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NEAR LATTICES

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In order to describe the posets by algebraic structures which are similar to lattices we introduce the near lattices with two binary operations denoted by $\bar{\wedge}$ and $\underline{\vee}$. In comparison with lattices only weaker forms of the associative and the commutative laws hold and also the correspondence between posets and near lattices is not unique. We discuss the axioms and show that a near lattice is a lattice if and only if the commutative laws hold. Furthermore we characterize the minimal non trivial subvarieties of the variety of near lattices. Finally we study the associative distributive near lattice and describe the simple algebras of this variety.

1. Near lattices and posets

Definition 1.1. *The algebra* $(V;\ \bar{\wedge},\ \underline{\vee})$ *is called a near lattice if the following equations are fulfilled:*

1) $x\,\bar{\wedge}(y\wedge z)=(x\wedge y)\wedge(y\,\bar{\wedge}\,z)$	1′) $(x\,\underline{\vee}\,y)\underline{\vee}\,z=(x\,\underline{\vee}\,y)\underline{\vee}(y\,\underline{\vee}\,z)$
2) $x\,\bar{\wedge}(x\wedge y)=x\wedge y$	2′) $(x\,\underline{\vee}\,y)\underline{\vee}\,y=x\,\underline{\vee}\,y$
3) $x\,\bar{\wedge}\,x=x$	3′) $x\,\underline{\vee}\,x=x$
4) $x\wedge y=x\wedge(y\wedge x)$	4′) $x\,\underline{\vee}\,y=(y\,\underline{\vee}\,x)\underline{\vee}\,y$
5) $x\,\bar{\wedge}(x\,\underline{\vee}\,y)=x$	5) $x\,\underline{\vee}(x\,\bar{\wedge}\,y)=x$
6) $(y\,\underline{\vee}\,x)\wedge x=x$	6′) $(y\,\bar{\wedge}\,x)\underline{\vee}\,x=x$

Proposition 1.2. *In every near lattice* $(V;\ \bar{\wedge},\ \underline{\vee})$ *the following laws* 6b) $x\wedge(y\,\underline{\vee}\,x)=x$ *and* 6′b) $(x\,\bar{\wedge}\,y)\underline{\vee}\,x=x$ *hold.*

Lemma 1.3. *To every near lattice* $(V;\ \wedge,\ \underline{\vee})$ *these corresponds a poset* $(V;\ \leqq)$ *defined by* $a\leqq b$ *iff* $b\,\bar{\wedge}\,a=a$.

Proof. Reflexivity $a\leqq a$ because $a\wedge a=a$ by 3). Antisymmetry $a\leqq b$ and $b\leqq a$. We have $b\,\bar{\wedge}\,a=a$ and $a\wedge b=b$;

$$b=b\,\underline{\vee}(b\,\bar{\wedge}\,a)=b\,\underline{\vee}\,a\qquad a=a\wedge(b\,\underline{\vee}\,a)=a\,\bar{\wedge}\,b=b.$$

Transitivity $a\leq b$ and $b\leqq c$. We have $b\,\bar{\wedge}\,a=a$, $b=c\,\bar{\wedge}\,b$

$$c\,\bar{\wedge}\,a=c\,\bar{\wedge}(b\wedge a)=(c\,\bar{\wedge}\,b)\wedge(b\wedge a)=b\,\bar{\wedge}\,a=a.$$

Remark 1: We note that $b \wedge a = a$ iff $b \veebar a = b$.

Remark 2: As the operations $\wedge$ and $\veebar$ are not commutative one has to define the antiautomorphism of a near lattice $(V;\ \wedge,\ \veebar)$ in the following way: The bijective map σ: $V \rightarrow V$ is called an antiautomorphism if $\sigma(x \wedge y) = \sigma(y) \veebar \sigma(x)$ and $\sigma(x \veebar y) = \sigma(y) \wedge \sigma(x)$.

Proposition 1.4. *If $x \wedge y < y$, then the element $x \wedge y$ is the infimum of x, y and has the property $x \wedge y = y \wedge x$.*

Proof We have $y \wedge (x \wedge y) = x \wedge y$ and by 4) $y \wedge (x \wedge y) = y \wedge x$. Since $x \wedge (x \wedge y) = x \wedge y$ we have $x \wedge y \leq x$ and $x \wedge y$ is a lower bound of x, y. If r is another lower bound, then $x \wedge r = r = y \wedge r$. We have $r = (x \wedge (y \wedge r) = (x \wedge y) \wedge (y \wedge r) = (x \wedge y) \wedge r$. Therefore $r \leq x \wedge y$.

We notice that from $x < x \veebar y$ it follows that $x \veebar y$ is the supremum of x, y and has the property $x \veebar y = y \veebar x$.

Theorem 1.5. *To every poset $(V;\ \leq)$ there corresponds a near lattice $(V;\ \wedge,\ \veebar)$ which has this poset as order relation.*

Proof:

We define

$$x \wedge y = \begin{cases} y & \text{if } y < x \\ x & \text{else} \end{cases}$$

and

$$x \vee y = \begin{cases} x & \text{if } x > y \\ y & \text{else} \end{cases}$$

The axioms can be verified by direct computation. We prove here only the axiom 1) $x \wedge (y \wedge z) = (x \wedge y) \wedge (y \wedge z)$. Consider the cases 1a) $y \wedge z = y$ and $x \wedge y = x$. Then $A = x \wedge (y \wedge z) = x$ and $B = (x \wedge y) \wedge (y \wedge z) = x \wedge y = x$. 1b) $y \wedge z = y$ and $x \wedge y = y$; then $A = y = B$. 1c) $y \wedge z = z$ and $x \wedge z = z$; then $B = z$ for $x \wedge y = x$ or $x \wedge y = y$. 1d) $y \wedge z = z$ and $x \wedge z = x$; then $B = z$ for $x \wedge y = x$. For $x \wedge y = y$ we have $z \leqq x$ and therefore $x = z$.

Proposition 1.6. *The following weak associative laws hold in a near lattice:*

1b) $(x \wedge y) \wedge z = (x \wedge y) \wedge (z \wedge x)$ 1b') $x \veebar (y \veebar z) = (z \veebar x) \veebar (y \veebar z)$

Proof.

$$\begin{aligned} (x \wedge y) \wedge z &= (x \wedge y) \wedge [z \wedge (x \wedge y)] && \text{by 4)} \\ &= (x \wedge y) \wedge [(z \wedge x) \wedge (x \wedge y)] && \text{by 1)} \\ &= (x \wedge y) \wedge (z \wedge x) && \text{by 4).} \end{aligned}$$

Theorem 1.7. *The near lattice $(V;\ \wedge,\ \veebar)$ is a lattice if and only if the commutative laws hold.*

Proof. We have only to show that the associative laws hold

$$
\begin{aligned}
x\,\bar{\wedge}\,(y\,\bar{\wedge}\,z) &= (x\,\bar{\wedge}\,y)\,\bar{\wedge}\,(y\,\bar{\wedge}\,z) && \text{by 1)}\\
&= (y\,\bar{\wedge}\,x)\,\bar{\wedge}\,(z\,\bar{\wedge}\,y) && \text{by commutativity}\\
&= (y\,\bar{\wedge}\,x)\,\bar{\wedge}\,z && \text{by 1b)}\\
&= (x\,\bar{\wedge}\,y)\,\bar{\wedge}\,z && \text{by commutativity}
\end{aligned}
$$

2. Subvarieties of near lattices

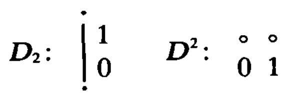
Notations. D_2 is the two-element lattice. By D^2 we denote the algebra $(0, 1; \bar{\wedge}, \underline{\vee})$ defined by

$\bar{\wedge}$	0	1
0	0	0
1	1	1

and

$\underline{\vee}$	0	1
0	0	1
1	0	1

The Hassediagrams are:



Theorem 2.1. *Every near lattice with more than one element has at least D_2 or D^2 as a subalgebra.*

Proof. If $(V; \bar{\wedge}, \underline{\vee})$ has two comparable elements a, b with $a<b$, then a, b generates D_2. We have $b\,\bar{\wedge}\,a=a$, $b\,\underline{\vee}\,a=b$ and $a\,\bar{\wedge}\,b=a\,\bar{\wedge}\,(b\,\bar{\wedge}\,a)=a$, $a\,\underline{\vee}\,b=(b\,\underline{\vee}\,a)\,\underline{\vee}\,b = b\,\underline{\vee}\,b=b$.

If $(V; \bar{\wedge}, \underline{\vee})$ consists only of incomparable elements, then we consider $a, b \in V$. We have $a\,\bar{\wedge}\,b \leqq a$ and hence $a\,\bar{\wedge}\,b=a$. Similarly we have $b\,\bar{\wedge}\,a=b$. We have $b \leqq a\,\underline{\vee}\,b$ and hence $b=a\,\underline{\vee}\,b$. Similarly $b\,\underline{\vee}\,a=a$.

Definition 2.2. *A near lattice is called distributive if the following laws hold:*

6a) $x\,\bar{\wedge}\,(y\,\underline{\vee}\,z)=(x\,\bar{\wedge}\,y)\,\underline{\vee}\,(x\,\bar{\wedge}\,z)$

6b) $(x\,\underline{\vee}\,y)\,\bar{\wedge}\,z=(x\,\bar{\wedge}\,z)\,\underline{\vee}\,(y\,\bar{\wedge}\,z)$

6a′) $x\,\underline{\vee}\,(y\,\bar{\wedge}\,z)=(x\,\underline{\vee}\,y)\,\bar{\wedge}\,(x\,\underline{\vee}\,z)$

6b′) $(x\,\bar{\wedge}\,y)\,\underline{\vee}\,z=(x\,\underline{\vee}\,z)\,\bar{\wedge}\,(y\,\underline{\vee}\,z)$

Lemma 2.3. *If $(V; \bar{\wedge}, \underline{\vee})$ is a associative distributive near lattice, then $f(x)=(a\,\bar{\wedge}\,x)\,\underline{\vee}\,b$ is an endomorphism.*

Proof.

$$
\begin{aligned}
f(x\,\bar{\wedge}\,y) &= [a\,\bar{\wedge}\,(x\,\bar{\wedge}\,y)]\,\underline{\vee}\,b\\
&= [(a\,\bar{\wedge}\,x)\,\bar{\wedge}\,y]\,\underline{\vee}\,b && \text{by associativity}\\
&= [(a\,\bar{\wedge}\,(x\,\bar{\wedge}\,a))\,\bar{\wedge}\,y]\,\underline{\vee}\,b && \text{by 4)}\\
&= [(a\,\bar{\wedge}\,x)\,\bar{\wedge}\,(a\,\bar{\wedge}\,y)]\,\underline{\vee}\,b && \text{by associativity}\\
&= [(a\,\bar{\wedge}\,x)\,\underline{\vee}\,b]\,\bar{\wedge}\,[(a\,\bar{\wedge}\,y]\,\underline{\vee}\,b && \text{by distributivity}\\
&= f(x)\,\bar{\wedge}\,f(y)
\end{aligned}
$$

$f(x\,\underline{\vee}\,y)=f(x)\,\underline{\vee}\,f(y)$ in a similar way.

Lemma 2.4. *A simple associative distributive near lattice cannot contain a chain with more then two elements or a chain with two elements a, b and an element incomparable to a, b.*

Proof. Assume that C is a chain with 3 elements $a<b<c$. The congruence relation θ defined by $(x, y)\in\theta$ iff $(b\wedge x)\vee a = (b\wedge y)\vee a$ is not trivial.

Assume that C is a two-element chain a, b and c an element incomparable to a, b, $a<b$. Again we have $(b\wedge c)\vee a = (b\vee a)\wedge(c\vee a) = b\bar{\wedge}(c\vee a)\leq b$. On the other hand we have $f(a)-(b\wedge a)\vee a=a$ and $f(b)=(b\wedge b)\vee a = b\vee a=b$. Therefore θ is not trivial

Theorem 2.5. *The associative and distributive near lattice V is simple if and only if V is isomorphic to D_2 or D^2.*

Proof. Obviously D_2 and D^2 are simple. We have to show that D^2 is associative and distributive. As the following equations hold $x\wedge y=x$ and $x\vee y=y$, we have $x\wedge(y\wedge z) = (x\wedge y)\wedge z$. Furthermore $x\bar{\wedge}(y\vee z) = (x\wedge y)\vee(x\wedge z)$ and $(x\vee y)\wedge z - (x\wedge z)\vee(y\wedge z)$. On the other hand we have by lemma 2.4 that V is isomorph to D_2 if V has two comparable elements. But V is only simple if V has only two elements.

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ПОЧТИ РЕШЕТКИ

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Резюме

В статье вводится понятие почти решетки с двумя бинарными операциями $\wedge$, $\underline{\vee}$ с целью описать частично упорядоченные множества как алгебраические структуры, похожие решеткам. В почти решетках имеют место только более слабые ассоциативные и коммутативные законы, а также взаимное соответствие между частично упорядоченными множествами и почти решетками не однозначно. Кроме того дана характеризация минимальных не тривиальных подмногообразиий многообразия почти решеток. Приведено тоже описание простых алгебр многообразия ассоциативных дистрибутивных почти решеток.