Mathematica Slovaca

Melvin F. Janowitz

Note on the center of a lattice

Mathematica Slovaca, Vol. 28 (1978), No. 3, 235--242

Persistent URL: http://dml.cz/dmlcz/136179

Math. Slovaca 28, 1978, No. 3, 235—242

NOTE ON THE CENTER OF A LATTICE

M. F. JANOWITZ

1. Preliminaries. When is the center of a complete lattice L a *closed* sublattice of L? Several authors have considered this problem and have arrived at sufficient conditions. Among such conditions are:

(a) L is a continuous geometry (von Neumann [18]).
(b) L is an orthocomplement modular lattice (Kaplansky [16]).
c) L is an orthomodular lattice (Foulis [2], Holland [6]).
(d) L is a relatively semi-orthocomplemented lattice (Maeda [17]).
(e) L is a relatively complemented lattice (Janowitz [11]).
(f) L is both section and dual section semicomplemented (Janowitz [13]).
(g) (i) L has permutable congruence relations, and
(ii) if $a, b, c, d \in L$, $a \leqslant b \leqslant c \leqslant d$, $b\theta c$ for some congruence relation θ on L, then there are elements $b_1, c_1 \in L$ such that $a < b_1 \leqslant d$, $a \leqslant c_1 < d$, $a\theta b_1$, $c_1\theta d$ (Jakubík [10]).

Condition (e) generalized (a) through (d), and both (f) and (g) are generalizations of (e). It is our purpose here to further generalize the above results by showing (among other things) that the condition on permutability can be omitted from (g).

In [11] we remarked that it was not known if the center of a complete lattice L was a closed sublattice of L. At that time we were not aware of [8] in which an example is given of a complete nondistributive lattice whose center fails to be closed. In [12] we produced an example of an upper continuous distributive lattice whose center is not closed. Indeed, we used results of Hashimoto [5] to establish that the center of the lattice of congruences of a bounded distributive lattice L is a closed sublattice if and only if every chain of L is finite. Such a result also follows immediately from [3], Theorem 2, p. 86. An easy example along these lines can be obtained by considering the ideal lattice $I(L)$ of a Boolean algebra L. The fact here is that the center of $I(L)$ is a closed sublattice of $I(L)$ if and only if L is finite. Similar examples were produced independently by Jakubík [9], who also obtained necessary and sufficient conditions for the center of a bounded distributive lattice to be a closed sublattice.

The basic notation we have been using is that of Birkhoff [1]. In what follows, we shall also be using notation that is defined in [4], [14], and [15]. L will denote a bounded lattice with smallest element 0, largest element 1, and axiom (X*) will denote the dual of axiom (X). In [15] we introduced axiom

(A) *$a/0 \to c/d$ with $c > d$ implies $c/d \to a_1/a_2$ for suitable a_1, a_2 with $a \geqslant a_1 > a_2$.*

We then proved that (A) is equivalent to the assertion that for every congruence relation θ on L, $a\theta^*0$ if and only if the interval $[0, a]$ contains only trivial congruence classes modulo θ. More generally, Grätzer and Schmidt [4] say that a lattice is *weakly modular* if $a/b \to c/d$ with $c > d$ implies that $c/d \to a_1/b_1$ for some a_1, b_1 with $a \geqslant a_1 > b_1 \geqslant b$. By [7], Theorem 4, p. 230 L is weakly modular if and only if for each congruence relation θ on L, $a\theta^*b$ is equivalent to the assertion that the interval $[a \wedge b, a \vee b]$ has only trivial congruence classes modulo θ.

We next introduce the following axioms:

(B′) *If $b, c, d \in L$, $b < c \leqslant d$, $b\theta c$ for some congruence relation θ on L, there is an element $c_1 \in L$ such that $c_1 < d$ and $c_1 \theta d$.*

(C′) *If $b, c, d \in L$, $b < c \leqslant d$, $b\theta c$ for some congruence relation θ on L, there is an element $c_1 \in L$ such that $b \leqslant c_1 < d$ and $c_1 \theta d$.*

The significance of axiom (B′) can best be understood by assuming that every congruence relation on L is the minimal one generated by a dual distributive filter.[1]) Suppose then that $b < c \leqslant d$ and $b\theta c$ for the congruence relation θ. We must have $b \wedge t = c \wedge t$ for some t such that $t\theta 1$. Then $t \ngeqslant d$, so $t \wedge d < d$ and $t \wedge d \theta d$. If every congruence relation θ of L has the property that $\{t \in L: t\theta 1\}$ is a principal filter, then by the dual of [14], Theorem 5.2, p. 295, axiom (B′) is equivalent to the assertion that each congruence relation of L is the minimal one generated by a dual distributive filter. Similarly, if each congruence relation of L is the minimal one generated by a dual standard filter,[1]) and if $b < c \leqslant d$ with $b\theta c$, we may write $b = c \wedge t$ with $t\theta 1$. Consequently $t \geqslant b$, $t \ngeqslant d$, so $b \leqslant t \wedge d < d$ with $t \wedge d \theta d$, and (C′) holds. If each congruence relation θ has the property that $\{t \in L: t\theta 1\}$ is a principal filter, then by [14], Theorem 5.3, p. 296, axiom (C′) is equivalent to the assertion that each congruence relation is the minimal one generated by a dual standard filter. Evidently the second part of (g) is equivalent to the validity of axioms (C′) and (C′*).

We shall also have occasion to invoke the following axioms:

(P) *For each congruence relation θ, θ^* is permutable with θ^{**}.*

(S) *The lattice of congruence relations of l is a Stone lattice.*

2. Results. We are finally ready to consider the center of L. In [10], Theorem 1, p. 342 it is assumed that $\{z_i\}$ is a family of central elements, that z_i' is the component of z_i, and that *both* $z = \bigwedge_i z_i$ and $z' = \bigvee_i z_i'$ exist. In the presence of (g) it

[1]) For this notion see [14]

is shown that z is central with z' its complement. Our first goal is to show the above result to be true if *either* $z=\bigwedge_i z_i$ or $z'=\bigvee_i z'_i$ exist. Before we can do this, we need a preliminary result.

Lemma 1. *If L satisfies both* (C′) *and* (C′*), *then L is weakly modular.*

Proof: In [14], p. 290 we noted that for each congruence relation θ of L, $a\theta^*b$ holds iff $a\vee b/a\wedge b \to c/d$ with $c\theta d$ implies $c=d$. If we could establish that when a/b has trivial congruence classes modulo θ and $a/b\to c/d$, the same is true for c/d, it would follow that $a\theta^*b$ iff $a\vee b/a\wedge b$ has trivial congruence classes modulo θ. By [7], Theorem 4, p. 230, it would follow that L is wealky modular.

Thus we need only establish that if a/b has trivial congruence classes modulo θ, the same is true for $a\vee x/b\vee x$ and $a\wedge x/b\wedge x$. By duality, we need only consider $a\vee x/b\vee x$. If $a\vee x\geqslant a_1>b_1\geqslant b\vee x$ with $a_1\theta b_1$ then by (C′) there is an element $c\in L$ such that $a\vee x>c\geqslant b_1$ and $a\vee x\theta c$. Now $c\geqslant b$ but $c\ngeqslant a$, since $c\geqslant a$ would force $c\geqslant a\vee x$, contrary to $a\vee x>c$. But $c\theta a\vee x$ implies $c\wedge a\theta(a\vee x)\wedge a=a$, so we have the contradictory assertion that $a>c\wedge a\geqslant b$ with $a\theta c\wedge a$. We deduce that $a\vee x/b\vee x$ has only trivial congruence classes modulo θ, thereby establishing the lemma.

It is immediate from Lemma 1 that the condition on weak modularity may be omitted from the hypothesis of [15], Theorem 7, p. 181. What is more pertinent is that it enables us to prove

Theorem 2. *Let $\{z_i\}$ be a family of central elements of L. For each index i, let z'_i denote the complement of z_i. Suppose that $z=\bigwedge_i z_i$ exists in L. Each of the following conditions is sufficient to guarantee that z be central:*

(1) (P), (C′) *and* (C′*).

(2) (P), (A*) *and* (B′*).

(3) (P) *and* (S).

If in addition $z'=\bigvee_i z'_i$ exists in L, then each of the following is sufficient to guarantee that z be central with z' its complement:

(4) (C′) *and* (C′*).

(5) (A), (B′) *and* (B′*).

Proof: For each index i, define θ_i by $x\theta_i y$ iff $x\vee z_i=y\vee z_i$. Then each θ_i is a central element of the lattice of congruences of L, and $\theta=\bigcap_i\theta_i$ has kernel $[0,z]$. Before proceeding, we note that by Lemma 1, (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5). Because of this, we need not establish the sufficiency of either condition (1) or condition (4). Assuming that $z=\bigwedge_i z_i$ exists, we now proceed to establish the sufficiency of (2) and (3).

(2) Assume (P), (A*) and (B′*). Let $a>b$ with $a\theta b$. If $a\vee z>b\vee z$, then by (B′*) there is an element $b_1>z$ such that $b_1\theta z$. But $b_1\theta z$ forces $b_1\leqslant z$. From this contradiction we deduce that $a\vee z=b\vee z$, whence θ is the minimal congruence generated by the distributive element z. By [15], Lemma 2, p. 278, z is in fact neutral. It is now immediate that θ has a complement θ' given by the rule $x\theta' y$ iff $x\wedge z = y\wedge z$. Using (P) and the argument given by Jakubík ([10], p. 340) we see that z has a complement, so it is central.

(3) Assume (P) and (S). We may use the fact that the center of *any* Stone lattice is closed under the formation of arbitrary existing infima to deduce that θ is a central element of the lattice of congruences of L. By (P) and the argument given in [10], p. 342, z is a central element of L.

Assume now that $z'=\bigvee_i z'_i$ exists in L.

(5) Assume (A), (B′) and (B′*). We begin by noting that $x\theta_i y$ is equivalent ot both $x\vee z_i=y\vee z_i$ and $x\wedge z'_i = y\wedge z'_i$. We already know that $[0, z]$ is the kernel of θ, and a dual argument shows that $[z', 1] = \{t\in L\colon t\theta 1\}$. Suppose $z'\geqslant c>b$ with $b\theta c$. Then by (B′), $z'\theta c_1$ for some $c_1<z'$. But $c_1\theta z'$ implies $c_1\geqslant z'$. Thus $[0, z']$ has only trivial congruence classes modulo θ. By [15], Lemma 1, p. 177, $z'\theta^*0$. Suppose $a\theta^*0$. Then $a\wedge z'\theta a$ with $a\wedge z'\leqslant a$ forces $a=a\wedge z'\leqslant z'$. Thus $[0, z']$ is the kernel of θ^*, so $z\wedge z'=0$. Suppose $z\vee z'<1$. Then $z\vee z'\theta 1$, so by (B′*), we can find an element $t>z$ for which $t\theta z$. But $t\theta z$ implies $t\leqslant z$. From this we deduce that $z\vee z'=1$. We now invoke [14], Theorem 7.4, p. 300 to see that z is central with z' its complement.

To make further contact with [10], we assume that (2) or (3) of Theorem 2 holds, and that $z=\bigwedge_i z_i$ and $z'=\bigvee_i z'_i$ exist. By Theorem 2, z is central. We claim that z' is necessarily the complement of z. We recall from the proof of the sufficiency of (2) and (3) that $\theta=\bigcap_i \theta_i$ is the minimal congruence relation whose kernel is $[0, z]$. Suppose z^* is the complement of z. Then $z\wedge z'_i=0$ forces $z'_i\leqslant z^*$, so $z'=\bigvee_i z'_i\leqslant z^*$; hence $z\wedge z'=0$. Dually, $z'\vee z_i=1$ implies that $z'\theta_i 1$ for all i, so $z'\theta 1$ and $z'\vee z=1$. This leads immediately to

Corollary 3. *Each of the conditions of Theorem 2 is sufficient for the center of a complete lattice to be a closed sublattice.*

3. Examples. This final section is devoted to the consideration of some examples. These examples serve to provide nontrivial illustrations of the various conditions that occur in Theorem 2.

Example 1. To prove the sufficiency of (4) and (5), we assumed in Theorem 2 that *both* $z=\bigwedge_i z_i$ and $z'=\bigvee_i z'_i$ exist in L. If only $z=\bigwedge_i z_i$ exists, it need not follow that z is central. To see this, let X be an infinite set. Take A to be an infinite subset of X whose complement is also infinite. Let L be the set of those subsets of X of the form $M\cup F$ with F finite and M either a subset of A, or having a finite complement. If L is partially ordered by set inclusion, it is easy to see that L forms a bounded distributive lattice that is both section and dual section semicomplemented. Hence L satisfies (C′) and (C′*). The center of L consists of those subsets of X that are either finite or have finite complements. Notice that A is both the join and the meet of a family of central elements, but A is not itself central. Thus L does not satisfy (P). From this example we see that neither (4) nor (5) implies any of the first three conditions of Theorem 2.

Example 2. Take F to be the lattice of finite dimensional subspaces of an infinite dimensional vector space. Then F is a simple relatively complemented modular lattice with 0 that does not have a largest element. Now let $L=F\cup F^*$, where F^* denotes the dual of F. Partially order L by the rules $a\leqslant b$ if

(*i*) $a\leqslant b$ in F or F^*, or

(*ii*) $a=0$, or

(*iii*) $b=1$.

Then L becomes a lattice in which $a>0$ in F, $b<1$ in F^* together imply a, b are complements. It is easy to show that the only nontrivial congruence relation on L is the congruence θ whose congruence classes are F and F^*. It is immediate that L satisfies (B′) and (B′*). Routine verification even shows L to be weakly modular, so it satisfies (A) and (A*). To see that (C′) fails, let $b<c$ in F. Then $b<c<1$ and $b\theta c$. But there is no element c_1 such that $b\leqslant c_1<1$ and $c_1\theta 1$. Dually, (C′*) also fails. Note that this example even satisfies (S).

Example 3. Take F^* as in Example 2, and adjoin a zero element. The lattice of congruences of the resulting lattice L is a 3 element chain. It is easy to verify that L satisfies (P), (S) (B′), and is weakly modular; yet L does not satisfy (B′*). To see this, consider the congruence relation whose classes are $\{0\}$ and F^*.

Example 4. Let L be an incomplete Boolean algebra. Then L satisfies (P), (C′), (C′*), but not (S).

The table in Fig. 1 illustrates the various possible implications among the conditions of Theorem 2. Though the question of the center being a closed sublattice is nonexistent, it may prove illuminating to consider the conditions of Theorem 2 for the case of a finite lattice L. Suppose such a lattice satisfies (A), (B′) and (B′*). Let θ be a congruence relation on L. By (B′), if $[z, 1] = \{t\in L\colon t\theta 1\}$ then z is a dual distributive element of L, and θ is the minimal congruence relation whose cokernel is $[z, 1]$. Let $[0, z']$ denote the kernel of θ. If $z\vee z'<1$, then $z\leqslant z\vee z'<1$ with $z\vee z'\theta 1$ shows that there must be an element t such that $t>z$ and $t\theta z$, a contradiction. We deduce that $z\vee z'=1$, and a dual argument shows that $z \wedge z' = 0$. By [15], Lemma 2, p. 178, z is neutral; consequently, z is central with z' its complement. Thus every congruence relation is the minimal one generated by a central element of L. It is immediate that L is the direct product of simple lattices, so L satisfies (C′), (C′*), as well as (P), (S). It now follows that (1), (4), (5) are equivalent, and that any one of them implies (3).

(1)→(2)	True
(1)→(3)	False Example 4
(1)→(4)	True
(1)→(5)	True
(2)→(1)	False Example 2
(2)→(3)	False Example 4
(2)→(4)	False Dual of Example 3
(2)→(5)	False Dual of Example 3
(3)→(1)	False Example 2
(3)→(2)	False Example 3
(3)→(4)	False Example 3
(3)→(5)	False Example 3
(4)→(1)	False Example 1
(4)→(2)	False Example 1
(4)→(3)	False Example 1
(4)→(5)	True
(5)→(1)	False Example 1
(5)→(2)	False Example 1
(5)→(3)	False Example 1
(5)→(4)	False Example 2

Table illustrating the various possible implications among the conditions of Theorem 2.

Fig. 1

Suppose now that (P), (A*) and (B′*) hold. Let $[0, z]$ be the kernel of the congruence relation θ, and use (B′*) to show that z is distributive with θ the minimal congruence relation generated by z. Then use (A*) to show that z is neutral, and (P) to show that it is central. It follows as before that L satisfies (C′) and (C′*), so we have established the equivalence of conditions (1), (2), (4) and (5) of Theorem 2. In order to show that (3) is not equivalent to these conditions, we consider the lattice whose Hasse diagram appears in Fig. 2. This is a section complemented lattice, so every congruence relation is the minimal one generated by a standard element. The only standard elements are 0, k, and 1, so the lattice of congruences is a 3 element chain, hence a Stone lattice. Thus L satisfies (P) and (S), but it does not satisfy (C′). To see this, note that if θ is the congruence generated by k, then $k\theta f$ and $f<k<1$; yet there is no element t such that $f \leqslant t < 1$ and $t\theta 1$.

We close by mentioning that (C) od [15] implies (C'). If for each congruence relation θ of L, $\{t \in L: t\theta 1\}$ is a principal filter, then ([14], Theorem 5.3, p. 296)

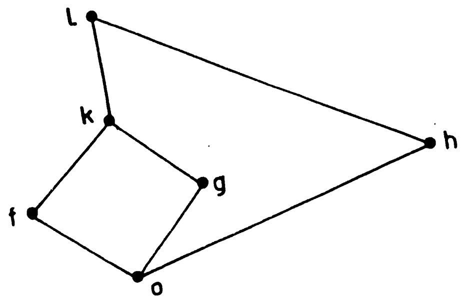

Fig. 2

the two conditions are equivalent. It is natural to conjecture that (C') need not imply (C), and it might be of some interest to produce an example to illustrate this. A similar situation occurs with (B) of [15] and (B').

REFERENCES

[1] BIRKHOFF, G.: Lattice theory, third edition. Amer. Math. Soc. Colloquium Publications Vol. 25, Providence 1967.

[2] FOULIS, D. J.: A note on orthomodular lattices. Portugal. Math., 21, 1962, 65—72.

[3] GRÄTZER, G.—SCHMIDT, E. T.: Two notes on lattice-congruences. Budapest. Tudomány-Egyetem. Annales. Sectio Math., 1, 1958, 83—87.

[4] GRÄTZER, G.—SCHMIDT, E. T.: Ideals and congruence relations in lattices. Acta Math. Acad. Sci. Hung., 9, 1958, 137—175.

[5] HASHIMOTO, J.: Ideal theory for lattices. Math. Japon., 2, 1952, 149—186.

[6] HOLLAND, Jr., S. S.: A Radon-Nikodym theorem in dimension lattices. Trans. Amer. Math. Soc., 108, 1963, 66—87.

[7] IQBALUNNISA: Normal, simple and neutral congruences on lattices. Illinois J. Math., 10, 1966, 227—234.

[8] JAKUBÍK, J.: Centrum nekonečne distributívnych zväzov. Mat. ffz. Čas., 8, 1957, 116—120.

[9] JAKUBÍK, J.: Center of a complete lattice. Czechoslovak Math. J., 23, 1973, 125—138.

[10] JAKUBÍK, J.: Center of a bounded lattice. Mat. Čas., 25, 1975, 339—343.

[11] JANOWITZ, M. F.: The center of a complete relatively complemented lattice is a complete sublattice. Proc. Amer. Math. Soc., 18, 1967, 189—190.

[12] JANOWITZ, M. F.: On the center of a complete distributive lattice. Univ. of New Mexico Tech. Report No. 135, 1967.

[13] JANOWITZ, M. F.: Section semicomplemented lattices. Math. Z., 108, 1968, 63—76.
[14] JANOWITZ, M. F.: A characterisation of standard ideals. Acta Math. Acad. Sci. Hung., 16, 1965, 289—301.
[15] JANOWITZ, M. F.: On a paper by Iqbalunissa. Fund. Math., 78, 1973, 177—182.
[16] KAPLANSKY, I.: Any orthocomplemented complete modular lattice is a continuous geometry. Ann. Math., 61, 1955, 524—541.
[17] MAEDA, S.: On relatively semi-orthocomplemented lattices. Hiroshima Univ. J. Sci., Ser. A, 24, 1960, 155—161.
[18] von NEUMANN, J.: Continuous geometry. Princeton, 1960.

Received May 7, 1976

Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA, 01002, U.S.A.