Slávka Bodjanová

Some basic notions of mathematical analysis in oriented metric spaces

*Mathematica Slovaca*, Vol. 31 (1981), No. 3, 277--289

Persistent URL: `http://dml.cz/dmlcz/136270`

Math. Slovaca 31, 1981, No. 3, 277—289

# SOME BASIC NOTIONS OF MATHEMATICAL ANALYSIS IN ORIENTED METRIC SPACES

SLÁVKA BODJANOVÁ

Our purpose is to develop some basic notions of mathematical analysis in oriented metric spaces (denoted by OMS). We obtain the notion of OMS from the usual metric space if we do not assume the distance function to be symmetric (that means, the distance from a point $x$ to a point $y$ may be different from the distance from the point $y$ to the point $x$). These oriented distances can be useful in practical applications. For instance, in a hilly country, it makes a difference whether an automobile climbs from a locality $A$ to a locality $B$ or goes down from $B$ to $A$, considering the cost of transport.

We give a survey of some concepts and results from the theory of oriented metric spaces which are analogous to the concepts and results from the usual metric spaces theory and some new results. Analogous results are stated without proofs which can be found in the standard monographs, e. g. Kolmogorov—Fomin [1].

## 1. Oriented metric spaces

**Definition 1.1.** *Let $M$ be a nonempty set. A nonnegative function $\varrho$ defined on the Cartesian produck $M \times M$ is called an oriented metric if it satisfies the following axioms:*

1. *for each $x, y \in M$ $\varrho(x, y) = 0$ if and only if $x = y$*
2. *for each $x, y, z \in M$ $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$.*

*The pair $(M, \varrho)$ is called an oriented metric space (OMS in abbrevation).*

Example 1.2. We obtain an oriented metric $\varrho$ on the set of all real numbers $R$, e. g. in this manner:

$\varrho: R \times R \to R$

$\varrho(x, y) = |y| - |x|$ for $|y| > |x|$

$\varrho(x, y) = 0$ for $x = y$

$\varrho(x, y) = |x|$ for $|x| \geq |y|$ and $y \neq x$.

In general $\varrho(x, y) \neq \varrho(y, x)$

Example 1.3. Let $E$, $F$, $G$ be pairwise disjoint sets which can be indexed by one-one mapping as follows:

$$E\{e_a;\, a\in(0,1\rangle\},\quad F=\{f_a;\, a\in(0,1\rangle\},\quad G=\{g_a;\, a\in(1,2\rangle\}.$$

Define a nonnegative function $\varrho\colon M\times M\to R$ as follows:

$$\varrho(e_a, e_a)=\varrho(f_a, f_a)=\varrho(g_c, g_c)=0$$
$$\text{for } a\in(0,1\rangle \text{ and for } c\in(1,2\rangle.$$

$$\varrho(e_a, f_a)=\varrho(f_a, e_a)=\varrho(e_a, f_b)=\varrho(f_a, e_b)=\varrho(e_a, g_c)=$$
$$=\varrho(f_a, g_c)=\varrho(e_a, e_b)=\varrho(f_a, f_b)=a$$
$$\text{for } a, b\in(0,1\rangle \text{ and } a\neq b \text{ and for } c\in(1,2\rangle,\ \varrho(g_c, e_a)=\varrho(g_c, f_a)=c-a,$$
$$\text{for } c\in(1,2\rangle\ a\in(0,1\rangle.$$

$$\varrho(g_c, g_d)=c-d \text{ if } c>d,\ \varrho(g_c, g_d)=c \text{ if } c<d,\ d\in(1,2\rangle.$$

This function satisfies the axioms from Definition 1.1.

Example 1.4. Let $M$ be the half-open interval $(0,2\rangle$. It is easy to show that a function $\varrho\colon M\times M\to R$ defined by $\varrho(x,y)=0$ for $x=y$, $\varrho(x,y) = x-y+1$ for $1<y<x+1\leq 2$, $\varrho(x,y)=x$ in the other cases, is an oriented metric on $(0,2\rangle$ and therefore $(M,\varrho)$ is an OMS.

## 2. Topologies induced by an oriented metric

**Definition 2.1.** *Let $(M,\varrho)$ be an oriented metric space, $x\in M$, $\varepsilon>0$. The set $L_\varepsilon(x) = \{y\in M;\ \varrho(y,x)<\varepsilon\}$ will be called an l-neighbourhood of $x$. The set $R_\varepsilon(x) = \{y\in M;\ \varrho(x,y)<\varepsilon\}$ will be called an r-neighbourhood of $x$.*

We shall first describe some examples of $l$-($r$-)neighbourhoods which will be helpful in the sequel.

Example 2.2. Let us consider the OMS from Example 1.2. Let $a, -a\in M$ and $0<\varepsilon<a$. Then

$$L_\varepsilon(a)=(-a,-a+\varepsilon)\cup(a-\varepsilon,a\rangle,$$
$$L_\varepsilon(-a)=\langle -a,-a+\varepsilon)\cup(a-\varepsilon,a)$$

$$R_\varepsilon(a)=(-a-\varepsilon,-a)\cup\langle a,a+\varepsilon),$$
$$R_\varepsilon(-a)=(-a-\varepsilon,-a\rangle\cup(a,a+\varepsilon).$$

Example 2.3. Let us consider the OMS from Example 1.3. Let $0<\varepsilon<a$, where $a\in(0,1\rangle$ and $c\in(1,2\rangle$. Then

$$L_\varepsilon(e_a)=\{e_x;\, x\in(0,\varepsilon)\}\cup\{f_x;\, x\in(0,\varepsilon)\}\cup\{g_x;\, x\in(1,a+\varepsilon)\}\cup$$
$$\cup\{e_a\},\ R_\varepsilon(e_a)=\{e_a\}$$

$$L_\varepsilon(f_a)=\{e_x;\, x\in(0,\varepsilon)\}\cup\{f_x;\, x\in(0,\varepsilon)\}\cup\{g_x;\, x\in(1,a+\varepsilon)\}\cup$$
$$\cup\{f_a\},\ R_\varepsilon(f_a)=\{f_a\}$$

$$L_\varepsilon(g_c)=\{e_x\,;\,x\in(0,\,\varepsilon)\}\cup\{f_x\,;\,x\in(0,\,\varepsilon)\}\cup\{g_x\,;\,x\in\langle c,\,c+\varepsilon)\cap(1,\,2\rangle\}$$

$$R_\varepsilon(g_c)=\{g_x\,;\,x\in(c-\varepsilon,\,c\rangle\cap(1,\,2\rangle\}.$$

Example 2.4. Consider the OMS from Example 1.4. Let $a\in M$ and $\varepsilon<a$. If $0<a\leqslant 1$, then $L_\varepsilon(a) = (0,\,\varepsilon)\cup\{a\}$ and $R_\varepsilon(a) = (\max\{a+1-\varepsilon,\,1\},\,a+1)\cup\{a\}$. If $1<a\leqslant 2$, then $L_\varepsilon(a) = (0,\,\varepsilon)\cup(a-1,\,\min\{1,\,a-1+\varepsilon\})\cup\{a\}$ and $R_\varepsilon(a)=\{a\}$.

**Theorem 2.5.** *The collection $L(x)=\{L_\varepsilon(x);\ \varepsilon\in R,\ \varepsilon>0\}$ is a neighbourhood system of $x$.*

Proof: It is easy to show that the collection $L(x)$ has the following properties:

1. $U\in L(x)\Rightarrow x\in U$
2. $U\in L(x)\wedge V\in L(x)\Rightarrow \exists W\in L(x)\colon W\subset U\cap V$
3. $U\in L(x)\wedge z\in U\Rightarrow \exists V\in L(z)\colon V\subset U$

Quite analogously the collection $R(x) = \{R_\varepsilon(x);\ \varepsilon\in R,\ \varepsilon>0\}$ is a neighbourhood system of $x$.

Remark 2.6. In general, the collection $U(x)=L(x)\cup R(x)$ is not a neighbourhood system of $x$. To show this, consider Example 2.2. Assume that $U(x)=L(x)\cup R(x)$ is a neighbourhood system of $x$. This assumption implies that for each $U_1,\ U_2\in U(x)$ there exists a set $U_3\in U(x)$ such that $U_3\subset U_1\cap U_2$. We have $L(x)\in U(x)$, $R(x)\in U(x)$ and $L(x)\cap R(x)=\{x\}$. Since all the sets in $U(x)$ are infinite, there is no set $U_3\in U(x)$ such that $U_3\in U(x)$, which contradicts our assumption.

**Theorem 2.7.** *Let $(M,\,\varrho)$ be an OMS. The collection $B_L=\{L_\varepsilon(x);\ x\in M,\ \varepsilon>0\}$ is a base of the topology $T_L = \{\cup U;\ U\in B_L\}\cup\{\emptyset\}$ on $M$. Similarly the collection $B_R=\{R_\varepsilon(x);\ x\in M,\ \varepsilon>0\}$ is a base of the topology $T_R = \{\cup U,\ U\in B_R\}\cup\{\emptyset\}$ on $M$.*

Remark 2.8. In general, an OMS $(M,\,\varrho)$ with the topology $T_L$ (or $T_R$) is not a Hausdorff topological space. We can show this fact using Example 2.2.

Consider $a,\,-a\in M$, $a\neq 0$. Obviously $a\neq -a$. We have $L_{\varepsilon_1}(a)\cap L_{\varepsilon_2}(-a) = (-a,\,\min\{-a+\varepsilon_1,\,-a+\varepsilon_2\})\cup(\max\{a-\varepsilon_1,\,a-\varepsilon_2\},\,a)\neq\emptyset$ for each $\varepsilon_1,\,\varepsilon_2>0$.

Remark 2.9. The following example shows that the topologies $T_L$ and $T_R$ may be incommensurable. Define an oriented metric $\varrho\colon M\times M\to R$, where $M=(0,\,\infty)$, as follows:

For $a,\,b\in M$ $a\leqslant b$ we put $\varrho(a,\,b)=b-a$
for $a,\,b\in M$ $a>b$ we put $\varrho(a,\,b)=b$.

Obviously we have: if $\varepsilon\leqslant a$: $L_\varepsilon(a) = (a-\varepsilon,\,a\rangle$, $R_\varepsilon(a) = (0,\,\varepsilon)\cup\langle a,\,a+\varepsilon)$. If $\varepsilon>a$: $L_\varepsilon(a) = (0,\,\infty)$, $R_\varepsilon(a) = (0,\,a+\varepsilon)$. We claim that $T_L\not\subset T_R$. It is sufficient to show that no $r$-neighbourhood $R_\varepsilon(a)$ can be written as a union of $l$-neighbourhoods of points of $M$. Suppose that $R_\varepsilon(a)=\bigcup_{\substack{x\in M\\ \delta>0}} L_\delta(x)$, $\varepsilon<a$. Then we can find an $l$-neighbourhood $L_\delta(b)$ such that $a\in L_\delta(b)$. Since $L_\delta(b)$ is the neighbourhood system of $b$, there exists $L_\xi(a) = (a-\xi, a\rangle$ such that $L_\xi(a)\subset L_\delta(b)$. But $L_\xi(a) = (a-\xi, a\rangle \not\subset R_\varepsilon(a) = (0, \varepsilon)\cup\langle a, a+\varepsilon)$, which is a contradiction. The proof that $T_R\not\subset T_L$ is very similar to the above one.

## 3. Convergence

**Definition 3.1.** *Let $(M, \varrho)$ be an oriented metric space and $\{x_n\}_{n=1}^\infty$ a sequence in $M$. We say that $x_n$ $l$-converges to $x\in M$ in $M$ (and write $x_n\rightarrow x$) if $\lim_{n\to\infty}\varrho(x_n, x)=0$, i. e. if for every $\varepsilon>0$ there exists an $n_0\in N$ such that $\varrho(x_n, x)<\varepsilon$ holds for all $n>n_0$.*

*We say that $x_n$ $r$-converges to $x\in M$ (and write $x\leftarrow x_n$) if $\lim_{n\to\infty}\varrho(x, x_n)=0$.*

We have shown that, in general, an OMS with the topology $T_L$ (or $T_R$) is not a Hausdorff topological space (see Remark 2.8) and therefore the $l$-convergence (or $r$-convergence) in OMS is not unique.

Example 3.2. Let $(M, \varrho)$ be an OMS from Example 1.3. The sequence $\{g_{1+\frac{1}{n}}\}_{n=1}^\infty$ $l$-converges to $e_1\in M$ and at the same time to $f_1\in M$, because $\lim_{n\to\infty}\varrho(g_{1+\frac{1}{n}}, e_1) = \lim_{n\to\infty}\varrho(g_{1+\frac{1}{n}}, f_1) = \lim_{n\to\infty}\left(1+\frac{1}{n}-1\right) = \lim_{n\to\infty}\frac{1}{n}=0$. The sequence $\{e_{\frac{1}{n}}\}_{n=1}^\infty$ $l$-converges to every point of $M$, because $\lim_{n\to\infty}\varrho(e_{\frac{1}{n}}, x) = \lim_{n\to\infty}\frac{1}{n}=0$ for every $x\in M$.

It will be useful to introduce the following sets:

$$[x_n]L=\{x\in M;\ x_n\rightarrow x\}\quad\text{and}\quad L[x_n]=\{x\in M;\ x\leftarrow x_n\}.$$

Then the results of Example 3.2 can be written as follows:

$$\left[g_1+\frac{1}{n}\right]L=\{e_1, f_1\},\ \ [e_{\frac{1}{n}}]L=M.$$

Remark 3.3. Let $\{x_n\}_{n=1}^\infty$ be a sequence in an oriented metric space $(M, \varrho)$ and $\{x_{n_k}\}_{k=1}^\infty$ a subsequence of it. Obviously $[x_n]L\subset[x_{n_k}]L$. But the converse inclusion, i. e. $[x_{n_k}]L\subset[x_n]L$ is not true in general. For instance, consider the OMS from Example 1.3. Let $\{x_n\}_{n=1}^\infty$ be defined by:

$$x_n=g_{1+\frac{1}{n}}\ \text{for}\ n\ \text{even}$$

$$x_n=e_{\frac{1}{n}}\ \text{for}\ n\ \text{odd}.$$

Obviously $[x_n]L=\{e_1, f_1\}$. If $x_{n_k}=x_n$ for $n$ odd, then $[x_{n_k}]L=M$, hence $[x_{n_k}]L\subset [x_n]L$ is not true.

In the OMS, analogously to the usual metric spaces, the following notions can be introduced:

1. $l$- ($r$-) closure of a subset $A$ of $M$ (denoted by $\bar{A}(L)$, $\bar{A}(R)$)
2. $l$- ($r$-) closed set $A$ in $M$ (*if* $A=\bar{A}(L)$, resp. $A=\bar{A}(R)$)
3. $l$- ($r$-) open set $A$ in $M$ (if $M-A=\overline{M-A}(L)$, $M-A=\overline{M-A}(R)$)
4. $l$- ($r$-) dense set $A$ in $M$ (if $\bar{A}(L)=M$, resp. $\bar{A}(R)=M$)
5. $l$- ($r$-) point of accumulation of a subset $A$ of $M$ and the set of all $l$- ($r$-) points of accumulation of $A$ (denoted by $A^d(L)$, resp. $A^d(R)$)
6. $l$- ($r$-) boundary point of $A$ and $l$- ($r$-) boundary of a subset $A$ of $M$
7. $l$- ($r$-) interior point of $A$.

Remark 3.4. To those in the usual metric spaces in the OMS analogous theorems are true for the $l$-notions and for the $r$-notions. Some differences arise if we consider an $l$-notion and $r$-notion at the same case. This fact is illustrated by the following example.

Example 3.5. Take the oriented metric $\varrho$ from Example 1.2 and consider an OMS $(M, \varrho_1)$, where $M=\langle 0, \infty)$ and $\varrho_1=\varrho|_{\langle 0, \infty)}$.

Let $0<a<b<\infty$. We can say:

an interval $\langle a, b)$ is $r$-open and $r$-closed but neither $l$-open nor $l$-closed; $(a, b\rangle$ is $l$-open and $l$-closed but neither $r$-open nor $r$-closed; $\langle a, b\rangle$ is $l$- ($r$-) closed but not $l$- ($r$-) open; $(a, b)$ is $l$- ($r$-) open set but not $l$- ($r$-) closed; $\langle 0, b\rangle$ is $l$- ($r$-) closed, $l$-open but not $r$-open; $(a, \infty)$ is $l$- ($r$-) open, $l$-closed but not $r$-closed.

Using these results we give:

$\langle 1, 3)$ is $r$-open, $(2, 4\rangle$ is $l$-open but the union $\langle 1, 3)\cup(2, 4\rangle = \langle 1, 4\rangle$ is neither $l$- nor $r$-open.

$\langle 1, 3)$ is $r$-closed, $(2, 4\rangle$ is $l$-closed but the intersection $\langle 1, 3)\cap(2, 4\rangle = (2, 3)$ is neither $r$- nor $l$-closed.

**Theorem 3.6.** *A subset $A\subset M$ is $l$-dense in $M$ if and only if for every $\varepsilon>0$ and for every $y\in M$ $L_\varepsilon(y)\cap A\neq\emptyset$, i. e., there exists a point $x\in A$ such that $x\in L_\varepsilon(y)$,* i. e. $\varrho(x, y)<\varepsilon$. *An analogous proposition holds true for an $r$-dense set in $M$.*

Example 3.7. a) Let us consider the OMS from Example 1.2. The set of all rational numbers is $l$-($r$-) dense in $R$. We omit the simple proof.

b) Consider the OMS from Example 1.3. Put $E'=\{e_n^1;\ n=1, 2, \dots\}$. Obviously $\bar{E}'(L)=M$ and therefore $E'$ is $l$-dense in $M$. We shall show that in this OMS the following proposition holds: if a subset $A\subset M$ is $r$-dense in $M=E\cup F\cup G$, then

$E \cup F \subset A$. Assume that $A$ is $r$-dense in $M$ and $E \cup F \not\subset A$. This assumption implies that there exists a point $e_a$ (or $f_a$) such that $e_a \in E \cup F$ and at the same time $e_a \notin A$. We can find a number $\varepsilon > 0$ (any $\varepsilon \leqslant a$ will do) and a point $y \in M (y = e_a)$ such that each point $x \in \dot{A}$ satisfies the inequality $\varrho(y, x) \geqslant \varepsilon$ (because $\varrho(e_a, x) = a \geqslant \varepsilon$). Then, in view of Theorem 3.6, the set $A$ is not $r$-dense in $M$, which is a contradiction proving the inclusion $E \cup F \subset A$.

## 4. Separable and complete spaces

**Definition 4.1.** *An oriented metric space $(M, \varrho)$ is said to be $l$-($r$-)separable if there exists a countable set $A \subset M$ which is $l$-($r$-)dense in $M$.*

Example 4.2. a) The OMS from Example 1.2. is an $l$-($r$-) separable space, because the countable set of all rational numbers is $l$-($r$-) dense in $R$.

b) Consider the OMS from Example 1.3. In Example 3.7b) we have shown that the countable set $E' = \{e_n^1; n = 1, 2, \dots\} \subset M$ is $l$-dense in $M$, hence this OMS is $l$-separable. But this OMS is not $r$-separable, because as Example 3.7b) shows, every $r$-dense subset of $M$ is uncoutable.

**Theorem 4.3.** *Any oriented metric space $(M, \varrho)$ with the topology $T_L$ (or $T_R$) is a first-countable topological space.*

Proof: For each $a \in M$ we can take $locB_a = \{L_n^1(a); n = 1, 2, \dots\}$ (or $locB_a = \{R_n^1(a); n = 1, 2, \dots\}$) where $locB_a$ denotes a local base of topology $T_L$ (or $T_R$) at $a$.

**Theorem 4.4.** *Let $(M, \varrho)$ be an OMS such that the topology $T_L$ (or $T_R$) is second-countable. Then $(M, \varrho)$ is an $l$-separable (or $r$-separable) space.*

Remark 4.5. In the usual theory of metric spaces the converse implication holds. In the theory of OMS, in general, it does not. For example, consider the OMS from Example 1.2. This OMS is $l$-($r$-)separable, but the base of the topology $T_R$ must contain all the intervals of the form $\langle a, b)$ for $a \in R$ and obviously it will be uncountable. Analogously the base of the topology $T_L$ must contain all the intervals of the form $(b, a\rangle$, $a \in R$ and therefore it cannot be countable. Hence the topology $T_L$ (or $T_R$) is not second-countable.

**Definition 4.6.** *A sequence $\{x_n\}_{n=1}^{\infty}$ in $M$ is said to be $l$-fundamental if $\lim\limits_{m>n\to\infty} \varrho(x_n, x_m) = 0$, i. e. if for every $\varepsilon > 0$ there exists an $n_0$ such that $\varrho(x_n, x_m) < \varepsilon$ whenever $m > n > n_0$. A sequence $\{x_n\}_{n=1}^{\infty}$ in $M$ is said to be $r$-fundamental if $\lim\limits_{m>n\to\infty} \varrho(x_m, x_n) = 0$.*

Example 4.7. Consider the OMS from Example 1.3. The sequence $\{e_n^1\}_{n=1}^{\infty}$ is

$l$-($r$-)fundamental, because $\lim\limits_{m>n\to\infty} \varrho(e^1_n, e^1_m) = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{m>n\to\infty} \varrho(e^1_m, e^1_n)$ $= \lim\limits_{m\to\infty} \frac{1}{m} = 0$.

The sequence $\{g_{1+\frac{1}{n}}\}_{n=1}^{\infty}$ is $l$-fundamental but not $r$-fundamental, because $\lim\limits_{m>n\to\infty} \varrho(g_{1+\frac{1}{n}}, g_{1+\frac{1}{m}}) = \lim\limits_{m>n\to\infty} \left(\frac{1}{n} - \frac{1}{m}\right) = 0$ and $\lim\limits_{m>n\to\infty} \varrho(g_{1+\frac{1}{m}}, g_{1+\frac{1}{n}})$ $= \lim\limits_{m>n\to\infty} \left(1 + \frac{1}{m}\right) = 1$.

In the usual metric spaces every convergent sequence is fundamental. As the next example shows, in the OMS there are some sequences which are $l$-($r$-)-convergent but not $l$-($r$-)fundamental.

Example 4.8. Consider the OMS from Example 1.3. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in $M$ in this manner:

$$x_n = g_{1+\frac{1}{n}} \text{ for } n \text{ odd}$$

$$x_n = e_{\frac{1}{n+1}} \text{ for } n \text{ even.}$$

Evidently $[x_n]L = \{e_1, f_1\}$. This sequence is not $l$-fundamental because there is an $\varepsilon > 0$ (e. g. $\varepsilon \leq 1$) such that for every $n_0$ there is a number $n > n_0 (n > n_0,\ n$ odd) and a number $m = n + 1$ such that $\varrho(x_n, x_m) = \varrho(g_{1+\frac{1}{n}}, e_{\frac{1}{n+1}}) = 1 + \frac{1}{n} - \frac{1}{n+1} > 1 \geq \varepsilon$. However, this sequence has an $l$-fundamental subsequence (containing only even members of $\{x_n\}_{n=1}^{\infty}$).

**Definition 4.9.** *An oriented metric space $(M, \varrho)$ is called an $F_l - (F_r$-) space if every $l$-($r$-) convergent sequence in $M$ contains an $l$-($r$-) fundamental subsequence.*

Example 4.10. a) The OMS from Example 1.2. and from Example 1.3 are $F_l$- ($F_r$-) spaces.

b) The OMS from Example 1.4 is neither an $F_l$-space nor an $F_r$-space (e. g. the sequence $\left\{\frac{1}{2} + \frac{1}{n}\right\}_{n=1}^{\infty} \to 1 + \frac{1}{2}$ but does not contain any $l$-fundamental subsequence). Analogously the $r$-convergent sequence $\left\{1 + \frac{1}{2} - \frac{1}{n}\right\}_{n=1}^{\infty}$ does not contain any $r$-fundamental subsequence.

c) Obviously the usual metric space is an $F_l$-($F_r$-)space.

**Definition 4.11.** *An oriented metric space $(M, \varrho)$ is said to be $l$-($r$-) complete if every $l$-($r$-) fundamental sequence in $M$ is $l$-($r$-) convergent, i. e. it has an $l$-($r$-) limit in $M$.*

Example 4.12. a) The OMS from Example 1.2. is $l$-($r$-)complete.

b) The OMS from Example 1.3 is $l$-complete but not $r$-complete (the sequence $\{e^1_n\}_{n=1}^{\infty}$ is $r$-fundamental but not $r$-convergent in $M$).

c) The OMS from Example 1.4 is neither $l$-complete nor $r$-complete $\Big($the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is $l$-($r$-) fundamental but it has neither an $l$-limit nor an $r$-limit in $M$$\Big)$.

**Theorem 4.13.** *Let $\{x_n\}_{n=1}^{\infty}$ be an $l$-($r$-) fundamental sequence in $M$ and let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$; if $x_{n_k} \to x$ ($x \leftarrow x_{n_k}$) in $M$, then $x_n \to x$ ($x \leftarrow x_n$) in $M$.*

**Theorem 4.14.** *An OMS $(M, \varrho)$ is $r$- ($l$-) separable if and only if for every $\varepsilon > 0$ there exists a countable subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} L_\varepsilon(x)$ $\left(M \subset \bigcup_{x \in A_\varepsilon} R_\varepsilon(x)\right)$*

Proof:

1. If $(M, \varrho)$ is $r$-separable then $M$ contains an $r$-dense countable subset $B \subset M$. Let $\varepsilon > 0$ be fixed. $R_\varepsilon(x) \cap B \neq 0$ for every $x \in M$, because $B$ is $r$-dense in $M$. A family $A_\varepsilon = \bigcup_{x \in M} (R_\varepsilon(x) \cap B)$ is obviously countable ($B$ is countable) and we shall prove that $M \subset \bigcup_{y \in A_\varepsilon} L_\varepsilon(y)$. If $x_0 \in M$, then there exists a point $y_0 \in A_\varepsilon$ such that $y_0 \in R_\varepsilon(x_0) \cap B$, i.e. $\varrho(x_0, y_0) < \varepsilon$ hence $x_0 \in L_\varepsilon(y_0) \subset \bigcup_{y \in A_\varepsilon} L(y)$.

2 If for every $\varepsilon > 0$ there exists a countable subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} L_\varepsilon(x)$, it is possible to construct a family $A$. $A = \bigcup_{n=1}^{\infty} A_{\varepsilon_n}$, where $\varepsilon_n = \frac{1}{n}$. The family obtained is obviously countable and we shall prove that it is also $r$-dense in $M$. Let $\varepsilon > 0$, $x_0 \in M$. Choose $\varepsilon_n = \frac{1}{n}$ such that $\varepsilon_n < \varepsilon$. $M \subset \bigcup_{y \in A_{\varepsilon_n}} L_{\varepsilon_n}(y)$, hence there exists a point $y_0 \in A_{\varepsilon_n} \subset A$ such that $x_0 \in L_{\varepsilon_n}(y_0)$, i.e. $\varrho(x_0, y_0) < \varepsilon_n < \varepsilon$ and $y_0 \in R_\varepsilon(x_0)$. We have shown that for every $R_\varepsilon(x) \subset M$: $R_\varepsilon(x) \cap A \neq \emptyset$, hence $A$ is $r$-dense in $M$.

Remark 4.15. a) If for every $\varepsilon > 0$ there exists a countable subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} R_\varepsilon(x)$, then $(M, \varrho)$ is not an $r$-separable OMS in general.

b) If $(M, \varrho)$ is $r$-separable, then for every $\varepsilon > 0$ there does not exist in general a countable set $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} R_\varepsilon(x)$.

Example 4.16. Put $M = (0, 1\rangle \cup ((1, 2) \cap Q)$. Define $\varrho$: $M \times M \to R$ as follows:

$$\varrho(x, y) = x - y + 1 \quad \text{if} \quad 0 < |x - y| < 1$$
$$\varrho(x, y) = x - y + 2 \quad \text{if} \quad |x - y| \geq 1$$
$$\varrho(x, y) = 0 \text{ if and only if } x = y.$$

$(M, \varrho)$ is obviously OMS. Let $\varepsilon<1$:

$$\begin{aligned}
&\text{if } x \leqq 1: \ R_\varepsilon(x)=\{x\} \cup ((x+1-\varepsilon, 1+x) \cup (x+2-\varepsilon, 2)) \cap Q \\
&\text{if } x>1: \ R_\varepsilon(x)=\{x\} \cup (x+1-\varepsilon, 2) \cap Q \ \text{ if } \ x-1<\varepsilon \\
&\qquad\qquad R_\varepsilon(x)=\{x\} \ \text{ if } \ x-1 \geqq \varepsilon
\end{aligned}$$

$(0, 2) \cap \mathrm{Q}$ is a countable, $r$-dense subset of $M$, hence $(M, \varrho)$ is $r$-separable. But for $\varepsilon<1$ there does not exist a countable subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} R_\varepsilon(x)$ ($A_\varepsilon$ must contain all points $x \in (0, 1)$).

Let $\varepsilon<x$:

if $x \leqq 1<x+\varepsilon$: $L_\varepsilon(x)=(0, \varepsilon+x-1) \cup \{x\}$
if $x \leqq 1$ and at the same time $x+\varepsilon \leqq 1$: $L_\varepsilon(x)=\{x\}$
if $x>1$ and at the same time $\varepsilon<x-1$:
$L_\varepsilon(x)=(x-1, \varepsilon+x-1) \cup \{x\}$ if $\varepsilon \leqq 2-x$
$L_\varepsilon(x)=(0, x+\varepsilon-2) \cup (x-1, 1\rangle \cup ((1, x+\varepsilon-1) \cap Q) \cup \{x\}$ if $\varepsilon>2-x$.

We have shown that $(M, \varrho)$ is $r$-separable, hence there exists for every $\varepsilon>0$ a countable subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{x \in A_\varepsilon} L_\varepsilon(x)$. But $(M, \varrho)$ is not $l$-separable because every $l$-dense set in $M$ must contain all points $x \in (0, 1)$ and therefore it is uncountable.

## 5. Compactness

Convention: Throughout this paragraph we shall consider only OMS with the topology $T_L$, but analogous statements can be established also for the OMS with the topology $T_R$.

If $(M, \varrho)$ is a usual metric space, then the following assertions are equivalent:

1. Every sequence in $M$ contains a convergent subsequence
2. $(M, \varrho)$ is a compact space
3. Every infinite subset $A \subset M$ has a point of accumulation
4. $(M, \varrho)$ is complete and totally bounded.

Let us discuss these assertions in the theory of OMS.

**Definition 5.0.** *An oriented metric space $(M, \varrho)$ is said to be $l$-($r$-)compact if from every $l$-($r$-)open covering of $M$ it is possible to choose a finite $l$-($r$-)open covering of $M$.*

**Theorem 5.1.** *Every sequence in $M$ contains an $l$-convergent subsequence if and only if every infinite subset $A \subset M$ has an $l$-point of accumulation.*

**Theorem 5.2.** *If $(M, \varrho)$ is an l-compact space, then every sequence in M contains an l-convergent subsequence.*

Problem: Is the converse of Theorem 5.2. true in general? We can easily prove only a weaker assertion:

**Theorem 5.3.** *If $(M, \varrho)$ is a second-countable space and if every sequence in M contains an l-convergent subsequence, then $(M, \varrho)$ is an l-compact space.*

**Definition 5.4.** *A set M is said to be l-totally bounded if for each $\varepsilon > 0$ there exists a finite set $A_\varepsilon \subset M$ such that $M \subset \bigcup_{y \in A_\varepsilon} L_\varepsilon(y)$.*

**Theorem 5.5.** *If $(M, \varrho)$ is an l-compact space, then it is also an l-complete and l-totally bounded space.*

Proof: Since the OMS is $l$-compact, by Theorems 5.2. and 4.13. it is also $l$-complete. Now we prove that $M$ is an $l$-totally bounded set. Let $\varepsilon > 0$ and $M \subset \bigcup_{y \in M} L_\varepsilon(y)$. The family $\{L_\varepsilon(y)\}_{y \in M}$ is obviously an $l$-open covering of $M$ and therefore it is possible to choose from $\{L_\varepsilon(y)\}_{y \in M}$ a finite $l$-open covering, i. e. there is a finite subset $A_\varepsilon \subset M$ such that $M \subset \bigcup_{y \in A_\varepsilon} L_\varepsilon(y)$. This proves the theorem.

Remark 5.6. The converse is not true in general. For example, let $M$ be an interval $(2, 3\rangle$ and $\varrho_1 = \varrho \,|\, (2, 3\rangle$, where $\varrho$ is the oriented metric from Example 1.2. Obviously $(M, \varrho_1)$ is an $l$-complete space. Now we shall prove that $M$ is an $l$-totally bounded set.

If $\varepsilon \geq 1$, $L_\varepsilon(3) \supset (2, 3\rangle$

if $0 < \varepsilon < 1$, there exists the smallest $k \in N$ such that

$3 - k\varepsilon \leq 2 + \varepsilon < 3 - (k-1)\varepsilon$.

$$
\begin{array}{lll}
\text{Choose:} & x_0 = 2 + \varepsilon, & \text{then } L_\varepsilon(x_0) = (2, 2 + \varepsilon\rangle \\
 & x_1 = 3 - \varepsilon, & \text{then } L_\varepsilon(x_1) = (3 - 2\varepsilon, 3 - \varepsilon\rangle \\
 & x_2 = 3 - 2\varepsilon, & \text{then } L_\varepsilon(x_2) = (3 - 3\varepsilon, 3 - 2\varepsilon\rangle \\
 & \vdots & \\
 & x_{k-1} = 3 - (k-1)\varepsilon, & \text{then } L_\varepsilon(x_{k-1}) = (3 - k\varepsilon, 3 - (k-1)\varepsilon\rangle \\
 & x_k = 3, & \text{then } L_\varepsilon(3) = (3 - \varepsilon, 3\rangle.
\end{array}
$$

Evidently the set $A_\varepsilon = \{x_0, x_1, \ldots, x_k\}$ is finite and $\bigcup_{x_i \in A_\varepsilon} L_\varepsilon(x_i) \supset M, i = 0, 1, \ldots, k$.

To show that $(M, \varrho_1)$ is not $l$-compact take

$L = \left\{\left(2 + \frac{1}{n}, 3\right\rangle; n = 1, 2, \ldots\right\}$ for an $l$-oven covering of $M$. Therefore $(M, \varrho_1)$ is not $l$-compact, because evidently no finite subfamily of $L$ covers $M$.

**Theorem 5.7.** *Let $(M, \varrho)$ be an $F_l$-space and let every sequence in $M$ contain an $l$-convergent subsequence. Then $M$ is an $r$-totally bounded set*

Proof.

Suppose that $M$ is not $r$-totally bounded. Hence there is a number $\varepsilon_0 > 0$ such that there does not exist a finite set $A_{\varepsilon_0} \subset M$ such that $M \subset \bigcup_{y \in A_{\varepsilon_0}} R_{\varepsilon_0}(y)$. Let $y_1 \in M$. Then there is a point $y_2 \in M$ such that $\varrho(y_1, y_2) \geqslant \varepsilon_0$ (in the other case $A_{\varepsilon_0} = \{y_1\}$).

Similarly there is a point $y_3 \in M$ such that $\varrho(y_j, y_3) \geqslant \varepsilon_0$, $j = 1, 2$ (in the other case $A_{\varepsilon_0} = \{y_1, y_2\}$). In this manner we can construct a sequence $y_1, y_2, \ldots, y_n, \ldots$ (1) in $M$ such that for each $m, n \in N$, $m > n$, $\varrho(y_n, y_m) \geqslant \varepsilon_0$, i. e. no subsequence of (1) is $l$-fundamental. By assumption the sequence (1) contains an $l$-convergent subsequence. But the $l$-convergent subsequence from (1) does not contain any $l$-fundamental subsequence, which contradicts the assumption that $(M, \varrho)$ is an $F_l$-space.

**Definition 5.8.** *A set $M$ is called totally bounded if for every $\varepsilon > 0$ there exists a finite set $A_\varepsilon \subset M$ such that*

$$M \subset \bigcup_{y \in A_\varepsilon} (L_\varepsilon(y) \cap R_\varepsilon(y)).$$

**Theorem 5.9.** *If an OMS is an $l$-complete and a totally bounded space, then it is also an $l$-compact space.*

This theorem can be proved analogously to the usual theory of metric spaces.

Remark. 5.10. As the next example shows, the converse is not true in general.

Put $M = \left\{\frac{1}{n};\ n = 1, 2, \ldots\right\}$. A nonnegative function $\varrho\colon M \times M \to R$ is defined as follows:

$$\varrho\left(\frac{1}{n}, \frac{1}{n}\right) = 0,\ \varrho\left(\frac{1}{n}, 1\right) = \frac{1}{n} \text{ for } n > 1,\ \varrho\left(\frac{1}{n}, \frac{1}{m}\right) = 1 + \frac{1}{n}$$

$$\text{for } n, m > 1 \text{ and } m \neq n,\ \varrho\left(1, \frac{1}{n}\right) = 1 \text{ for } n > 1.$$

Evidently

$$L_\varepsilon\left(\frac{1}{n}\right) = \left\{\frac{1}{n}\right\} \text{ for } n > 1,\ L_\varepsilon(1) = \left\{\frac{1}{n};\ n \in \left(\frac{1}{\varepsilon}, \infty\right)\right\}$$

$$R_\varepsilon\left(\frac{1}{n}\right) = \left\{\frac{1}{n}\right\} \text{ for } \varepsilon < 1.$$

This oriented metric space is $l$-compact and $l$-complete (the proof is trivial). For all $y \in M$ $L_\varepsilon(y) \cap R_\varepsilon(y) = \{y\}$. Therefore there exists no finite set $A_\varepsilon \subset M$ such that $\bigcup_{y \in A_\varepsilon} L_\varepsilon(y) \cap R_\varepsilon(y) \supset M$ ($M$ is infinite), hence the OMS is not totally bounded.

Remark 5.11. If an $l$-compact oriented metric space $(M, \varrho)$ satisfies the following condition: for each $y \in M$ and each $\varepsilon > 0$ the intersection $L_\varepsilon(y) \cap R\ (y)$ is an $l$-open set, then $(M, \varrho)$ is a totally bounded space. The proof is evident.

**Theorem 5.12.** *Every $l$-compact space is $r$-separable.*
Proof: from Theorems 4.14, 5.5.

**Theorem 5.13.** *Every $l$-compact $F_r$-space is $l$-separable.*
Proof: from Theorems 4.14, 5.7.

## 6. Mappings

**Definition 6.1.** *Let $(X, \varrho_1)$ and $(Y, \varrho_2)$ be oriented metric spaces and let $f$ be a mapping from $X$ into $Y$. Let $a$ be an $l$-point of accumulation of $X$. A point $b \in Y$ will be called an $lr$-limit of $f$ at the point $a$ if for every $r$-neighbourhood $R_\varepsilon(b)$ there is an $l$-neighbourhood $L_\delta(a)$ such that if $x \in L_\delta(a)$, then $f(x) \in R_\varepsilon(b)$. Analogously we define $rl$-($ll$- $rr$-) limits of $f$.*

Similarly as in the usual metric spaces, we can introduce continuous mappings and contractive mappings in the theory of OMS.

**Theorem 6.2.** *Let $(M, \varrho)$ be a nonempty $l$-complete oriented metric space and let $f\colon M \to M$ be a contractive mapping, let the topology $T_L$ on $M$ satisfy the Hausdorff condition. Then there exists exactly one point $x_0 \in M$ such that $x_0 = f(x_0)$. This point will be called an $l$-fixed point of $f$.*

Remark 6.3. The preceding theorem is not true if the topology $T_L$ is not assumed to be Hausdorff. To show this, consider the OMS from Example 1.3.

Put $M_1 = E$ and $\varrho_1 = \varrho\,|E.(M, \varrho_1)$ is an $l$-complete OMS and the topology $T_L$ is not Hausdorff. Define a contractive mapping

$$\varphi\colon (M_1, \varrho_1) \to (M_1, \varrho_1) \text{ by: } \varphi(e_a) = e_{a\,2}.$$

The mapping $\varphi$ has no $l$-fixed point.

REFERENCES


[1] KOLMOGOROV, A. N.—FOMIN, S. V.: Элементы теории функций и функционального анализа. Nauka, Moskva 1972.

[2] KELLEY, J. L.: General Topology (ruský preklad), Nauka, Moskva 1968.



Received February 9, 1979



*Katedra štatistiky FR VŠE*
*ul. Odbojárov 10*
*886 33 Bratislava*

# НЕКОТОРЫЕ ПОНЯТИЯ МАТЕМАТИЧЕСКОГО АНАЛИЗА В ОРИЕНТИРОВАННЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

Славка Бодйанова

Резюмэ

В работе рассматриваются некоторые основные понятия математического анализа в ориентированных метрических пространствах. Понятие ориентированного метрического пространства можно получить из понятия обыкновенного метрического пространства, если попустим предпосылку симметричности метрики.