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PRODUCTS OF VECTOR MEASURES BY MEANS OF FUBINI'S THEOREM

CHARLES SWARTZ

Let S, T be locally compact Hausdorff spaces with $\mathcal{B}(S)$, $\mathcal{B}(T)$ the Borel sets of S, T and let X, Y, Z be quasi-complete locally convex Hausdorff spaces with u: $X\times Y\to Z$ a separately continuous bilinear map. For convenience we write $u(x, y)=xy$ for $x\in X$, $y\in Y$. Let μ: $\mathcal{B}(S)\to X$ and ν: $\mathcal{B}(T)\to Y$ be countably additive set functions which are weakly regular in the sense that $x'\mu$ $(y'\nu)$ is regular for each $x'\in X'$ $(y'\in Y')$ ([5] Def. 2.6). We define the product measure $\mu\times\nu$ of μ and ν with respect to the bilinear map u by essentially using the idea which was employed by M. Duchoň in [9] to define the product for two vector measures of bounded variation (cf. also [22]). That is, $\mu(\nu)$ induces a continuous linear operator M: $C_0(S)\to X$ via $Mf=\int_S f\,\mathrm{d}\mu$ (N: $C_0(T)\to Y$ by $Ng=\int_T g\,\mathrm{d}\nu$), where $C_0(S)$ denotes the B-space of continuous real-valued functions on S which vanish at ∞ equipped with the sup-norm (we assume all vector spaces are real for convenience). The integral here is understood to be that of D. R. Lewis [17]. The product $\mu\times\nu$ is constructed by defining a continuous linear map P: $C_0(S\times T)\to Z$ by means of the iterated integral $Ph=\int_T\int_S h(s, t)\ \mathrm{d}\mu(s)\ \mathrm{d}\nu(t)$; the integral used here is indicated briefly in section 1. By a generalization of the Riesz-Representation Theorem ([5] Theorem 2.2; [24]) there is a unique finitely additive set function λ: $\mathcal{B}(S\times T)\to Z''$ such that $Ph=\int_{S\times T}h\,\mathrm{d}\lambda$, $h\in C_0(S\times T)$, (λ has other properties which we indicate later) and we define the product of μ and ν to be λ (compare with [9]).

One unpleasant feature of this approach to the product measure is that the product measure λ takes its values in Z'', the bidual of Z (see, however, [13], [16] or [19]; this phenomena is not uncommon in vector measures [8]). In Theorem 3 we give sufficient conditions for λ to actually have its range in Z and then we present an example to show that at least in some situations this condition is also necessary.

1. Integration of vector functions with respect to vector measures

In this section we very briefly outline the integral required to define the operator P above. It is only necessary that we integrate bounded functions so we do not attempt to give a complete discussion of the integral. (Recall that even integrating bounded vector functions with respect to vector measures is non-trivial, [6] Example 1.)

Let Σ be a σ-algebra of subsets of some set Ω and $\nu: \Sigma \to Y$ countably additive. Let $\mathcal{S}(\Sigma, X)$ be the space of all X-valued Σ-simple functions. If $s = \sum_{i=1}^{n} c_{E_i} x_i \in \mathcal{S}(\Sigma, X)$, where c_E denotes the characteristic function of E and the $\{E_i\}$ are disjoint, the ν-integral of s over A is given by

$$\textstyle\int_A s \, \mathrm{d}\nu = \sum_{i=1}^{n} x_i \nu(A \cap E_i) \tag{1}$$

(as usual the definition is independent of the representation of s). We say that ν has bounded u-semi-variation if for each continuous semi-norm r on Z there is a continuous semi-norm p on X such that

$$\tilde{\nu}_{p,r}(\Omega) = \sup\left\{ r\left(\sum_{i=1}^{n} x_i \nu(E_i)\right) : \{E_i : 1 \leqslant i \leqslant n\} \text{ partition of } \Omega,\ p(x_i) \leqslant 1 \right\} < \infty \tag{2}$$

($\tilde{\nu}_{p,r}$ is the p, r semi-variation of ν; see [21] §1). From (1) and (2), we obtain

$$r\left(\textstyle\int_\Omega s \, \mathrm{d}\nu\right) \leqslant \sup\{p(x_i) : 1 \leqslant i \leqslant n\} \tilde{\nu}_{p,r}(\Omega). \tag{3}$$

If we equip $\mathcal{S}(\Sigma, X)$ with the topology of uniform convergence on Ω, then (2) implies that the integral with respect to ν is a continuous linear map from $\mathcal{S}(\Sigma, X)$ into Z and if we let $B(\Sigma, X)$ be the closure of $\mathcal{S}(\Sigma, X)$ in the topology of uniform convergence on Ω, the integral has a unique linear extension to $B(\Sigma, X)$. Thus if $f \in B(\Sigma, X)$, there is a net of simple functions $\{s_\alpha\}$ converging uniformly to f and $\int_\Omega f \, \mathrm{d}\nu = \lim \int_\Omega s_\alpha \, \mathrm{d}\nu$, $r(\int_\Omega f \, \mathrm{d}\nu) \leqslant \tilde{\nu}_{p,r}(\Omega) \sup_{t \in \Omega} p(f(t))$. In particular, every function $f \in C_0(T, X)$ belongs to $B(\mathcal{B}(T), X)$ and therefore is integrable with respect to $\nu: \mathcal{B}(T) \to Y$ ([20] Prop. 1).

2. The product measure

Let $\mu: \mathcal{B}(S) \to X$, $\nu: \mathcal{B}(T) \to Y$ be countably additive and weakly regular ([5] Def. 2.6.) with ν being of bounded u-semi-variation. Then $\mu(\nu)$ induces a continuous linear map M: $C_0(S) \to X$ (N: $C_0(T) \to Y$) via $Mf = \int_S f \, \mathrm{d}\mu$ ($Ng = \int_T g \, \mathrm{d}\nu$) ([17]). Define a continuous linear map P: $C_0(S \times T) \to Z$ by $Ph = \int_T \int_S h(s, t) \mathrm{d}\mu(s) \mathrm{d}\nu(t)$. Note that since the function $t \to \int_S h(s, t) \mathrm{d}\mu(s)$ is continuous and vanishes at ∞, the iterated integral defining P exists, and for any continuous semi-norm p on X $p(\int_S h(s, t) \ \mathrm{d}\mu(s)) \leqslant \sup \ \{|h(s, t)| : s \in S, t \in T\} \tilde{\mu}_p(S)$, where $\tilde{\mu}_p$ is the scalar semi-variation of μ with respect to p ([14] IV.

10.3), so from (2) and (3) it follows that P is continuous. By a generalization of the Riesz Representation Theorem ([5] Theorem 2.2; [24]), there is a unique finitely additive set function λ: $\mathscr{B}(S \times T) \to Z''$ such that λ has finite semi-variation, λ is weakly regular (i.e., $\langle \lambda(\cdot), z' \rangle$ is regular for each $z' \in Z'$) and $Ph = \int_{S \times T} h \, d\lambda$ for $h \in C_0(S \times T)$. We define the product of μ and ν (with respect to u) to be λ and write $\lambda = \mu \times \nu$. Z'' carries the topology of uniform convergence on equicontinuous subsets of Z'.

To see that definition above is reasonable, consider the case where u: $X \times Y \to X \hat{\otimes}_\varepsilon Y$, the completion of $X \otimes Y$ with respect to the inductive tensor topology ([23] §43). For $x' \in X'$, $y' \in Y'$ we have $\langle x' \otimes y', \int_{S \times T} \varphi \otimes \psi d\lambda \rangle = \int_{S \times T} \varphi \otimes \psi d\langle x' \otimes y', \lambda \rangle = \int_{S \times T} \varphi \otimes \psi d x' \mu \times y' \nu$ for $\varphi \in C_0(S)$, $\psi \in C_0(T)$. Thus $\langle x' \otimes y', \lambda \rangle = x' \mu \times y' \nu$ and since $\{x' \otimes y' : x' \in X', y' \in Y'\}$ separates points in $X \otimes Y$, $\lambda(A \times B) = \mu(A) \otimes \mu(B)$ for $A \in \mathscr{B}(S)$, $B \in \mathscr{B}(T)$. This agrees with the previous definition of the tensor product measure as given in [10], [11], [12].

It should be pointed out that one shortcoming of this approach to the product measure via an iterated integral is that one must assume that the measure ν has bounded semi-variation ([20] § 1) in order to insure that the ν-integral exists. However, spectral measures in general do not have bounded semi-variation but a meaningful and useful theory of products of spectral measures can be developed ([4]).

Another difficulty with this approach to the product measure is that the product has values in Z'' and not in Z; this is, of course, unavoidable as several examples illustrate ([13], [16], [19]; see also [8], Th. 1.). We would like to obtain reasonably broad conditions on the measures μ and ν which will guarantee that $\mu \times \nu$ has values in Z (and is then countably additive with respect to the topology of Z). Of course, if Z is semi-reflexive, λ has values in Z and is countably additive. Now $\mu \times \nu$ has values in Z exactly when the operator P corresponding to $\mu \times \nu$ is weakly compact ([5] Th. 4.4) and in this case $\mu \times \nu$ is countably additive with respect to the original topology of Z. To show P is weakly compact (under appropriate conditions), we employ some results of Pelczynski on weakly compact and unconditional converging operators ([18]).

Recall that a continuous linear operator $U: X \to Y$ is unconditionally converging (u.c.) if U carries weakly unconditional Cauchy series (w.u.c. series) into unconditionally converging series (u.c. series) [18]. (A series Σx_n is w.u.c. if $\Sigma |\langle x', x_n \rangle| < \infty$ for $x' \in X'$ and is u.c. if every rearrangement converges [18], [15].) Pelczynski ([18]) showed that $C(S)$, S compact Hausdorff, has the property that a continuous linear operator $U: C(S) \to Y$, Y locally convex, is weakly compact if and only if U is u.c. (Pelczynski's result is stated for Y a B-space but the proofs are valid for locally convex spaces, see [15]). It is easily seen that $C_0(S)$ has the same property since $C_0(S)$ is a complemented subspace of $C(S^*)$, where S^* is the one-point compactification of S. Thus to show that the operator P above is weakly compact it suffices to show that P is u.c.

We now impose a condition on the measure v which is sufficient to guarantee that P is u.c. Since the condition is related to the bounded convergence theorem, we also further restrict S, T, X, Y and Z. Henceforth, we assume that S and T are σ-compact and X, Y, Z are Fréchet. Thus if $f: T \to X$ is continuous and vanishes at ∞, there is a sequence $\{s_n\}$ in $\mathcal{S}(\mathcal{B}(T), X)$ which converges uniformly to f on T and $\int_T s_n \, dv \to \int_T f \, dv$. In this situation we extend the v-integral from $B(\mathcal{B}(T), X)$ by means of

Definition 1. *A function $f: T \to X$ is v-integrable iff*

(i) *there is a sequence $\{s_n\}$ of $\mathcal{B}(T)$-simple X-valued functions such that $\{s_n\}$ converges to f pointwise on T*

(ii) *$\{\int_T s_n \, dv\}$ is Cauchy in Z. The integral of f is defined to be $\int_T f \, dv = \lim \int_T s_n \, dv$.*

Then the definition is independent of the sequence $\{s_n\}$ and due to the additional restrictions agrees with the previous definition of the v-integral.

The condition imposed on v is given in

Definition 2. *The measure $v: \Sigma \to Y$ has property* B C T *(with respect to u) if v satisfies the conclusion of the bounded convergence theorem for X-valued functions. That is, if $f_n: \Omega \to X$ is a sequence from $B(\Sigma, X)$ converging pointwise to a function f and if $\{f_n(t): n \geq 1, t \in \Omega\}$ is bounded in X, then f is v-integrable and $\int_\Omega f_n \, dv \to \int_\Omega f \, dv$.*

For the case when X and Z are B-spaces and $Y = L(X, Z)$, Dobrakov gives sufficient conditions for the bounded convergence theorem to hold ([6]). We will list conditions equivalent to the B C T condition in Theorem 4 below. Pertaining to $\mu \times v$, we have

Theorem 3. *Suppose v has* B C T. *Then P is* u.c. *(and hence weakly compact).*

Proof: Suppose Σh_k is w.u.c. in $C_0(S \times T)$. Then $\left\{ \sum_{k \in \sigma} h_k : \sigma \text{ a finite subset of the positive integers } \mathbf{N} \right\}$ is a bounded subset of $C_0(S \times T)$ ([15]). Thus for each $t \in T$, $\sum_{k=1}^{\infty} h_k(\cdot, t)$ is w.u.c. in $C_0(S)$ ([15]). Now M is weakly compact so $\sum_{k=1}^{\infty} M h_k(\cdot, t) = \sum_{k=1}^{\infty} \int_S h_k(s, t) \, d\mu(s) = H(t)$ is u.c. in X (with limit $H(t)$). Since the partial sums $\left\{ \sum_{k=1}^{N} M h_k(\cdot, t) \right\}$ are bounded and converge pointwise to H, the B C T condition implies $\sum_{k=1}^{\infty} \int_T \int_S h_k(s, t) \, d\mu(s) \, dv(t) = \int_T H \, dv = \sum_{k=1}^{\infty} P h_k$, where the series converges in Z. Since the argument above is obviously applicable to any rearrangement of the series Σh_k, the series ΣPh_k is u.c. and P is u.c.

From the remarks above, when v has B C T, the product measure $\mu \times v$ has values in Z and is countably additive.

We now give some conditions on v which are equivalent to B C T and which are often more easily checked than B C T. These conditions have been treated by several different authors for normed spaces; their methods can be easily adapted to treat the locally convex case so we only indicate appropriate references.

Theorem 4. *Let $v:\Sigma \to Y$. The following are equivalent:*

(i) *v has* B C T

(ii) *for each disjoint sequence $\{A_j\} \subset \Sigma$ and bounded sequence $\{x_j\} \subset X$, the series $\Sigma x_j v(A_j)$ converges in Z*

(iii) *v is dominated,* i.e., *for each continuous semi-norm r on Z there is a continuous semi-norm p on X and a positive measure β on Σ such that $\beta(A) \to 0$ implies $\tilde{v}_{p,r}(A) \to 0$* ([21])

(iv) *for each continuous semi-norm r on Z there is a continuous semi-norm p on X such that $\tilde{v}_{p,r}$ is continuous at $\emptyset$,* i.e., *$A_j \in \Sigma$, $A_j \downarrow \emptyset$ implies $\tilde{v}_{p,r}(A_j) \to 0$.*

Proof: (i) implies (ii): The sequence of partial sums $\sum_{k=1}^{n} c_{A_k} x_k$ converges pointwise and is bounded in X so by B C T the series $\sum_{k=1}^{\infty} x_k v(A_k)$ converges.

That (ii) implies (iii) follows from the argument that F_2 implies F_3 in the proof of Theorem 6 of [3] and a familiar criteria for weak compactness in ca(Σ) ([14] IV. 9.2).

That (iii) and (iv) are equivalent follows from the proof of Lemma 2 of [7].

Finally (iv) implies (i) follows from the bounded convergence theorem (again adapted to the more general situation) developed by either Bartle ([2]) or Dobrakov ([6]).

Remark 5. See [3], Theorem 6, for even further conditions equivalent to the bounded multiplier condition (ii).

In [21], Theorem 6, it is shown that if $v:\Sigma \to Y$ is dominated (as in (iii)), then v has a countably additive Z-valued product with respect to any X-valued measure μ. It is interesting to see that this condition arises naturally in the form of the bounded convergence theorem when the iterated integral approach to the product measure as above is used.

Also if $Z = X \hat{\otimes}_\varepsilon Y$ and u is the tensor map, then it is shown in [21], Theorem 11, that any measure $v:\Sigma \to Y$ is dominated with respect to u. This fact along with Theorem 3 yields the result of E. Thomas on ε-tensor products of vector measures ([22]).

It should also be pointed out that the approach to the product measure above gives a "measure" on the Borel sets of $S \times T$ whereas the more standard construction ([12]) gives a product measure only defined on, the σ-algebra generated by $\mathscr{B}(S) \times \mathscr{B}(T)$, in general a smaller σ-algebra than $\mathscr{B}(S \times T)$ (see also Theorem 3, [8]).

We now give some indication as to the necessity of the property B C T for the existence of the product measure.

Example 6. ([1]) Let $X = c_0$, $Y = l^1$ and $Z = l^1(c_0) = l^1 \hat{\otimes}_\pi c_0$ ([23] 44.2) with u the tensor map. Let $S = T = \mathbf{N}$ equipped with the discrete topology. Suppose the series Σy_n is u.c. in l^1, $y_n = \{y_{ni}\}_{i=1}^{\infty}$. Then the series induces a measure v on $\mathscr{B}(\mathbf{N})$ via $v(E) = \sum_{n \in E} y_n$, $E \subset \mathbf{N}$. We show that if v is such that the product $\mu \times v$ with any measure $\mu : \mathscr{B}(\mathbf{N}) \to c_0$ has values in $l^1(c_0)$, then Σy_n is absolutely convergent or v has bounded variation.

Let $\{t_n\} \in c_0$ and $e_n \in c_0$ be the sequence $\{\delta_{nj}\}_{j=1}^{\infty}$. Define a measure $\mu : \mathscr{B}(\mathbf{N}) \to c_0$ by $\mu(E) = \sum_{n \in E} t_n e_n$. If $\mu \times v$ is countably additive and has values in $l^1(c_0)$, then $\sum_{n,m} \mu \times v(m, n)$ is u.c. Thus, if $\zeta = \{e_n\}_{n=1}^{\infty} \in l^\infty(l^1) = (l^1(c_0))'$, then

$$\sum_{n,m} |\langle \zeta, \mu \times v(m, n) \rangle| = \sum_{m,n} |y_{nm} t_m| = \sum_m |t_m| \sum_n |y_{nm}| < \infty .$$

Since $\{t_n\} \in c_0$ is arbitrary, this gives $\sum_{nm} |y_{nm}| = \sum_n \|y_n\|_1 < \infty$.

Now any vector measure of bounded variation is dominated with respect to any bilinear map ([21] Prop. 8) so it follows that v satisfies B C T. This shows that the B C T condition of Theorem 3 is necessary at least in this particular situation.

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ПРОИЗВЕДЕНИЯ ВЕКТОРНЫХ МЕР ПРИ ПОМОЩИ ТЕОРЕМЫ ФУБИНИ

Чарлз Сварц

Резюме

Пусть S и T – локально компактные хаусдорфовы пространства, X, Y и Z – локально выпуклые пространства и u – отдельно непрерывное билинейное отображение из $X \times Y$ в

Z ($u(x, y) = xy$). Пусть $\mu(\nu)$ – регулярная X-значная ($У$-значная) мера на борелевских множествах в $S(T)$. При надлежащих условиях определяется непрерывный линейный оператор P на пространстве непрерывных функций на $S \times T$ идущих к нулю в ∞, $C_0(S \times T)$, при помощи инерированного интеграла $Ph = \int_T \int_S h(s, t) \mathrm{d}\mu(s) \mathrm{d}\nu(t)$. Используя обобщение теоремы Рисса о представлении, доказывается существование Z''-значной меры λ на борелевских множествах в $S \times T$; λ называется произведением μ и ν относительно u. Приведены некоторые достаточные условия для того, чтобы мера-произведение λ на самом деле принимала свои значения из Z.