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REPRESENTATIONS OF FINITE LATTICES BY ORDERS ON FINITE SETS

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1. Introduction

We are going to characterize finite lattices which can be represented by orders on some finite sets. This problem was put forward by Schein [2], where the following representation theorem was proved.

Theorem 1.1. *Every algebra of the form $(F, \circ, \cap)$, where F is a set of orders (reflexive, antisymmetric and transitive binary relations) on some set closed under the relative product $\circ$ and the set-theoretical intersection $\cap$ is a lattice, and every lattice is isomorphic to a lattice of this form.*

The construction used by Schein in the proof of Theorem 1.1 gives for finite lattices representations by orders on infinite countable sets.

Lemma 1.1. *Let (A, τ) be an ordered set (it means, A is a set and τ is an order on A) and let $S(A, \tau)$ be the set of all orders η on A such that $\eta \subseteq \tau$ ordered by the set-theoretical inclusion. Then $S(A, \tau)$ is a lattice with the operations $\vee$ (transitive span of the union) and $\cap$ (intersection).*

Definition 1.1. *Let L be a lattice and let (A, τ) be an ordered set. Any monomorphism of lattices $L \to S(A, \tau)$ will be called a representation of the lattice L on the set A. This representation is said to be finite if L and A are finite, and it is said to be commutative if the images of any two elements of L commute (under the operation $\circ$).*

The lattice L will be called finitely (commutatively, finitely commutatively) representable if it has a finite (commutative, finite commutative) representation. The class of all finitely (finitely commutatively) representable lattices will be denoted by FR (FCR).

By Theorem 1.1, every lattice has some commutative representation. Schein's problem can be formulated in the following way: Which finite lattices are finitely commutatively representable?

Lemma 1.2. *The classes FR, FCR are closed under isomorphisms and formation of sublattices.*

Lemma 1.3. *The classes FR, FCR are closed under finite direct products.*

Proof. Let L_1, L_2 be finitely representable lattices. Then there exist representations r_i of L_i on some finite sets A_i. We can assume A_i to be disjoint. Then the assignment

$$(x_1, x_2) \mapsto r_1(x_1) \cup r_2(x_2), \quad x_i \in L_i$$

defines a finite representation of the lattice $L_1 \times L_2$ on the set $A_1 \cup A_2$. This representation is commutative if r_i are commutative.

2. Small congruences on lattices

The notion of small congruence will allow us to characterize the class FR.

Definition 2.1. *Let Θ be a congruence on a lattice L. We call Θ small if there exists a homomorphism of semilattices φ: $(L, \wedge) \to \{0,1\}$ such that its restriction to each class of the congruence Θ is injective.*

Lemma 2.1. *Let Θ be a small congruence on a lattice, A its class and φ the homomorphism of semilattices corresponding to Θ. Then either A is a singleton, or $A = \{a_0, a_1\}$, where $a_1 > a_0$, $\varphi(a_0) = 0$, $\varphi(a_1) = 1$.*

Lemma 2.2. *Let Θ be a non-trivial (not equal to the diagonal ω) small congruence on a finite lattice L, φ the corresponding homomorphism of semilattices, D_1 the set of all elements $x \in L$ such that $\varphi(x) = 1$ and the class $[x]\Theta$ has two elements. Then:*

(i) *$\varphi^{-1}(1)$ is a filter in L.*

(ii) *D_1 is a subsemilattice of $\varphi^{-1}(1)$.*

(iii) *$\varphi^{-1}(1) = \langle d_1, 1\rangle$, where d_1 is the least element of D_1 and 1 is the greatest element of L.*

(iv) *For all $x \in D_1$, $\langle d_1, x\rangle \subseteq D_1$.*

(v) *Θ is an atom in* Con(L), *the lattice of all congruences on L.*

Proof. As φ is a homomorphism of semilattices, (i) and (ii) trivially hold. Choose $x \in \varphi^{-1}(1)$. Then $\varphi(x) = 1$ and $1 = \varphi(x) = \varphi(x) \wedge \varphi(d_1) = \varphi(x \wedge d_1)$. If $[d_1]\Theta = \{d_1, d_0\}$, then $\varphi(d_0) = 0$, $\varphi(x \wedge d_0) = 0$, therefore $\{x \wedge d_0, x \wedge d_1\}$ is a two-element class of Θ and $x \wedge d_1 \in D_1$. As d_1 is the least element of D_1, $x \wedge d_1 = d_1$ and $x \in \langle d_1, 1\rangle$. We proved $\varphi^{-1}(1) \subseteq \langle d_1, 1\rangle$.

Choose $x \in \langle d_1, 1\rangle$, then $\varphi(x) = \varphi(x) \wedge 1 = \varphi(x) \wedge \varphi(d_1) = \varphi(x \wedge d_1) = \varphi(d_1) = 1$. We proved $\langle d_1, 1\rangle \subseteq \varphi^{-1}(1)$.

Choose $x \in D_1$, $y \in \langle d_1, x\rangle$. There exists $x' \in L$ such that $\varphi(x') = 0$, $[x]\Theta =$

$\{x, x'\}$. Then $\varphi(x' \wedge y)=0$, $x \wedge y = y$, $\varphi(y)=1$, therefore $x' \wedge y$ is the second element of $[y]\Theta$ and $y \in D_1$. We proved (iv).

Choose $(x, y) \in \Theta$ such that $x < y$. As $x \wedge d_1 \ \Theta \ y \wedge d_1 = d_1$ and $\varphi(x \wedge d_1)=0$, $x \wedge d_1 = d_0$ and $x \geqslant d_0$. Similarly, $x \vee d_1 \ \Theta \ x \vee d_0 = x$ and $\varphi(x \vee d_1)=1$, therefore $x \vee d_1 = y$. (See fig. 1) This holds for each $(x, y) \in \Theta$, $x < y$, therefore Θ is an atom.

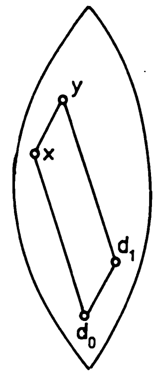

Fig. 1

By Lemma 2.2, if Θ is a non-trivial small congruence on a finite lattice, then the corresponding homomorphism of semilattices is uniquely determined. In fact, the set D_1 can be defined without using φ.

Lemma 2.3. *Let L be a finite lattice, $D \subseteq L$ a subsemilattice, d its least element and $\langle d, x \rangle \subseteq D$ for all $x \in D$. Form the following subset of $L \times \{0,1\}$:*

$$L' = [\langle d, 1 \rangle \times \{1\}] \cup [(L - (\langle d, 1 \rangle - D)) \times \{0\}].$$

This set with a termwise order is a lattice with the following operations:

$$(x,1) \vee (y,1) = (x,1) \vee (y,0) = (x,0) \vee (y,1) = (x \vee y, 1),$$

$$(x,0) \vee (y,0) = (x \vee y, 1), \text{ if } x \vee y \in \langle d, 1 \rangle - D,$$

$$(x,0) \vee (y,0) = (x \vee y, 0), \text{ if } x \vee y \in (L - \langle d, 1 \rangle) \cup D,$$

$$(x,i) \wedge (y,j) = (x \wedge y, i \wedge j).$$

The relation $\Theta = \{((x,0), (x,1)) | x \in D\} \cup \{((x,1), (x,0)) | x \in D\} \cup \omega_{L'}$ is a small congruence on L', the corresponding homomorphism of semilattices is the projection $(x,i) \mapsto i$ and the factor lattice L'/Θ is isomorphic to L.

The proof is trivial. Note that if D is an interval in L, then the just described construction is identical with the "interval construction" of A. Day [1].

Lemma 2.4. *Let L be a finite lattice, Θ a non-trivial small congruence on L. Then the construction described in Lemma 2.3 used for the lattice L/Θ and its subsemilattice $\{[x]\Theta \mid x \in D_1\}$ gives a lattice isomorphic to L.*

Lemma 2.5. *Let L be a finite lattice, Θ and λ two different non-trivial small congruences on L, φ the homomorphism of semilattices corresponding to Θ. Then φ is constant on each class of the congruence λ.*

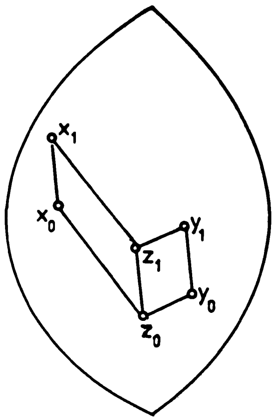

Fig. 2

Proof. Let d_1, d_0 be the same elements as in Lemma 2.2 and its proof, $\{a, b\}$ a class of the congruence λ and $a<b$. If φ is not constant on the class $\{a, b\}$, then $\varphi(a)=0$, $\varphi(b)=1$. The elements $a\wedge d_0$, $a\wedge d_1$ are in the same class of the congruence Θ and $\varphi(a\wedge d_0) = \varphi(a\wedge d_1)=0$, therefore $a\wedge d_0=a\wedge d_1$. As $\varphi(b)=1$, $b\wedge d_1=d_1$ and we have:

$$a\wedge d_1=a\wedge d_0\leqslant d_0<d_1=b\wedge d_1, \quad (a\wedge d_1, b\wedge d_1)\in\lambda,$$

therefore $(d_0, d_1)\in\lambda$, a contradiction, as by Lemma 2.2 (v), $\lambda\cap\Theta=\omega$.

Lemma 2.6. *Let L be a finite lattice, Θ a small and λ any congruence on L. Let $\{x_0, x_1\}$ and $\{y_0, y_1\}$ be classes of Θ, $x_0<x_1$, $y_0<y_1$. Then $(x_0, y_0)\in\lambda$ iff $(x_1, y_1)\in\lambda$.*

Proof. Let φ be the homomorphism of semilattices corresponding to Θ, $z_i=x_i\wedge y_i$. Then $\{z_0, z_1\}$ is a class of Θ, $\varphi(z_i)=i$. The intervals $\langle z_0, x_0\rangle$ and $\langle z_1, x_1\rangle$ are transposed, $\langle z_0, y_0\rangle$ and $\langle z_1, y_1\rangle$ too.

Assume $(x_0, y_0)\in\lambda$. Then $(z_0, x_0)\in\lambda$, $(z_0, y_0)\in\lambda$, therefore $(z_1, x_1)\in\lambda$, $(z_1, y_1)\in\lambda$. (See fig. 2) We proved that $(x_0, y_0)\in\lambda$ implies $(x_1, y_1)\in\lambda$. Similarly, $(x_1, y_1)\in\lambda$ implies $(x_0, y_0)\in\lambda$.

Lemma 2.7. *Let Θ, λ be small congruences on a finite lattice L. Then* $\Theta\circ\lambda\circ\Theta\subseteq\lambda\circ\Theta\circ\lambda$.

Proof. Assume $(a, b)\in\Theta\circ\lambda\circ\Theta-\lambda\circ\Theta\circ\lambda$. Then Θ is not trivial. Let φ be the corresponding homomorphism of semilattices. There exist elements $c, d\in L$ such that $(a, c)\in\Theta$, $(c, d)\in\lambda$, $(d, b)\in\Theta$. Trivially, $a\neq c$, $b\neq d$, so $\varphi(c)=1-\varphi(a)$, $\varphi(d)=1-\varphi(b)$. By Lemma 2.5, $\varphi(c)=\varphi(d)$, therefore $\varphi(a)=\varphi(b)$. The sets $\{a, c\}$ and $\{b, d\}$ are classes of Θ and either $a<c$, $b<d$, or $a>c$, $b>d$. By Lemma 2.6, $(a, b)\in\lambda$, a contradiction.

Lemma 2.8. *Let Θ, λ be small congruences on a finite lattice L. Then* $\Theta\vee\lambda = \lambda\cup[\lambda\circ(\Theta-\omega)\circ\lambda]$.

Lemma 2.9. *Let Θ, λ be small congruences on a finite lattice L. Then $(\Theta\vee\lambda)/\Theta$ is a small congruence on L/Θ.*

Proof. Recall the notion of factor congruence, If $\Theta_1\subseteq\Theta_2$ are congruences on a lattice L, we can define the projection $L/\Theta_1\to L/\Theta_2$, $[x]\Theta_1\mapsto[x]\Theta_2$. Its kernel is the factor congruence $\Theta_2/\Theta_1 = \{([a]\Theta_1, [b]\Theta_1)|(a, b)\in\Theta_2\}\in\mathrm{Con}(L/\Theta_1)$.

The lemma trivially holds if Θ or λ is trivial and if $\Theta=\lambda$. Assume Θ, λ to be different and non-trivial. There exist a homomorphism of semilattices $\varphi\colon L\to\{0,1\}$ injective on each class of λ. Define the mapping

$$\varepsilon\colon L/\Theta\to\{0,1\},\quad [x]\Theta\to\varphi(x).$$

By Lemma 2.5, this definition is correct, Trivially, ε is a homomorphism of semilattices. There suffices to prove that it is injective on each class of $(\Theta\vee\lambda)/\Theta$. Assume $(a, b)\in\Theta\vee\lambda$, $[a]\Theta\neq[b]\Theta$, but $\varepsilon([a]\Theta) = \varepsilon([b]\Theta)$. Then $\varphi(a)=\varphi(b)$. As $(a, b)\in\Theta\vee\lambda$, by Lemma 2.8, $(a, b)\in\Theta\circ(\lambda-\omega)\circ\Theta$, so there exist $c, d\in L$ such that $(a, c)\in\Theta$, $(c, d)\in\lambda-\omega$, $(d, b)\in\Theta$. By Lemma 2.5, $\varphi(c)=\varphi(a)$, $\varphi(d)=\varphi(b)$, a contradiction, as $(c, d)\in\lambda-\omega$ implies $\varphi(c)\neq\varphi(d)$.

3. Finitely representable lattices

In this paragraph we give several characterizations of the class *FR*.

Lemma 3.1. *Let $r\colon x\mapsto r_x$ be a representation of a lattice L with the greatest element 1 on a set A. Assume that A has an r_1-least element a, $B=\langle b_0, b_1\rangle$ is some r_1-interval in A, B' is a set disjoint with A and there exists a bijection $B\to B'$, $b\mapsto b'$. Define the orders $\bar{r}_x$ on the set $A\cup B'$ in the following way:*

$$\bar{r}_x=r_x\vee(r_x|B)'\vee\{(b_1', a)\},\quad x\in L,$$

where $\vee$ is the transitive span of the union and $r_x|B$ is the restriction of r_x to B. The assignment $x\mapsto\bar{r}_x$ defines a representation of L on the set $A\cup B'$.

Proof. Trivially, $\bar{r}_x$ are orders and $x \mapsto \bar{r}_x$ is monotone, therefore $\bar{r}_{x \vee y} \supseteq \bar{r}_x \vee \bar{r}_y$ and $\bar{r}_{x \wedge y} \subseteq \bar{r}_x \cap \bar{r}_y$. We shall prove the inverse inclusions.

First assume $(u, v) \in \bar{r}_{x \vee y}$, we want to prove $(u, v) \in \bar{r}_x \vee \bar{r}_y$. It trivially holds if u, v are in the same of the sets A, B'. The case $u \in A$, $v \in B'$ gives a contradiction.

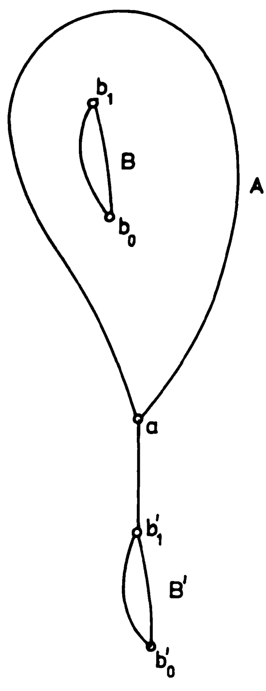

Fig. 3

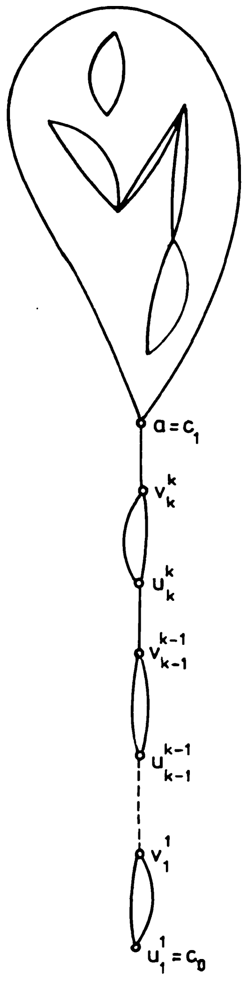

Fig. 4

There remains only the case $u \in B'$, $v \in A$. (See fig. 3.) Then $(u, b') \in (r_{x \vee y} | B)'$ $= (r_x \vee r_y | B)'$, but as B is r_1-convex, $r_x \vee r_y | B = (r_x | B) \vee (r_y | B)$, therefore $(u, b_1') \in (r_x | B)' \vee (r_y | B)'$. As $(a, v) \in r_{x \vee y} = r_x \vee r_y$, we have:

$$(u, v) \in (r_x | B)' \vee (r_y | B)' \vee \{(b_1', a)\} \vee r_x \vee r_y = \bar{r}_x \vee \bar{r}_y .$$

Now assume $(u, v) \in \bar{r}_x \cap \bar{r}_y$. There suffices to consider the case $u \in B'$, $v \in A$ again. In this case we have $(u, b'_1) \in (r_x|B)' \cap (r_y|B)' = (r_x \cap r_y|B)' = (r_{x\wedge y}|B)'$, $(a, v) \in r_x \cap r_y = r_{x\wedge y}$, therefore

$$(u, v) \in (r_{x\wedge y}|B)' \vee \{(b'_1, a)\} \vee r_{x\wedge y} = \bar{r}_{x\wedge y}.$$

Lemma 3.2. *Let L be a finitely representable lattice, $f \in L$. Then there exists a finite representation r of L on some set A and elements $c_0, c_1 \in A$ such that*

(i) *c_0 is the r_1-least element of A, where 1 is the greatest element of L,*

(ii) *for all $x \in L$, $(c_0, c_1) \in r_x$ iff $x \geqslant f$.*

Proof. Choose a representation r' of L on some finite set A'. We can assume that A' has an r'_1-least element a. Let F be the set of all ordered pairs $(u, v) \in A' \times A'$ such that u is r'_f-covered by v. If $F = \emptyset$, f is the least element of L and the representation r' with $c_0 = c_1 = a$ satisfies the conditions (i), (ii). Assume $F = \{(u_1, v_1), \ldots, (u_k, v_k)\}$. Let $\langle u_i, v_i \rangle$ be r'_1-intervals in A'. For each i we find a set B^i and a bijection $\langle u_i, v_i \rangle \to B^i$, $w \mapsto w^i$, in such a way that A' and B^i are pairwise disjoint. Then we define the orders r_x on the set $A' \cup B^1 \cup \ldots \cup B^k$ in the following way:

$$r_x = r'_x \vee (r'_x|\langle u_1, v_1 \rangle)^1 \vee \ldots \vee (r'_k|\langle u_k, v_k \rangle)^k \vee$$
$$\vee \{(v^1_1, u^2_2), (v^2_2, u^3_3), \ldots, (v^{k-1}_{k-1}, u^k_k), (v^k_k, a)\}.$$

(See fig. 4.) By Lemma 3.1 used k-times, r is a finite representation of L. The conditions (i), (ii) are satisfied for $c_0 = u^1_1$, $c_1 = a$.

Lemma 3.3. *Let the assumptions of Lemma 2.3 be satisfied and $L \in FR$. Then $L' \in FR$.*

Proof. Let F be the set of all minimal elements of the set $\langle d, 1 \rangle - D$. If $F = \emptyset$, L' is isomorphic to a sublattice of $L \times \{0,1\}$, and by Lemma 1.1 and Lemma 1.2, $L' \in FR$. Assume $F = \{f^1, \ldots, f^k\}$. Then for all $x \in L$, $x \in \langle d, 1 \rangle - D$ iff $x \geqslant f^i$ for some i. By Lemma 3.2, we can find for each i a representation r^i on a finite set A^i and elements $c^i_0, c^i_1 \in A^i$ such that

(i) c^i_0 is the r^i_1-least element of A^i,

(ii) for all $x \in L$, $(c^i_0, c^i_1) \in r^i_x$ iff $x \geqslant f^i$.

We can assume A^i to be pairwise disjoint. Let a, a' be two different elements not in A^i, $A = A^1 \cup \ldots \cup A^k \cup \{a, a'\}$. We define the orders r_x on A in the following way:

$$r_x = [r^1_x \vee \ldots \vee r^k_x \vee (\{a\} \times \{c^1_0, \ldots, c^k_0\}) \vee$$
$$\vee (\{c^1_1, \ldots, c^k_1\} \times \{a'\})] \cup \{(a, a')\}.$$

(See fig. 5.) It can be simply proved that r is a finite representation of L. Let us define the mapping s: $L' \to S(A, r_1)$, $(x, 0) \mapsto r_x - \{(a, a')\}$, $(x, 1) \mapsto r_x$.

This definition is correct: if $(x, 0) \in L'$, x cannot be in the set $\langle d, 1\rangle - D$, so $(c_0^i, c_1^i) \in r_x^i$ for no i and $r_x - \{(a, a')\}$ is transitive. Trivially, s is a monomorphism of semilattices. Therefore $s((x, i) \vee (y, j)) \supseteq s(x, i) \vee s(y, j)$. It suffices to prove that in this inclusion equality holds.

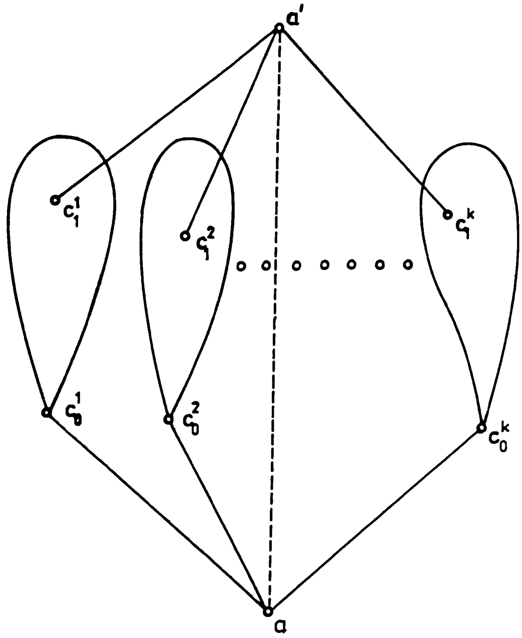

Fig. 5

Assume $s((x, i) \vee (y, j)) > s(x, i) \vee s(y, j)$. Then the difference has to be exactly $\{(a, a')\}$, and so $i = j = 0$, $(x, 0) \vee (y, 0) = (x \vee y, 1)$, $(a, a') \notin s(x, 0) \vee s(y, 0)$. As $x \vee y \in \langle d, 1\rangle - D$, for some i we have $x \vee y \geqslant f^i$, $(c_0^i, c_1^i) \in r_{x\vee y}^i = r_x^i \vee r_y^i \subseteq (r_x - \{(a, a')\}) \vee (r_y - \{(a, a')\}) = s(x, 0) \vee s(y, 0)$. Therefore $(a, a') \in s(x, 0) \vee s(y, 0)$, a contradiction.

Theorem 3.1. *Let L be an at least two-element finite lattice. Then $L \in FR$ iff (a) or (b) holds:*

a) *There exist non-trivial congruences Θ_1, Θ_2 on L such that $\Theta_1 \cap \Theta_2 = \omega$ and L/Θ_1, $L/\Theta_2 \in FR$.*

b) *There exists a non-trivial small congruence Θ on L such that $L/\Theta \in FR$.*

Proof. First assume $L \in FR$. Choose any finite representation r of L on some finite set A with a minimal possible cardinality. As L has at least two elements, we can choose on r_1-minimal element $a_1 \in A$ and one r_1-maximal element $a_2 \in A$ such that $a_1 \neq a_2$. The assignments $x \mapsto r_x | (A - \{a_i\})$, $i = 1, 2$, define homomorphisms of lattices since the sets $A-\{a_i\}$ are r_1-convex. As A has a minimal possible cardinality, these homomorphisms are not injective. Let Θ_i be their kernels, then $L/\Theta_i \in FR$. The congruence $\Theta=\Theta_1 \cap \Theta_2$ is small, the corresponding homomorphism of semilattices is the following one:

$$\varphi(x)=0, \quad \text{if} \quad (a_1, a_2) \notin r_x,$$

$$\varphi(x)=1, \quad \text{if} \quad (a_1, a_2) \in r_x.$$

If $\Theta=\omega$, (a) holds. Assume Θ to be non-trivial. Then there exists a monomorphism of lattices $L/\Theta \to L/\Theta_1 \times L/\Theta_2$ and by Lemma 1.2 and Lemma 1.3, $L/\Theta \in FR$ and (b) holds.

Now assume that (a) holds. As there exists a monomorphism of lattices $L \to L/\Theta_1 \times L/\Theta_2$, there suffices to use Lemma 1.2 and Lemma 1.3.

More interesting is the case (b). Let Θ be a nontrivial small congruence on L and $L/\Theta \in FR$. By Lemma 2.4, L is isomorphic to some lattice arising from L/Θ by the construction described in Lemma 2.3. By Lemma 3.3 and Lemma 1.2, $L \in FR$.

Lemma 3.4. *Let L be a finitely representable lattice, $\Theta \in \operatorname{Con}(L)$ an atom. Then Θ is small.*

Proof. Induction on the cardinality of L. Assume that lemma holds for all lattices with cardinality less than n and choose a lattice $L \in FR$ with the cardinality n and an atom $\Theta \in \operatorname{Con}(L)$. First we prove the following statement:

(*) Let $\lambda \in \operatorname{Con}(L)$ be not comparable with Θ and $L/\lambda \in FR$. Then Θ is small.

As $\operatorname{Con}(L)$ is distributive, $\Theta \vee \lambda$ covers λ, and so $(\Theta \vee \lambda)/\lambda$ is an atom in $\operatorname{Con}(L/\lambda)$. By the induction assumption $\Theta \vee \lambda/\lambda$ is small, so there exists a homomorphism of semilattices φ: $L/\lambda \to \{0,1\}$ injective on each class of $\Theta \vee \lambda/\lambda$. Define the mapping ε: $L \to \{0,1\}$, $x \mapsto \varphi([x]\lambda)$. Then ε is a homomorphism of semilattices, we want to prove it is injective on each class of Θ. Choose $(x, y) \in \Theta$ such that $\varepsilon(x)=\varepsilon(y)$. As $\varphi([x]\lambda)=\varphi([y]\lambda)$ and $[x]\lambda$, $[y]\lambda$ are in the same class of the congruence $\Theta \vee \lambda/\lambda$, $(x, y) \in \lambda$. Therefore $(x, y) \in \Theta \cap \lambda=\omega$.

Let us continue the proof of the lemma. By Theorem 3.1 there are two possibilities:

a) There exist non-trivial congruences Θ_1, Θ_2 on L such that $\Theta_1 \cap \Theta_2=\omega$ and $L/\Theta_i \in FR$. As Θ is an atom, some of Θ_i is not comparable with Θ and it suffices to use (*).

b) There exists a non-trivial small congruence λ on L such that $L/\lambda \in FR$. The case $\Theta=\lambda$ is trivial. If $\Theta \neq \lambda$, Θ and λ are not comparable and it suffices to use (*).

Lemma 3.5. *Let L be a finitely representable lattice, $\Theta \in \operatorname{Con}(L)$ an atom. Then $L/\Theta \in FR$.*

Proof. Induction on the cardinality of L again. By Theorem 3.1 there are two possibilities:

a) There exist non-trivial congruences Θ_1, Θ_2 on L such that $\Theta_1 \cap \Theta_2 = \omega$ and $L/\Theta_i \in FR$. Each of the congruences $\Theta \vee \Theta_i/\Theta_i$ is either trivial or an atom. By the induction assumption, $(L/\Theta_i)/(\Theta \vee \Theta_i/\Theta_i) \in FR$. By the definition of factor congruences, these factor lattices are isomorphic to $L/\Theta \vee \Theta_i$. As $\mathrm{Con}(L)$ is distributive, $(\Theta \vee \Theta_1) \cap (\Theta \vee \Theta_2) = \Theta \vee (\Theta_1 \cap \Theta_2) = \Theta$, therefore there exists a monomorphism of lattices $L/\Theta \to (L/\Theta \vee \Theta_1) \times (L/\Theta \vee \Theta_2)$ and it suffices to use Lemma 1.2 and Lemma 1.3.

b) There exists a non-trivial small congruence λ on L such that $L/\lambda \in FR$. We can assume $\Theta \neq \lambda$. As Θ is an atom, $\Theta \vee \lambda/\lambda$ is also one and by the induction assumption, $(L/\lambda)/(\Theta \vee \lambda/\lambda) \in FR$. This factor lattice is isomorphic to $L/\Theta \vee \lambda$. By Lemma 3.4, Θ is small, and by Lemma 2.9, $\Theta \vee \lambda/\Theta$ is small. As $L/\Theta \vee \lambda$ is isomorphic to $(L/\Theta)/(\Theta \vee \lambda/\Theta)$, by Theorem 3.1, $L/\Theta \in FR$.

Theorem 3.2. *The class FR is closed under factorisation.*

Proof. Simple induction on the cardinality of the lattice on the base of Lemma 3.5.

Theorem 3.3. *Let L be a finite lattice. The following three statements are equivalent:*

(i) $L \in FR$;

(ii) $\Theta_2 > \Theta_1$ *in* $\mathrm{Con}(L) \Rightarrow \Theta_2/\Theta_1$ *is small;*

(iii) $\langle \Theta_1, 1 \rangle = \{\Theta_1\} \cup \langle \Theta_2, 1 \rangle$ *in* $\mathrm{Con}(L) \Rightarrow \Theta_2/\Theta_1$ *is small, where* 1 *is the greatest element of* $\mathrm{Con}(L)$.

Proof. Theorem 3.2 and Lemma 3.4 imply (i)$\Rightarrow$(ii). The implication (ii)$\Rightarrow$(iii) is trivial. The implication (iii)$\Rightarrow$(i) will be proved by induction on the cardinality of L. Assume that it holds for all lattices with cardinality less than $n > 1$ and choose a lattice L with the cardinality n satisfying (iii). If there exist two different atoms Θ_1, Θ_2 in $\mathrm{Con}(L)$, then by the induction assumption $L/\Theta_i \in FR$, and it suffices to use Theorem 3.1. If $\mathrm{Con}(L)$ has exactly one atom Θ, then $\langle \omega, 1 \rangle = \{\omega\} \cup \langle \Theta, 1 \rangle$, thus by (iii) Θ is small. By the induction assumption, $L/\Theta \in FR$. It suffices to use Theorem 3.1 again.

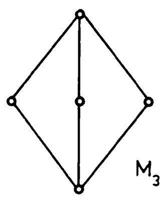

Fig. 6

Corollary. *A finite modular lattice is finitely representable iff it is distributive.*

Proof. The lattice M_3 (see fig. 6) is not finitely representable since it has no small congruence $\neq \omega$, therefore each modular finitely representable lattice is distributive. Conversely, if L is a finite distributive lattice and Θ_1, Θ_2 satisfy the assumptions of (iii), then either $\Theta_1=\Theta_2$ or L/Θ_1 is subdirectly irreducible and Θ_2/Θ_1 is the atom of its congruence lattice, therefore L/Θ_1 is the two-element chain and Θ_2/Θ_1 is small.

4. A characterization of finitely commutatively representable lattices (Schein's problem)

We know that $FCR \subseteq FR$ and FCR is closed under isomorphisms, formation of sublattices and finite direct products. In this paragraph we shall prove $FCR = FR$.

For any natural number n let C_n be the set $\{0, 1, \ldots, n\}$ and $\leqslant$ the natural order on C_n.

Theorem 4.1. *For each n, $S(C_n, \leqslant) \in FCR$.*

Proof. Form the set $D_n = \{X | 0 \in X \subseteq C_n\}$. Let us define the mapping j: $S(C_n, \leqslant) \to S(D_n, \subseteq)$ in the following way: for any $\alpha \in S(C_n, \leqslant)$ and $A, B \in D_n$, $(A, B) \in j(\alpha)$ iff

$$B = A \dot{\cup} (r_1, s_1\rangle \dot{\cup} (r_2, s_2\rangle \dot{\cup} \ldots \dot{\cup} (r_k, s_k\rangle$$

for some finite number of pairs $(r_i, s_i) \in \alpha$, where $\dot{\cup}$ is the symbol for the disjoint union and $(r_i, s_i\rangle$ are half-closed intervals in $(C_n, \leqslant)$.

The definition of j is correct, as $j(\alpha)$ are orders and $(A, B) \in j(\alpha)$ implies $A \subseteq B$. Moreover, j is monotone, hence $j(\alpha \vee \beta) \supseteq j(\alpha) \vee j(\beta)$, $j(\alpha \cap \beta) \subseteq j(\alpha) \cap j(\beta)$ for any α, β.

Lemma 4.1. *For any α, $\beta \in S(C_n, \leqslant)$, $j(\alpha \vee \beta) \subseteq j(\alpha) \vee j(\beta)$.*

Proof. Assume that $(A, B) \in j(\alpha \vee \beta)$; then

$$B = A \dot{\cup} (r_1, s_1\rangle \dot{\cup} \ldots \dot{\cup} (r_k, s_k\rangle,$$

where $(r_i, s_i) \in \alpha \vee \beta$, therefore there exist finite chains $r_i = t_{i,0} \leqslant t_{i,1} \leqslant \ldots \leqslant t_{i,m_i} = s_i$, $(t_{i,p-1}, t_{i,p}) \in \alpha \cup \beta$ for $p = 1, \ldots, m_i$, $i = 1, \ldots, k$. Then

$$(r_i, s_i\rangle = (t_{i,0}, t_{i,1}\rangle \dot{\cup} \ldots \dot{\cup} (t_{i,m_i-1}, t_{i,m_i}\rangle,$$

therefore we can assume $(r_i, s_i) \in \alpha \cup \beta$.

Let us define $A_0 = A$, $A_1 = A_0 \dot{\cup} (r_1, s_1\rangle$, $A_2 = A_1 \dot{\cup} (r_2, s_2\rangle$, $\ldots$, $A_k = A_{k-1} \dot{\cup} (r_k, s_k\rangle = B$. By the definition of j, $(A_{i-1}, A_i) \in j(\alpha) \cup j(\beta)$ for $i = 1, 2, \ldots, k$. As $A_0 = A$ and $A_k = B$, we have $(A, B) \in j(\alpha) \vee j(\beta)$.

Lemma 4.2. *For any α, $\beta \in S(C_n, \leqslant)$, $j(\alpha \cap \beta) \supseteq j(\alpha) \cap j(\beta)$.*

Proof. By the definition, $(A, B) \in j(\gamma)$ iff

$$B = A \dot{\cup} (r_1, s_1\rangle \dot{\cup} \ldots \dot{\cup} (r_k, s_k\rangle,$$

where $(r_i, s_i) \in \gamma$. We can assume $s_1 < s_2 < \ldots < s_k$. As the written union is disjoint, it is

$$r_1 \leqslant s_1 \leqslant r_2 \leqslant s_2 \leqslant \ldots \leqslant r_k \leqslant s_k .$$

If $r_i = s_i$, $(r_i, s_i\rangle = \emptyset$ and this interval can be omitted. If $s_i = r_{i+1}$, $(r_i, s_i\rangle \dot{\cup} (r_{i+1}, s_{i+1}\rangle = (r_i, s_{i+1}\rangle$ and the two intervals can be replaced by one interval $(r_i, s_{i+1}\rangle$, where $(r_i, s_{i+1}) \in \gamma$ since γ is transitive. Therefore we can assume

$$r_1 < s_1 < r_2 < s_2 < \ldots < r_k < s_k .$$

Let us have a pair $(A, B) \in j(\alpha) \cap j(\beta)$. As we have just proved

$$B = A \dot{\cup} (r_1, s_1\rangle \dot{\cup} \ldots \dot{\cup} (r_k, s_k\rangle =$$

$$= A \dot{\cup} (r'_1, s'_1\rangle \dot{\cup} \ldots \dot{\cup} (r'_{k'}, s'_{k'}\rangle ,$$

where $r_1 < s_1 < \ldots < r_k < s_k$, $r'_1 < s'_1 < \ldots < r'_{k'} < s'_{k'}$, $(r_i, s_i) \in \alpha$, $(r'_i, s'_i) \in \beta$, but then $k = k'$, $r_i = r'_i$, $s_i = s'_i$ and therefore each $(r_i, s_i) \in \alpha \cap \beta$, which gives $(A, B) \in j(\alpha \cap \beta)$.

Lemma 4.3. *j is injective.*

Proof. Choose two different orders α, $\beta \in S(C_n, \leqslant)$, then there exists say $(p, q) \in \beta$ such that $(p, q) \notin \alpha$. Let us denote $A = \{0, 1, \ldots, p\}$, $B = \{0, 1, \ldots, q\}$. Then $B = A \dot{\cup} (p, q\rangle$ and by the definition of j there is $(A, B) \in j(\beta)$. We shall prove $(p, q) \notin j(\alpha)$.

Assume $(A, B) \in j(\alpha)$; then $B = A \dot{\cup} (r_1, s_1\rangle \dot{\cup} \ldots \dot{\cup} (r_k, s_k\rangle$, where $(r_i, s_i) \in \alpha$. As in the proof of Lemma 4.2, we can assume that $r_1 < s_1 < \ldots < r_k < s_k$, but then the convexity of $B - A$ gives $k = 1$, $r_1 = p$, $s_1 = q$, $(p, q) = (r_1, s_1) \in \alpha$, a contradiction.

Let us continue the proof of Theorem 4.1. By the preceding lemmas, j is a finite representation of $S(C_n, \leqslant)$. We shall prove this representation to be commutative. Take any pair $(A, B) \in j(\alpha) \circ j(\beta)$, then there exists a set $U \in D_n$ such that $(A, U) \in j(\alpha)$, $(U, B) \in j(\beta)$. Form the set $V = A \cup (B - U)$. A simple set-theoretical calculation gives $(A, V) \in j(\beta)$, $(V, B) \in j(\alpha)$, therefore $(A, B) \in j(\beta) \circ j(\alpha)$.

Corollary. $FCR = FR$.

Proof. Choose a lattice $L \in FR$, L is isomorphic to a sublattice of some $S(A, \tau)$ with A finite. There exists an order γ on A such that $\tau \subseteq \gamma$ and (A, γ) is a chain. Then $S(A, \tau)$ is an ideal in $S(A, \gamma)$ and $S(A, \gamma)$ is isomorphic to $S(C_n, \leqslant)$, where $n + 1$ is the cardinality of A.

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ПРЕДСТАВЛЕНИЯ КОНЕЧНЫХ СТРУКТУР УПОРЯДОЧЕНИЯМИ НА КОНЕЧНЫХ МНОЖЕСТВАХ

Богуслав Цивак

Резюме

Для любого упорядоченного множества (A, τ) может быть построена структура $S(A, \tau)$ всех упорядочений η на множестве A, для которых $\eta \subseteq \tau$. Изоморфизм структуры L на подструктуру структуры $S(A, \tau)$ называется представлением структуры L упорядочениями на множестве A. Известно, что каждая структура обладает представлением взаимно перестановочными упорядочениями. Конечные структуры представляются упорядочениями на бесконечных счетных множествах.

Конечные структуры обладающие представлениями при помощи упорядочений на конечных множествах здесь характеризуются свойствами их структур конгруэнций. Оказывается, что такие представления существуют например для всех конечных дистрибутивных структур. Наконец доказано, что если конечная структура обладает таким представлением, то она обладает и представлением взаимно перестановочными упорядочениями на конечном множестве.