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REACHING PHASE ELIMINATION IN VARIABLE STRUCTURE CONTROL OF THE THIRD ORDER SYSTEM WITH STATE CONSTRAINTS

Andrzej Bartoszewicz and Aleksandra Nowacka

In this paper the design of a time varying switching plane for the sliding mode control of the third order system subject to the velocity and acceleration constraints is considered. Initially the plane passes through the system representative point in the error state space and then it moves with a constant velocity to the origin of the space. Having reached the origin the plane stops and remains motionless. The plane parameters (determining angles of inclination and the velocity of its motion) are selected to ensure the minimum integral absolute error without violating velocity and acceleration constraints. The optimal parameters of the plane for the system subject to the acceleration constraint are derived analytically, and it is strictly proved that when both the system velocity and acceleration are limited, the optimal parameters can be easily found using any standard numerical procedure for solving nonlinear equations. The equation to be solved is derived and the starting points for the numerical procedure are given.

Keywords: variable structure systems, sliding mode control, switching plane design

AMS Subject Classification: 93B12, 93C10

1. INTRODUCTION

In recent years much of the research in the area of control systems theory focused on the design of a discontinuous feedback which switches the structure of the system according to the evolution of its state vector. This technique, usually called sliding mode control, provides an effective and robust means of controlling nonlinear plants [6, 7, 8, 11]. The main advantage of this technique is that once the system state reaches a sliding surface, the system dynamics remain insensitive to a class of parameter variations and disturbances.

However, robust tracking is assured only after the system state hits the sliding surface, i. e. the robustness is not guaranteed during the reaching phase. Provided a conventional time-invariant sliding plane is considered, the advantage of the sliding mode control, namely the desired dynamic behaviour of the system, is not obtained for some time from the beginning of its motion. Furthermore, usually for the given initial conditions there is a trade-off between the short reaching phase and the fast system response in the sliding phase. In order to overcome these problems the idea of the time-varying switching lines applied for the sliding mode control of the second order systems was introduced in [3, 4, 5] and further discussed in [1] and [2]. The control algorithms proposed in the papers [1] and [2] eliminate the reaching phase and guarantee fast error convergence rate for the second order uncertain systems with arbitrary initial conditions. Further results on the application of the time-varying switching lines for the sliding mode control of the second order systems have recently been reported in [9] and [10]. In the paper [9] rotation of the straight switching line was considered in detail. In that paper the authors used a new coordinate frame to propose a feasible choice of the time-varying switching line slope. A similar approach has been adopted in [10] where a nonlinear, time varying switching line, i. e. a changing shape parabola, was introduced.

In this paper the sliding mode control of the third order, nonlinear and time-varying system subject to the velocity and acceleration constraints is considered. For that purpose a moving switching plane is introduced. At the initial time the plane passes through the system representative point in the error state space and then it moves with a constant velocity towards the origin of the space. Once the plane reaches the origin, it stops moving and remains time-invariant. The plane is characterised by the four parameters representing its initial offset, velocity and angles of inclination. The main contribution of this work is the procedure for the optimal selection of these parameters. The procedure determines the quadruple of the parameters which ensure the minimisation of the integral absolute error in the controlled system subject to the velocity and acceleration constraints. Furthermore, the proposed control strategy ensures tracking error convergence without oscillations and overshoots.

The remainder of this paper is organised as follows. Problem formulation, the proposed control strategy and the system performance when the strategy is applied are presented in Section 2. Then the details of the switching plane design are discussed in Section 3. The acceleration and velocity constraints are analysed and formulated in terms of the switching plane parameters. The optimal parameters of the plane when the system is subject to the acceleration constraint are analytically derived. Furthermore, the case of two constraints (i. e. limited acceleration and velocity) is studied and it is proved that the optimal parameters can be easily found using any standard numerical procedure for solving nonlinear equations. The equation to be solved is derived and the starting points for the procedure are strictly defined. The control strategy proposed in the paper is illustrated by a simulation example presented in Section 4. Finally, Section 5 comprises conclusions of the paper.

2. CONTROL STRATEGY

Let us consider the time varying and nonlinear, third order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= f(\boldsymbol{x},t) + \Delta f(\boldsymbol{x},t) + b(\boldsymbol{x},t)u + d(t) \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are the state variables of the system and $\boldsymbol{x}(t) = [x_1(t) \ \ x_2(t) \ \ x_3(t)]^T$ is the state vector, t denotes time, u is the input signal, b, $f-$ are a priori known, bounded functions of time and the system state, Δf and d are functions representing the system uncertainty and external disturbances respectively. Further in the paper, it is assumed that there exists a strictly positive constant δ which is the lower bound of $b(\boldsymbol{x}, t)$, i.e. $0 < \delta = \inf\{|b(\boldsymbol{x}, t)|\}$. Furthermore, functions Δf and d are unknown and bounded. Therefore, there exists a constant μ which for every pair $(\boldsymbol{x}, t)$ satisfies the following condition $|\Delta f \ (\boldsymbol{x}, t) + d(t)| \leq \mu$. Initial conditions of the system are denoted as x_{10}, x_{20}, x_{30}, where $x_{10} = x_1(t_0)$, $x_{20} = x_2(t_0)$, $x_{30} = x_3(t_0)$. The system (1) is supposed to track the demand trajectory given as a function of time $\boldsymbol{x}_d(t) = [x_{1d}(t) \ \ x_{2d}(t) \ \ x_{3d}(t)]^T$, where $x_{2d} = \dot{x}_{1d}(t)$, $x_{3d} = \dot{x}_{2d}(t)$ and $x_{3d}(t)$ is a differentiable function of time. The trajectory tracking error is defined by the following vector $\boldsymbol{e}(t) = [e_1(t) \ \ e_2(t) \ \ e_3(t)]^T = \boldsymbol{x}(t) - \boldsymbol{x}_d(t)$. Hence, we have $e_1(t) = x_1(t) - x_{1d}(t)$, $e_2(t) = x_2(t) - x_{2d}(t)$, $e_3(t) = x_3(t) - x_{3d}(t)$. In this paper it is assumed that at the initial time $t = t_0$, the tracking error and the error derivatives $e_1(t_0) = e_0$, $e_2(t_0) = 0$, $e_3(t_0) = 0$, where e_0 is an arbitrary real number. This assumption holds well for the most typical regulation problem of the system which is initially in a steady state and its desired output changes instantaneously between two constant values. An example of such a problem is regulation of the link positions in point-to-point robot arm control.

Let us consider a time varying switching plane with the constant angle of inclination. Originally the plane moves uniformly (i.e. with a constant velocity) in the state space and then it stops at the time instant t_f. Consequently, for any $t \leq t_f$ the switching plane is described by the following equation

$$s(\boldsymbol{e}, t) = 0 \quad \text{where} \quad s(\boldsymbol{e}, t) = e_3(t) + c_2 e_2(t) + c_1 e_1(t) + A + Bt \tag{2}$$

where c_1, c_2, A and B are some constants. The selection of these constants will be considered in the next section. Since the plane stops at the time t_f, for any $t \geq t_f$

$$s(\boldsymbol{e}, t) = 0 \quad \text{where} \quad s(\boldsymbol{e}, t) = e_3(t) + c_2 e_2(t) + c_1 e_1(t). \tag{3}$$

First, the constants c_1, c_2, A and B should be chosen in such a way that the representative point of the system at the initial time $t = t_0$ belongs to the switching plane. For that purpose, the following condition must be satisfied

$$s[\boldsymbol{e}(t_0), t_0] = e_3(t_0) + c_2 e_2(t_0) + c_1 e_1(t_0) + A + Bt_0 = 0. \tag{4}$$

Notice that the input signal

$$u = \frac{-f(\boldsymbol{x}, t) - c_2 e_3(t) - c_1 e_2(t) + \dot{x}_{3d}(t) - B - \gamma \operatorname{sgn}\left[s\left(\boldsymbol{e}, t\right)\right]}{b(\boldsymbol{x}, t)} \tag{5}$$

where $\gamma = \eta + \mu$ and η is a strictly positive constant, ensures the stability of the sliding motion on the switching plane (2). In order to verify this property we consider the product

$$s(\boldsymbol{e}, t)\dot{s}(\boldsymbol{e}, t) = s(\boldsymbol{e}, t)\left[\dot{e}_3(t) + c_2 e_3(t) + c_1 e_2(t) + B\right]. \tag{6}$$

Substituting relations (1), (5) and the expression defining the trajectory tracking error into (6) we get

$$s(\boldsymbol{e},t)\dot{s}(\boldsymbol{e},t) = s(\boldsymbol{e},t)\left\{\Delta f(\boldsymbol{x},t) + d(t) - \gamma \,\mathrm{sgn}[s(\boldsymbol{e},t)]\right\} \leq -\eta|s(\boldsymbol{e},t)| \tag{7}$$

which proves the existence and stability of the sliding motion on the plane described by equations (2) and (3). Consequently, for any time $t \in \langle 0, t_f \rangle$ the system dynamics is described by equation (2) with the assumed initial conditions. Therefore, we consider the following equation

$$e_3(t) + c_2 e_2(t) + c_1 e_1(t) + A + Bt = 0. \tag{8}$$

In order to solve it, we consider

$$e_3(t) + c_2 e_2(t) + c_1 e_1(t) = 0. \tag{9}$$

Since the tracking error convergence to zero without oscillations is required, the characteristic polynomial of equation (9) should have one, double real root [8]. Hence, we get

$$c_2 = 2\sqrt{c_1}. \tag{10}$$

Furthermore, the parameters c_1 and c_2 must be strictly positive to make the system (1) stable in the sliding mode. Solving equation (8) with condition (10) and assuming for the sake of clarity that $t_0 = 0$ we get the tracking error and its derivatives for the time $t \in \langle 0, t_f \rangle$. Taking into account condition (4) and the assumption that $t_0 = 0$ we have

$$A = -c_1 e_0. \tag{11}$$

Then the tracking error and its derivatives can be written as

$$e_1(t) = \left(-\frac{2B\sqrt{c_1}}{c_1^2} - \frac{B}{c_1}t\right) e^{-\sqrt{c_1}t} + \frac{2B\sqrt{c_1}}{c_1^2} + e_0 - \frac{B}{c_1}t \tag{12}$$

$$e_2(t) = \frac{B}{c_1}\left(1 + \sqrt{c_1}t\right) e^{-\sqrt{c_1}t} - \frac{B}{c_1} \tag{13}$$

$$e_3(t) = -Bte^{-\sqrt{c_1}t}. \tag{14}$$

Next, we will analyse the behaviour of the system in the second phase of its motion, that is when the switching plane does not move. Notice that for the time $t \geq t_f$ the switching plane is fixed and passes through the origin of the error state space. This leads to the condition

$$A + Bt_f = 0. \tag{15}$$

From this equation and relation (11) we have that

$$t_f = \frac{e_0 c_1}{B}. \tag{16}$$

The time invariant switching plane is described by relation (3), which is equivalent to equation (9). The initial conditions which are necessary to solve equation (9) can be determined from equations (12), (13) and (14) whose values are evaluated at time t_f. Solving equation (9) and using relation (11) we get three equations describing the tracking error for any time $t \geq t_f$

$$e_1(t) = e^{-\sqrt{c_1}t}\left[-\frac{2B\sqrt{c_1}}{c_1^2} + \frac{2B\sqrt{c_1}}{c_1^2}e^k - e_0e^k + \left(-\frac{B}{c_1} + \frac{B}{c_1}e^k\right)t\right] \tag{17}$$

$$e_2(t) = e^{-\sqrt{c_1}t}\left[\frac{B}{c_1} - \frac{B}{c_1}e^k + e_0e^k\sqrt{c_1} - \left(-\frac{B}{\sqrt{c_1}} + \frac{B}{\sqrt{c_1}}e^k\right)t\right] \tag{18}$$

$$e_3(t) = e^{-\sqrt{c_1}t}\left[-e_0e^kc_1 + B(e^k - 1)t\right]. \tag{19}$$

In these equations

$$k = \frac{e_0c_1\sqrt{c_1}}{B}. \tag{20}$$

Notice that k is a strictly positive constant. From the above equation we have

$$c_1 = \left(\frac{kB}{e_0}\right)^{2/3}. \tag{21}$$

3. SWITCHING PLANE DESIGN

In the sequel a method of choosing the switching plane parameters will be proposed. Let us consider the following control quality criterion

$$J = \int_{t_0}^{\infty} |e_1(t)|\,\mathrm{d}t. \tag{22}$$

Because the tracking error described by equations (12) and (17) does not exhibit any overshoots and it converges monotonically in the considered system, criterion (22) is equivalent to

$$J = \left|\int_0^{\infty} e_1(t)\,\mathrm{d}t\right|. \tag{23}$$

Substituting equations (12) and (17) into this expression the following relation is obtained

$$\begin{aligned} J &= \left|\int_0^{t_f} e_1(t)\,\mathrm{d}t + \int_{t_f}^{\infty} e_1(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^{\frac{e_0c_1}{B}} \left[\left(-\frac{2B\sqrt{c_1}}{c_1^2} - \frac{B}{c_1}t\right)e^{-\sqrt{c_1}t} + \frac{2B\sqrt{c_1}}{c_1^2} + e_0 - \frac{B}{c_1}t\right]\mathrm{d}t\right. \\ &\quad \left. + \int_{\frac{e_0c_1}{B}}^{\infty} \left\{e^{-\sqrt{c_1}t}\left[\frac{2B\sqrt{c_1}}{c_1^2}e^k - \frac{2B\sqrt{c_1}}{c_1^2} - e_0e^k + \left(-\frac{B}{c_1} + \frac{B}{c_1}e^k\right)t\right]\right\}\mathrm{d}t\right|. \end{aligned} \tag{24}$$

Then calculating appropriate integrals we get

$$J = \frac{2\,|e_0|}{\sqrt{c_1}} + \frac{e_0^2 c_1}{2\,|B|}. \tag{25}$$

In order to calculate the parameters B and c_1 of the switching plane, further in the paper, criterion (25) with state constraints will be minimised. For that purpose we express criterion (25) as a function of variables k and B, rather than c_1 and B. Finding c_1 from equation (21) and substituting this parameter into relation (25) we obtain the following form of the considered criterion

$$J(k, B) = \frac{|e_0|^{4/3}}{|B|^{1/3}} \left(2k^{-1/3} + \frac{1}{2}k^{2/3}\right). \tag{26}$$

This formulation facilitates the minimisation procedure. We begin this procedure with expressing the acceleration constraint in terms of variables k and B.

3.1. Acceleration constraint

Equation (14) represents the system acceleration for $t \le t_f$, that is before the switching plane stops moving. The extreme value of this function is achieved at

$$t_{ma1} = \frac{1}{\sqrt{c_1}} \tag{27}$$

and it is equal to

$$e_3(t_{ma1}) = -\frac{B}{e\sqrt{c_1}}. \tag{28}$$

Further in this section, we consider two cases: one when $t_{ma1} < t_f$ and the other when $t_{ma1} \ge t_f$.

Case 1. $(t_{ma1} < t_f)$ If $t_{ma1} < t_f$, then the absolute value of the right-hand side of equation (28) is the greatest acceleration/deceleration of the system both when the plane moves and after it stops. In order to show this property, it is necessary to compare this quantity with the extreme values of the function on the right-hand side of equation (19). That function reaches its extreme value, equal

$$e_3(t_{ma2}) = \exp[-ke^k/(e^k - 1)](e^k - 1)B/e\sqrt{c_1} \tag{29}$$

at the time instant

$$t_{ma2} = 1/\sqrt{c_1} + e_0 e^k c_1/B(e^k - 1). \tag{30}$$

Since $k > 0$, then $\exp[-ke^k/(e^k-1)](e^k-1) < 1$. Consequently, inequality $|e_3(t_{ma1})| > |e_3(t_{ma2})|$ is always satisfied. Therefore, the absolute value of $e_3(t_{ma1})$ is actually the greatest acceleration of the considered system for any time $t \ge 0$. As a result, we get the following constraint for the considered minimisation of criterion (25)

$$|B| \le a_{\max} e\sqrt{c_1} \tag{31}$$

where $a_{\max}$ is the maximum admissible value of the system acceleration. Taking into account equation (21) we get the following form of constraint (31)

$$|B| \leq \left(\frac{a_{\max} e k^{1/3}}{|e_0|^{1/3}} \right)^{3/2}. \tag{32}$$

Notice that in the analysed case $t_{ma1} < t_f$ and consequently

$$\frac{c_1 e_0}{B} > \frac{1}{\sqrt{c_1}}. \tag{33}$$

This is equivalent to the situation when $k > 1$. Thus we conclude that for any $k > 1$ the acceleration constraint is expressed by inequality (32).

Case 2. $(t_f \leq t_{ma1})$ In this case we have $|B| \geq |e_0| \sqrt{c_1} c_1$ which is equivalent to $k \leq 1$. Now the greatest system acceleration/deceleration is given by the absolute value of

$$e_3(t_f) = -e^{-k} c_1 e_0. \tag{34}$$

Therefore, the explicit constraint formulation can be found from

$$a_{\max} \geq e^{-k} c_1 |e_0|. \tag{35}$$

From this relation and equation (21) we get

$$|B| \leq \left(\frac{a_{\max}}{e^{-k} |e_0|^{1/3} k^{2/3}} \right)^{3/2}. \tag{36}$$

This inequality represents the acceleration constraints for any $k \leq 1$.

3.2. Minimisation of criterion J subject to acceleration constraint

Let us take into account control quality criterion (26). In this section we will minimise this criterion subject to the controlled system acceleration constraint. Since criterion (26) decreases with the increasing absolute value of B, the minimisation of the two variables function $J(k, B)$ with the constraint may be accomplished minimising a single variable function without constraints. For that purpose the maximum admissible value of $|B|$ is derived from relations (32) and (36). For this value, criterion (26) can be expressed as

$$J_a(k) = \begin{cases} \frac{|e_0|^{3/2}}{\sqrt{a_{\max}}} \left(2e^{-k/2} + \frac{k}{2} e^{-k/2}\right) & \text{for } k \leq 1 \\ \\ \frac{|e_0|^{3/2}}{\sqrt{a_{\max} e}} \left(2k^{-1/2} + \frac{1}{2} k^{1/2}\right) & \text{for } k > 1 \end{cases} \tag{37}$$

and its derivative is

$$\frac{\mathrm{d}J_a(k)}{\mathrm{d}k} = \begin{cases} \frac{-|e_0|^{3/2} e^{-k/2}}{2\sqrt{a_{\max}}} \left(1 + \frac{k}{2}\right) & \text{for } k \leq 1 \\ \\ \frac{|e_0|^{3/2}}{\sqrt{a_{\max} e k}} \left(-k^{-1} + \frac{1}{4}\right) & \text{for } k > 1. \end{cases} \tag{38}$$

Checking the sign of this derivative we conclude that $J_a(k)$ decreases for $k \in (0,\, 4)$ and it increases for any $k > 4$. Thus for $k_{a\,\mathrm{opt}} = 4$ this function reaches its minimum. For that value of k we get the optimal parameter B from the following formula

$$|B| = \left(\frac{a_{\max} e k^{1/3}}{|e_0|^{1/3}} \right)^{3/2}. \tag{39}$$

Consequently

$$B_{a\,\mathrm{opt}} = 2(a_{\max} e)^{3/2} \operatorname{sgn}(e_0) \Big/ \sqrt{|e_0|}\,. \tag{40}$$

The pair $(k_{a\,\mathrm{opt}},\, B_{a\,\mathrm{opt}})$ is the optimal solution of the criterion J minimisation task subject to the acceleration constraint.

3.3. Minimisation of criterion J subject to acceleration and velocity constraints

In this section we will consider system (1) subject to two constraints, i. e. now both the system acceleration and velocity are limited. The acceleration cannot be greater than $a_{\max}$ and the maximum admissible velocity is $v_{\max}$.

For any time $t \le t_f$ the system velocity is described by equation (13) and for the time $t \ge t_f$ by equation (18). The extreme value of the velocity

$$e_2(t_{mv}) = \exp[-ke^k/(e^k-1)](1-e^k)B/c_1 \tag{41}$$

is achieved at the time instant $t_{mv} = e_0 c_1 e^k / B(e^k - 1)$. Notice that from relation (21) we have

$$e_2\left(t_{mv}\right) = \exp\left(\frac{-ke^k}{e^k-1} \right) \left(1-e^k\right) \frac{|B|^{1/3}\, e_0^{2/3}}{k^{2/3}}. \tag{42}$$

We require that

$$\left| \exp\left(\frac{-ke^k}{e^k-1} \right) \left(1-e^k\right) \frac{|B|^{1/3}\, e_0^{2/3}}{k^{2/3}} \right| \le v_{\max}. \tag{43}$$

We begin with checking if for the parameters $k_{a\,\mathrm{opt}}$ and $B_{a\,\mathrm{opt}}$ derived in the previous section the following condition is satisfied

$$\begin{aligned} \max_t \left|e_2\left(t\right)\right| &= \left| \exp\left(\tfrac{-k_{a\,\mathrm{opt}} e^{k_{a\,\mathrm{opt}}}}{e^{k_{a\,\mathrm{opt}}}-1} \right) \left(1-e^{k_{a\,\mathrm{opt}}}\right) \tfrac{|B_{a\,\mathrm{opt}}|^{1/3} e_0^{2/3}}{k_{a\,\mathrm{opt}}^{2/3}} \right| \\ &= \left| \exp\left(\tfrac{-4e^4}{e^4-1} \right) \left(1-e^4\right) \tfrac{\sqrt{a_{\max} e |e_0|}}{2} \right| \le v_{\max}. \end{aligned} \tag{44}$$

If this is the case, then the pair $k = k_{a\,\mathrm{opt}}$ and $B = B_{a\,\mathrm{opt}}$ is also the optimal solution of the optimisation problem considered in this section. Otherwise, i. e. if inequality (44) does not hold then the pair does not belong to the admissible set, and in the sequel we will find another solution of this problem.

Taking into account velocity constraint (43) only, we find the maximum admissible value of $|B|$, and substituting this value into expression (26) we get

$$J_v(k) = \frac{e_0^2}{v_{\max}} \exp\left(\frac{-ke^k}{e^k - 1}\right)\left(\frac{2}{k} + \frac{1}{2}\right)\left(e^k - 1\right). \tag{45}$$

This function, for any k expresses the minimum value of criterion $J(k, B)$ which can be achieved when velocity constraint (43) is satisfied. On the other hand, $J_a(k)$ shows the minimum value of the same criterion $J(k, B)$ achievable subject to the acceleration constraint. Therefore, for any k, the minimum value of $J(k, B)$ subject to both velocity and acceleration constraints, (i. e. when simultaneously both velocity and acceleration are limited) is given by $J(k) = \max[J_a(k),\ J_v(k)]$. Consequently, the optimal solution of the minimisation of criterion $J(k,\ B)$ is such a value of the argument k, for which

$$J(k_{av\,\mathrm{opt}}) = \min_{k\in[0,\infty)} \{J(k)\} = \min_{k\in[0,\infty)} \{\max[J_a(k), J_v(k)]\} \tag{46}$$

and a respective value of B. In the sequel, we will find the optimal parameter $k_{av\,\mathrm{opt}}$. For that purpose, first we formulate and prove essential properties of function $J_v(k)$. These properties are given by the following lemma.

Lemma. There exists such a number $k_{v\max} \in (2,\ 2.5)$ that for any $k \in [0, k_{v\max})$ function $J_v(k)$ increases and it decreases for any $k \in (k_{v\max}, \infty)$.

P r o o f. First we calculate the derivative of $J_v(k)$

$$\begin{aligned} \tfrac{\mathrm{d}J_v(k)}{\mathrm{d}k} &= \tfrac{e_0^2}{v_{\max}} \exp\left(\tfrac{-ke^k}{e^k-1}\right) \tfrac{1}{2k^2(e^k-1)} \left(4k^2e^k + k^3e^k + 8e^k - 4e^{2k} - 4\right) \\ &= \tfrac{e_0^2}{v_{\max}} \exp\left(\tfrac{-ke^k}{e^k-1}\right) \tfrac{e^k}{2k^2(e^k-1)} \left(4k^2 + k^3 + 8 - 4e^k - 4e^{-k}\right) \end{aligned} \tag{47}$$

and then, using sequence expansion, we get

$$\frac{\mathrm{d}J_v(k)}{\mathrm{d}k} = \frac{e_0^2}{v_{\max}} \exp\left(\frac{-ke^k}{e^k - 1}\right) \frac{ke^k}{2\left(e^k - 1\right)} \left(1 - 8\frac{k}{4!} - 8\frac{k^3}{6!} - 8\frac{k^5}{8!} - 8\frac{k^7}{10!} - ...\right). \tag{48}$$

This derivative has exactly one root, and at this point with the increase of the argument k, the sign of the derivative changes from positive to negative. Consequently, $J_v(k)$ has a single maximum. Let us denote as $k_{v\max}$ such a value of the argument k that $J_v'(k) = 0$. Because it follows from relation (47) that $J_v'(2) > 0$ and $J_v'(2.5) < 0$, we get $2 < k_{v\max} < 2.5$. Consequently, we conclude that there actually exists such a number $k_{v\max} \in (2,\ 2.5)$ that for any $k < k_{v\max}$ function $J_v(k)$ increases and for any $k > k_{v\max}$ it decreases with increasing argument k. This conclusion ends the proof of Lemma. □

Theorem. If condition (44) is not satisfied, then criterion $J(k) = \max[J_a(k),\ J_v(k)]$ achieves its minimum value at a point $k_{av\,\mathrm{opt}}$ which belongs to the open interval from 4 to $|e_0|a_{\max}e/(v_{\max})^2$.

P r o o f . Since inequality (44) does not hold, we have $J_v(4) > J_a(4)$. Furthermore, for any $k > 4$ $J_v(k)$ is a decreasing function of its argument and for any $k > 4$ $J_a(k)$ is an increasing function of k. Furthermore $\lim_{k\to\infty} J_a k) = \infty$, so there exists such a number $k_\alpha \in (4, \infty)$ that $J_v(k_\alpha) = J_a(k_\alpha)$. We will now demonstrate that function $J(k)$ achieves its minimum value at the point $k = k_\alpha$. For that purpose we will consider three situations, i. e. the first case when $k_\alpha \in (4,\ 6.4)$, the second when $k_\alpha \in [6.4,\ 8.5)$, and the third one when $k_\alpha \geq 8.5$.

i) case one $k_\alpha \in (4,\ 6.4)$

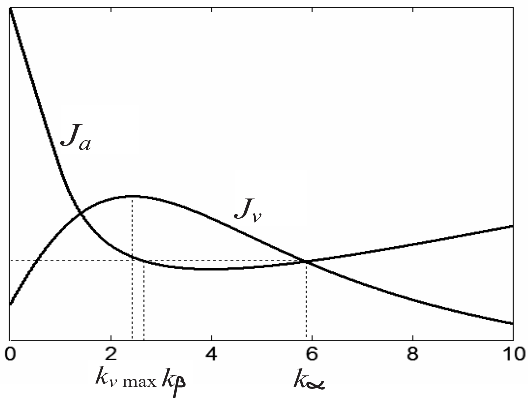

Fig. 1. Functions $J_a(k)$ and $J_v(k)$ for $k_\alpha \in (4,\ 6.4)$.

It follows from equation (37) that if $k_\alpha \leq 16$, then there exists such a number $k_\beta = 16/k_\alpha \geq 1$, that $J_a(k_\beta) = J_a(k_\alpha)$. Notice that $k_\alpha > 4$ and consequently k_β is smaller than 4. Then for any $k \notin (k_\beta,\ k_\alpha)$, $J_a(k) > J_a(k_\alpha) = J_a(k_\beta)$. For any $k \in (k_{v\,\max},\ k_\alpha)$ $J_v(k) > J_v(k_\alpha) = J_a(k_\alpha)$ which means that if $k_\beta > k_{v\,\max}$, then function $J(k)$ has its minimum at the point k_α. This situation takes place when $k_\alpha < 16/k_{v\,\max}$, and taking into account inequality $k_{v\,\max} < 2.5$, we conclude that if $k_\alpha < 16/2.5 = 6.4$ then function $J(k)$ must achieve its minimum value at the point k_α. This situation is illustrated in Figure 1.

ii) case two $k_\alpha \in [6.4,\ 8.5)$

Now we consider the situation when $k_\alpha \in [6.4,\ 8.5)$, which implies that $k_\beta \in (16/8.5,\ 2.5]$. Function $J_v(k_\beta)$ increases for any $k_\beta \in (16/8.5,\ k_{v\,\max})$ and decreases for any $k_\beta \in (k_{v\,\max},\ 2.5]$. Consequently, the minimum value of $J_v(k_\beta)$ in the considered interval $k_\beta \in (16/8.5,\ 2.5]$

$$\min_{k_\beta \in (16/8.5,\ 2.5]} J_v(k_\beta) = \min\left[J_v\left(16/8.5\right);\ J_v(2.5)\right] = J_v(16/8.5) > \frac{e_0^2}{v_{\max}} 0.9447 \tag{49}$$

is greater than the biggest value

$$\max_{k_\alpha \in [6.4,\, 8.5)} J_v(k_\alpha) = J_v(6.4) < \frac{e_0^2}{v_{\max}} 0.803 \tag{50}$$

which implies that $J(k)$ achieves its minimum value at the point k_α. Figure 2 shows the plots of $J_a(k)$ and $J_v(k)$ in the considered case.

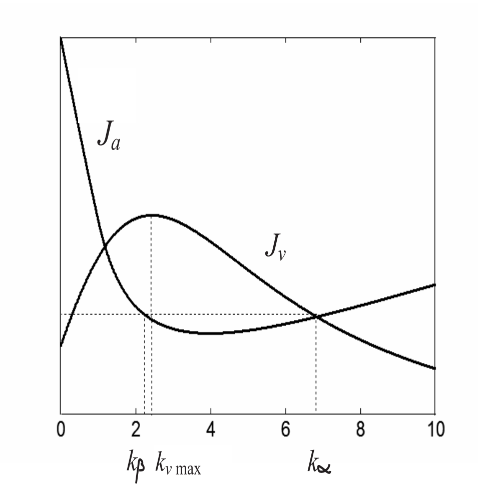

Fig. 2. Functions $J_a(k)$ and $J_v(k)$ for $k_\alpha \in [6.4, 8.5)$.

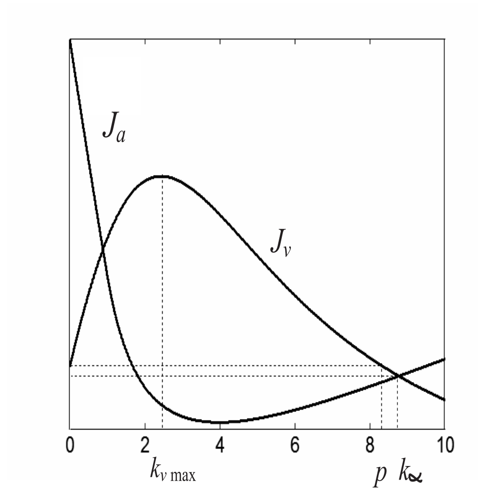

Fig. 3. Functions $J_a(k)$ and $J_v(k)$ for $k_\alpha \geq 8.5$.

iii) case three $k_\alpha \geq 8.5$

Function $J_v(k)$ is continuous and decreasing for any $k > k_{v\,\max}$. Furthermore, $k_{v\,\max} < 4$, $J_v(4) > 2/e\, e_0^2/v_{\max}$ and $\lim_{k\to\infty} J_v(k) = 1/2\, e_0^2/v_{\max}$. Consequently, there exists exactly one number $p \in (4, \infty)$, such that for $k \in (4, \infty)$ we have $J_v(k) < 2/e\, e_0^2/v_{\max} \Leftrightarrow k > p$. Moreover, $J_v(0) = 2/e\, e_0^2/v_{\max}$ and from Lemma it follows that $J_v(k)$ increases for any $k \in [0,\ k_{v\,\max})$. Consequently if $k_\alpha > p$, then for any $k \in [0,\ k_{v\,\max})$ $J_v(k) > J_v(k_\alpha)$. On the other hand, $J_v(k)$ decreases for any $k \in (k_{v\,\max},\ \infty)$, so for any $k \in [k_{v\,\max},\ k_\alpha)$ $\quad J_v(k) > J_v(k_\alpha)$. This implies, that if $k_\alpha > p$, then $J_v(k)$ achieves its minimum value at the point k_α. It is easy to verify that $p < 8.5$. Indeed $J_v(8.5) \approx 0.734\, e_0^2/v_{\max} < 2/e\, e_0^2/v_{\max}$. Therefore, if $k_\alpha \geq 8.5$, then function $J_v(k)$ has its minimum value at the point k_α. The scenario considered in this case is presented in Figure 3.

It follows from the analysis presented above that $J(k)$ achieves its minimum value at a point $k_\alpha > 4$. In the sequel, we will show that $k_\alpha < a_{\max} e|e_0|/v_{\max}^2$. For that purpose we define the following function

$$\tilde{J}_v(k) = \frac{e_0^2}{v_{\max}}\left(\frac{2}{k} + \frac{1}{2}\right). \tag{51}$$

For any $k > 0$, the following relation holds

$$\begin{aligned}\tilde{J}_v(k) &= \tfrac{e_0^2}{v_{\max}}\left(\tfrac{2}{k}+\tfrac{1}{2}\right) > \tfrac{e_0^2}{v_{\max}}\exp\left(\tfrac{-k}{e^k-1}\right)\left(\tfrac{2}{k}+\tfrac{1}{2}\right)\left(1-e^{-k}\right)\\ &= \tfrac{e_0^2}{v_{\max}}\exp\left(-k-\tfrac{k}{e^k-1}\right)\left(\tfrac{2}{k}+\tfrac{1}{2}\right)\left(e^k-1\right)\\ &= \tfrac{e_0^2}{v_{\max}}\exp\left(\tfrac{-ke^k}{e^k-1}\right)\left(\tfrac{2}{k}+\tfrac{1}{2}\right)\left(e^k-1\right) = J_v(k).\end{aligned} \tag{52}$$

This implies that for any $k > 0$ $\tilde{J}_v(k)$ dominates $J_v(k)$. Therefore, taking into account that $J_v(4) > J_a(4)$, $\lim_{k\to\infty} J_a(k) = \infty$ and $\lim_{k\to\infty} \tilde{J}_v(k) = 1/2\, e_0^2/v_{\max}$, we conclude that there exists such a number $k_\gamma > 4$, that $\tilde{J}_v(k_\gamma) = J_a(k_\gamma)$. Solving equation

$$\frac{e_0^2}{v_{\max}}\left(\frac{2}{k_\gamma}+\frac{1}{2}\right) = \frac{|e_0|^{3/2}}{\sqrt{a_{\max}e}}\left(\frac{2}{\sqrt{k_\gamma}}+\frac{\sqrt{k_\gamma}}{2}\right) \tag{53}$$

we get

$$k_\gamma = \frac{a_{\max}e\,|e_0|}{v_{\max}^2}. \tag{54}$$

As for any $k \in (0,\ \infty)$ $\tilde{J}_v(k) > J_v(k)$, it is easy to find that $k_\gamma > k_\alpha$. Thus if inequality (44) does not hold, then a number $k_{av\,\mathrm{opt}} \in (4,\ k_\gamma)$ is the optimal solution of the criterion $J(k) = \max[J_a(k),\ J_v(k)]$ minimisation task. This conclusion ends the proof of the theorem. □

The theorem presented above shows that the optimal value $k_{av\,\mathrm{opt}}$ of the parameter k belongs to the interval $(4,\ a_{\max}e|e_0|/v_{\max}^2)$. At this point $J_a(k_{av\,\mathrm{opt}}) = J_v(k_{av\,\mathrm{opt}})$. Moreover, for any $k \in (4,\ a_{\max}e|e_0|/v_{\max}^2)$ $J_a(k)$ is an increasing function of k and $J_v(k)$ is a decreasing function of its argument. Therefore, in order to find the optimal value $k_{av\,\mathrm{opt}}$ we introduce the following function

$$\begin{aligned}f(k) &= J_v(k) - J_a(k)\\ &= \tfrac{|e_0|^{3/2}}{v_{\max}\sqrt{a_{\max}e}}\left[\sqrt{a_{\max}e\,|e_0|}\exp\left(\tfrac{-ke^k}{e^k-1}\right)\left(e^k-1\right) - v_{\max}\sqrt{k}\right].\end{aligned} \tag{55}$$

Clearly, $f(k)$ is monotonic in the considered interval and $f(4)\cdot f(a_{\max}e|e_0|/v_{\max}^2) < 0$. Thus $k_{av\,\mathrm{opt}}$ which is the only root of equation $f(k) = 0$ in the interval can be easily found using any standard numerical procedure (for example bisection or falsi rule). The respective optimal value $B_{av\,\mathrm{opt}}$ is then determined by the following formula

$$B_{av\,\mathrm{opt}} = \frac{v_{\max}^3 k_{av\,\mathrm{opt}}^2 \operatorname{sgn}(e_0)}{e_0^2\exp\left(\frac{-3k_{av\,\mathrm{opt}}e^{k_{av\,\mathrm{opt}}}}{e^{k_{av\,\mathrm{opt}}}-1}\right)\left(e^{k_{av\,\mathrm{opt}}}-1\right)^3}. \tag{56}$$

The parameters determined in this way ensure the optimal performance of the controlled system together with satisfaction of both acceleration and velocity constraints.

Finally, let us remark that in the trivial case when only velocity constraint is taken into account (i. e. the maximum admissible acceleration tends to infinity), then $k_{v\,\mathrm{opt}} \to \infty$ and $\operatorname{sgn}(e_0) \cdot B_{v\,\mathrm{opt}} \to \infty$.

4. SIMULATION EXAMPLE

In order to verify the performance of the sliding mode control method proposed in this paper the following uncertain third order system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -3x_3 - 4x_2 - 2x_1 + x_1x_2 + \Delta f(\boldsymbol{x}, t) + d\,(t) + u. \end{aligned} \tag{57}$$

The uncertainty $\Delta f(\boldsymbol{x}, t) = 0.39 \sin\left(x_1x_2 + x_3\sqrt{t}\right)$ and the system is subject to the external disturbance $d(t) = 0.6\sin(10t)$. Consequently, γ has been chosen as $\gamma = 1$. The initial conditions of the system are $x_{10} = 10$, $x_{20} = 1$ and $x_{30} = 0$. The demand trajectory is determined as $x_{1d} = \sin t$. The maximum admissible acceleration $a_{\max} = 2$. Then we get the following values of the switching plane parameters $c_1 \approx 2.17$, $c_2 \approx 2.95$, $B \approx 8.01$, $A \approx -21.7$. For these parameters the switching plane stops at time $t_f \approx 2.71$. The tracking error and its derivatives are shown in Figures 4 and 5. The tracking error converges to zero monotonically without overshooting and the acceleration constraint is satisfied in the controlled system. Notice that, the maximum absolute value of the system velocity for the presented switching plane parameters equals $|v| = 3.35$.

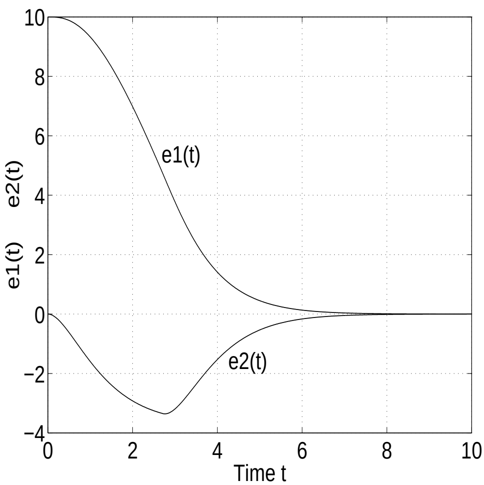

Fig. 4. The tracking error $e_1(t)$ and its derivative $e_2(t)$.

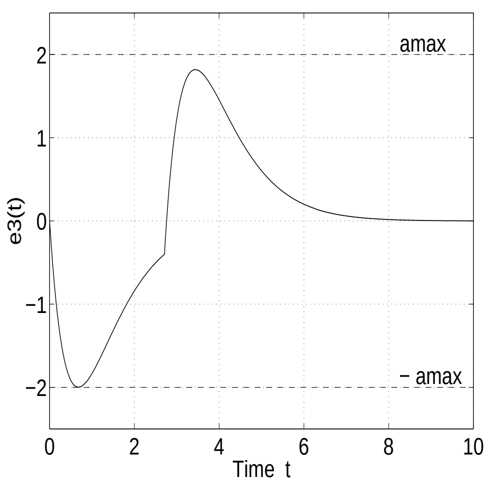

Fig. 5. The second derivative of the tracking error.

Now we require the maximum velocity of the controlled system not to exceed $v_{\max} = 3$. The parameters calculated above do not satisfy this condition because

$|v| > v_{\max}$. Therefore, another set of optimal parameters has to be determined. The first estimate of the parameter k is equal to $k_\gamma = 6.04$. Consequently, the optimal parameter k belongs to the interval (4, 6.04) and it has been found numerically to be $k_{av\,\mathrm{opt}} \approx 5.79$. Next we calculate the other parameters of the switching plane, which satisfy the two constraints considered in the paper and ensure minimisation of the integral of the absolute value of the tracking error. The switching plane designed in this way has the following parameters $c_1 \approx 3.15$, $c_2 \approx 3.55$, $B \approx 9.65$ and $A \approx -31.5$. In this case the time when the plane stops is equal to $t_f \approx 3.26$. The tracking error and its derivatives for the above parameters are shown in Figures 6 and 7. It can be seen from these figures that both constraints $a_{\max} = 2$ and $v_{\max} = 3$ are satisfied, however the system converges slightly slower than in the case considered previously. In both of the simulation examples, the tracking error converges to zero without oscillations and overshoots. This is a direct consequence of taking into account condition (10) in the design process.

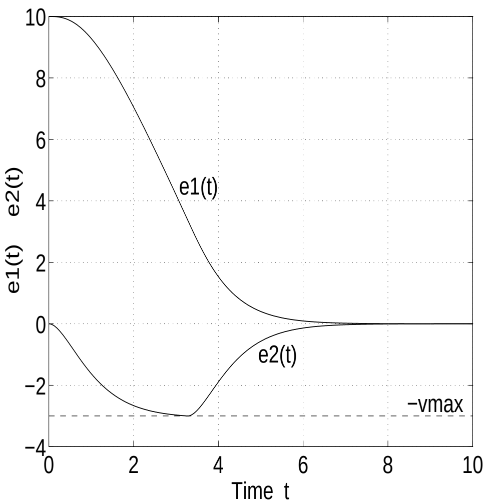

Fig. 6. The tracking error $e_1(t)$ and its derivative $e_2(t)$.

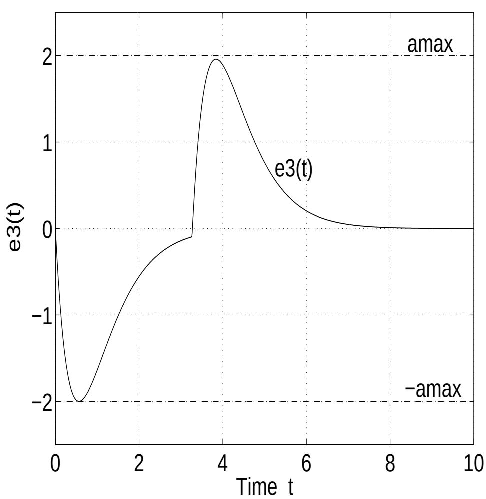

Fig. 7. The second derivative of the tracking error.

5. CONCLUSIONS

In this paper sliding mode control of the third order dynamic system subject to velocity and acceleration constraints has been considered. For that purpose we introduced a new time varying switching plane. Application of the plane ensures insensitivity of the system with respect to external disturbance and the plant uncertainty from the very beginning of its motion. At the initial time the plane passes through the system representative point. Then the plane moves towards the origin of the error state space and having (in finite time) reached the origin it stops moving. The optimal, in the sense of the integral absolute error, parameters of the plane are strictly determined. If the system is subject to the acceleration constraint the parameters are given explicitly, and in the case of two constraints (i. e. velocity and acceleration constraints) the parameters can be found solving a single nonlinear equation. This can be easily achieved using any standard numerical procedure. The equation to be solved is derived and the starting points for the numerical procedure are strictly determined. Even though, in this paper the switching plane moving with a constant velocity and a constant angle of inclination has been considered, the application of alternative switching surfaces – nonlinear and/or moving in a more sophisticated manner – could possibly ensure better dynamic properties of the controlled system. However, it is worth to point out that the controllers needed to obtain these favourable properties, would be computationally less efficient than our strategy. Moreover, they would require more adjustable parameters, which could make the design procedure laborious and time consuming.

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