a_2,\ b = b_1 b_2,\ \alpha = a_1 \alpha_2 + a_2 \alpha_1 - \alpha_1 \alpha_2,\ \beta = b_1 \beta_2 + b_2 \beta_1 + \beta_1 \beta_2, \tag{11}$$

and for the shapes L, R the following is valid:

$$a - \alpha L^{-1}(u) = (a_1 - \alpha_1 L_1^{-1}(u))(a_2 - \alpha_2 L_2^{-1}(u)),$$

which implies

$$L^{-1}(u) = \frac{1}{\alpha} \left(L_1^{-1}(u)\, a_2 \alpha_1 + L_2^{-1}(u)\, a_1 \alpha_2 - \alpha_1 \alpha_2 L_1^{-1}(u) L_2^{-1}(u) \right). \tag{12}$$

Similarly

$$R^{-1}(u) = \frac{1}{\beta} \left(R_1^{-1}(u)\, b_2 \beta_1 + R_2^{-1}(u)\, b_1 \beta_2 + \beta_1 \beta_2 R_1^{-1}(u) R_2^{-1}(u) \right).$$

Under the assumption that L_1, L_2 (R_1, R_2) possess finite negative derivatives on their domains, we can compute c_L, c_R.
First,

$$c_L = \int_0^1 h(L(u))\, \mathrm{d}u_{|_{L(u)=t}} = -\int_0^1 h(t)\left(L^{-1}(t)\right)'\, \mathrm{d}t$$

and after substitution (12) into the previous integral and simplifying we obtain

$$\begin{aligned} c_L &= \frac{a_2\alpha_1}{\alpha}\left(-\int_0^1 h(t)\left(L_1^{-1}(t)\right)'\, \mathrm{d}t\right) + \frac{a_1\alpha_2}{\alpha}\left(-\int_0^1 h(t)\left(L_2^{-1}(t)\right)'\, \mathrm{d}t\right) \\ &- \frac{\alpha_1\alpha_2}{\alpha}\left(-\int_0^1 h(t)\left(L_1^{-1}(t)\right)' L_2^{-1}(t)\, \mathrm{d}t - \int_0^1 h(t)\left(L_2^{-1}(t)\right)' L_1^{-1}(t)\, \mathrm{d}t\right). \end{aligned}$$

Moreover, it holds

$$-\int_0^1 h(t)\left(L_1^{-1}(t)\right)' L_2^{-1}(t)\, \mathrm{d}t_{|_{L_1^{-1}(t)=u}} = \int_0^1 h\left(L_1(u)\right) L_2^{-1}\left(L_1(u)\right)\, \mathrm{d}u = d_{L_2L_1},$$

and analogously the last member of the previous formula gives $d_{L_1L_2}$. Thus

$$c_L = \frac{a_2\alpha_1}{\alpha}\, c_{L_1} + \frac{a_1\alpha_2}{\alpha}\, c_{L_2} - \frac{\alpha_1\alpha_2}{\alpha}\left(d_{L_1L_2} + d_{L_2L_1}\right). \tag{13}$$

Similarly,

$$c_R = \frac{b_2\beta_1}{\beta}\, c_{R_1} + \frac{b_1\beta_2}{\beta}\, c_{R_2} + \frac{\beta_1\beta_2}{\beta}\left(d_{R_1R_2} + d_{R_2R_1}\right). \tag{14}$$

Theorem 3. Let $F_i = (a_i, b_i, \alpha_i, \beta_i)_{L_iR_i}$, $i = 1, 2$, be non–negative fuzzy numbers with strictly decreasing shape functions. Then

$$\begin{aligned} H(F_1 \otimes_{T_M} F_2) &= a_2\alpha_1\, c_{L_1} + a_1\alpha_2\, c_{L_2} - \alpha_1\alpha_2(d_{L_1L_2} + d_{L_2L_1}) \\ &+ b_2\beta_1\, c_{R_1} + b_1\beta_2\, c_{R_2} + \beta_1\beta_2(d_{R_1R_2} + d_{R_2R_1}). \end{aligned} \tag{15}$$

Proof. The claim follows from (13), (14) and Proposition 1. □

Remark 3. From the fact that all shapes L_i, R_i can be approximated uniformly by shapes satisfying the assumptions of Theorem 3, and because of the uniform continuity of all involved functions, we can extend Theorem 3 to the class of all non–negative L-R fuzzy numbers, see also Remark 2(ii).
Note that if a shape L (R) is not strictly decreasing we have to deal with its pseudo–inverse [15]

$$L^{(-1)} : [0,1] \to [0,1], \quad L^{(-1)}(u) = \sup\{x \in [0,1] \,|\, L(x) > u\}.$$

Thus Theorem 3 can be rephrased in the following way.

Corollary 3. For arbitrary non-negative L-R fuzzy numbers $F_i = (a_i, b_i, \alpha_i, \beta_i)_{L_i R_i}$, $i = 1, 2$, the entropy $H(F_1 \otimes_{T_M} F_2)$ is given by (15).

Let us still formulate several consequences of the previous theorem in some special cases.

Corollary 4.

(i) Let $F_i = (a_i, b_i, \alpha_i, \beta_i)_{KK}$, $i = 1, 2$, be non-negative fuzzy numbers with the same left and right shape functions.Then

$$H(F_1 \otimes_{T_M} F_2) = (a_2\alpha_1 + a_1\alpha_2 + b_2\beta_1 + b_1\beta_2)\, c_K + 2d_K(\beta_1\beta_2 - \alpha_1\alpha_2)\,, \tag{16}$$

where $d_K = d_{KK}$.

(ii) Let F_1, F_2 be linear fuzzy numbers. Then

$$H(F_1 \otimes_{T_M} F_2) = \alpha\, c_K + \beta\, c_K = (\alpha + \beta)\, c_K\,, \tag{17}$$

where α and β are spreads of the product given by (11).

Proof. (i) Let $L_i = K$, $i = 1, 2$. Then

$$d_{L_1L_2} = d_{L_2L_1} = d_{KK} = \int_0^1 h(K(u))K^{-1}(K(u))\,\mathrm{d}u = \int_0^1 uh(K(u))\,\mathrm{d}u = d_K\,,$$

and the same is valid for $R_i = K$, $i = 1, 2$. Now the claim follows already directly from (15).

(ii) Let $L_i(x) = R_i(x) = K(x) = 1 - x$, $x \in [0, 1]$. Then for any norm function h and the linear shape K we have

$$c_K = \int_0^1 h(K(u))\,\mathrm{d}u = \int_0^1 h(1-u)\,\mathrm{d}u = \int_0^1 h(u)\,\mathrm{d}u\,. \tag{18}$$

Next, by the previous step of the proof it holds

$$\begin{aligned} d_K &= \int_0^1 uh(K(u))\,\mathrm{d}u = \int_0^1 uh(1-u)\,\mathrm{d}u_{|_{1-u=v}} = \int_0^1 (1-v)h(v)\,\mathrm{d}v \\ &= \int_0^1 h(v)\,\mathrm{d}v - \int_0^1 vh(v)\,\mathrm{d}v = \int_0^1 h(v)\,\mathrm{d}v - \int_0^1 vh(1-v)\,\mathrm{d}v = c_K - d_K\,, \end{aligned}$$

which means that for linear shapes we have

$$d_K = c_K - d_K\,, \quad \text{i.e.,} \quad d_K = \frac{c_K}{2}\,.$$

Thus using (16) and (11) for the entropy of the T_M–product of two non–negative linear fuzzy numbers we obtain

$$\begin{aligned} H(F_1 \otimes_{T_M} F_2) &= (a_2\alpha_1 + a_1\alpha_2 + b_2\beta_1 + b_1\beta_2)\, c_K + 2\frac{c_K}{2}(\beta_1\beta_2 - \alpha_1\alpha_2) \\ &= ((a_2\alpha_1 + a_1\alpha_2 - \alpha_1\alpha_2) + (b_2\beta_1 + b_1\beta_2 + \beta_1\beta_2))\, c_K \\ &= (\alpha + \beta)\, c_K\,, \end{aligned}$$

which is the claim. □

Note that the previous result for linear fuzzy numbers is interesting, because although the shape functions of the T_M–product of two linear fuzzy numbers are not preserved, the entropy of the result is computed as for the linear fuzzy number $F = (a, b, \alpha, \beta)$. Let us illustrate it by the following example.

Example 7. Take linear fuzzy numbers $F_1 = F_2 = (1,1,1,1)$. It can be shown that

$$F(x) = F_1 \otimes_{T_M} F_2(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 2 - \sqrt{x} & \text{if } x \in [1,4] \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $F = (1,1,1,3)_{LR}$ with $L(x) = \sqrt{1-x}$ and $R(x) = 2 - \sqrt{3x+1}$, $x \in [0,1]$.

First, compute the entropy of F directly by definition (2) (for any norm function h):

$$H(F) = \int_0^1 h(\sqrt{x})\,\mathrm{d}x + \int_1^4 h(2-\sqrt{x})\,\mathrm{d}x\,.$$

Since

$$\int_0^1 h(\sqrt{x})\,\mathrm{d}x_{|\sqrt{x}=t} = 2\int_0^1 t\,h(t)\,\mathrm{d}t\,,$$

and

$$\int_1^4 h(2-\sqrt{x})\,\mathrm{d}x_{|2-\sqrt{x}=t} = 4\int_0^1 h(t)\,\mathrm{d}t - 2\int_0^1 t\,h(t)\,\mathrm{d}t\,,$$

for the entropy $H(F)$ we obtain

$$H(F) = 4\int_0^1 h(t)\,\mathrm{d}t\,.$$

Further, if we compute $H(F)$ by (17), for $\alpha = 1$ and $\beta = 3$ we have $H(F) = 4c_K$, where

$$c_K = \int_0^1 h(K(u))\,\mathrm{d}u = \int_0^1 h(1-u)\,\mathrm{d}u = \int_0^1 h(u)\,\mathrm{d}u\,.$$

Thus

$$H(F) = 4\int_0^1 h(u)\,\mathrm{d}u\,,$$

which is the same result as in the previous part.

Remark 4.

(i) Note that for the symmetric non-negative fuzzy numbers $F_1 = (a_1, b_1, \gamma, \gamma)_{KK}$ and $F_2 = (a_2, b_2, \delta, \delta)_{KK}$, the entropy is

$$H(F_1 \otimes_{T_M} F_2) = ((a_1 + a_2)\gamma + (b_1 + b_2)\delta)\ c_K\ ,$$

and for symmetric non-negative fuzzy numbers with the same shapes and spreads $\gamma = \delta = \omega$, the previous formula is of the form

$$H(F_1 \otimes_{T_M} F_2) = (a_1 + a_2 + b_1 + b_2)\,\omega\, c_K\ .$$

(ii) Applying (3) and Proposition 2, Theorem 3 can be immediately extended to the case of two L–R fuzzy numbers that do not have zero in their supports.

(iii) For the T_M–product of more than two non-negative L–R fuzzy numbers, by a formula related to (12), a generalization of (15) can be obtained.

6. CONCLUSION

Entropy measures for fuzzy quantities defined by means of norm functions and their properties in the case of L–R fuzzy numbers were studied. We have shown that for a given norm function h, computing the entropy of L–R fuzzy number can be reduced to using a simple formula which depends only on the shapes L, R and spreads α, β of the considered fuzzy numbers.
Moreover, the properties of the entropy of T–sums of L–R fuzzy numbers were studied. It was shown that the entropy of T_M–sum of any L–R fuzzy numbers (not necessarily with the same shapes) is T_M–additive, and so it can be computed as the sum of entropy of summands. In addition, also the results for the entropy of T–sums of special L–R fuzzy numbers for some other t–norms (Theorem 2) were derived.
Finally, the entropy of T_D– and T_M–products of non-negative L–R fuzzy numbers was discussed. We have also derived the formula for the entropy of T_M–product of any two non-negative L–R fuzzy numbers (with any shapes) which enables computing the entropy of the result only by means of parameters of incoming numbers and constants depending on their shapes.

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