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ON THE GENERALIZED PROPERTY (K)

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Let X be a complete separable dense in itself metric space.

For any family T of real-valued functions defined on X, let $B_0(T)=T$ and, for each ordinal number $\alpha>0$, let $B_\alpha(T)$ be the family of all pointwise limits of sequences with terms taken from $\bigcup_{\gamma<\alpha} B_\gamma(T)$. The first number α such that $B_\alpha(T)=B_{\alpha+1}(T)$ is called the Baire order of the family T and will be denoted by $r(T)$. Evidently, $r(T)\leqq\omega_1$ where ω_1 is the first uncountable ordinal number. If T is the family of all continuous functions on X, then $B_\alpha(T)$, $\alpha<\omega_1$, are the usual Baire classes which will be shortly denoted by B_α, $\alpha<\omega_1$.

In [4] Grande considered a Borel, G_δ — regular, complete and σ-finite measure μ on X such that $\mu(X)>0$. He introduced the following definition

Definition I. *A function $f\colon X\to\mathbf{R}$ is said to have the property (K) if and only if the set of points of continuity of the function $f|A$ is dense in A for each closed set A such that $U\cap A\neq\emptyset$ implies $\mu(U\cap A)>0$ for all open sets U.*

Denote by $\boldsymbol{K}$ the family of all functions possessing the property (K). Let $\boldsymbol{I}_\mu$ be the σ-ideal of all sets of measure μ zero. Denote by $C_{\boldsymbol{I}_\mu}$ the family of all functions whose sets of points of discontinuity belong to $\boldsymbol{I}_\mu$.

In [4] Grande proved that $\boldsymbol{K}\subset B_1(C_{\boldsymbol{I}_\mu})$, $B_1(\boldsymbol{K})=B_2(C_{\boldsymbol{I}_\mu})$. Using similar methods, we shall extend these results to the case when $\boldsymbol{I}_\mu$ is replaced by an arbitrary σ-ideal $\boldsymbol{I}$ with some natural properties.

Throughout the paper, we shall assume that $\boldsymbol{I}$ is a σ-ideal of subsets of X, all singletons $\{x\}$ belong to $\boldsymbol{I}$ and $\boldsymbol{I}$ does not contain open nonempty sets.

Definition II. (comp. [1]). *We say that a nonempty closed set A is $\boldsymbol{I}$-perfect if and only if $U\cap A\neq\emptyset$ implies $U\cap A\notin\boldsymbol{I}$ for all open sets U.*

Remarks. (a) Of course, every $\boldsymbol{I}$-perfect set A is perfect. Each nonempty set, open in $\boldsymbol{I}$-perfect A, does not belong to $\boldsymbol{I}$.

(b) For any σ-ideal $\boldsymbol{I}$, let $\boldsymbol{I}_1$ denote the σ-ideal of all sets which are contained in sets of type F_σ belonging to $\boldsymbol{I}$ (for example, if $\boldsymbol{I}$ is the σ-ideal of sets of the Lebesgue measure zero on $\boldsymbol{R}$, then $\boldsymbol{I}$, $\boldsymbol{I}_1$ are distinct). Observe that the notions of an $\boldsymbol{I}$-perfect set and an $\boldsymbol{I}_1$-perfect set are identical.

Definition III. ([5], [11]; comp. also [1], [7], [8]). *For any subset A of X, let*

$$A^* = \{x \in X\colon\ U \cap A \notin \boldsymbol{I} \quad \text{for each neighbourhood} \quad U \text{ of } x\}.$$

We give a few properties of the operation $A \mapsto A^*$:

Proposition I. For any subsets A, B of X, we have
(a) $A^* = \bar{A}^* = A^{**} \subset \bar{A}$;
(b) $A \setminus A^* \in \boldsymbol{I}$;
(c) $A \subset B$ *implies* $A^* \subset B^*$;
(d) $B \in \boldsymbol{I}$ *implies* $(A \cup B)^* = (A \setminus B)^* = A^*$;
(e) $A \in \boldsymbol{I}$ *if and only if* $A^* = \emptyset$;
(f) *if* $A \neq \emptyset$, *then* A *is* $\boldsymbol{I}$*-perfect if and only if* $A^* = A$;
(g) $A \notin \boldsymbol{I}$ *if and only if* A^* *is* $\boldsymbol{I}$*-perfect.*

Proof. Properties (a)—(e) were proved in [5], [11] and property (f) in [1]. It remains to show (g). Assume that $A \notin \boldsymbol{I}$. Then, by (e), we have $A^* \neq \emptyset$. In virtue of (a), $A^{**} = A^*$. So, by (f), A^* is $\boldsymbol{I}$-perfect. Assume now that $A \in \boldsymbol{I}$. Then, by (e), we have $A^* = \emptyset$, whence A^* cannot be $\boldsymbol{I}$-perfect.

Definition 1. *A function $f\colon X \to \mathbf{R}$ is said to have the property (K_I) if and only if the set of points of continuity of the function $f|A$ is dense in A for every $\boldsymbol{I}$-perfect set A.*

Remarks. (a) Assume that $\boldsymbol{I}$ contains a nonempty perfect set A. Then there exists a function which is nonborel and has the property (K_I) (comp. [2], Example 1). Indeed, let B be a subset of A such that B, $A \setminus B$ are totally imperfect (see [7], p. 422). We shall show that the characteristic function χ_B of B has the required properties. B is nonborel, so is χ_B. To prove that χ_B has the property (K_I), consider an $\boldsymbol{I}$-perfect set E. We easily check that χ_B restricted to E is continuous at each point of the set $E \setminus A$ which is dense in E since, by Proposition I (a), (d), (f), we have

$$\overline{E \setminus A} \supset (E \setminus A)^* = E^* = E.$$

(b) Assume that $\boldsymbol{I}$ does not contain any nonempty perfect set. Then, every nonempty perfect set is $\boldsymbol{I}$-perfect. Indeed, let A be a nonempty perfect set and suppose that A is not $\boldsymbol{I}$-perfect. There exists an open set U such that $U \cap A \neq \emptyset$ and $U \cap A \in \boldsymbol{I}$. But $U \cap A$ is uncountable, thus it contains a nonempty perfect set (see [7], p. 355), which is impossible. Thus, in this case, the notions of an $\boldsymbol{I}$-perfect set and a nonempty perfect set are identical. Consequently, a function has the property (K_I) if and only if it is in the Baire class 1 (see [7], p. 326).

Definition 1′. *A function $f\colon X \to \mathbf{R}$ is said to have the property (K_I') if and only if $f|A$ has a point of continuity for every $\boldsymbol{I}$-perfect set A.*

Let ω denote the ordinal type of the set of natural numbers.

Lemma 1. *Let $\varepsilon>0$ and assume that f has the property (K_I'). There are an ordinal number $\alpha<\omega_1$ and a sequence $\{U_n\}_{n<\alpha}$ of nonempty, open, pairwise disjoint sets, such that*

(1) $X_n\cap U_n\notin \boldsymbol{I}$ *for all* $n<\alpha$, *where* $X_n=\left(X\setminus\bigcup_{i<n}U_i\right)^*$;

(2) $\operatorname{osc} f\leqq\varepsilon$ *on* $X_n\cap U_n$ *for all* $n<\alpha$;

(3) $X\setminus\bigcup_{n<\alpha}U_n\in\boldsymbol{I}$.

Proof. Let $X_1=X^*$. Notice that X_1 is $\boldsymbol{I}$-perfect by Proposition I (g). From the assumption it follows that $f|X_1$ has a point of continuity we denote by x_1. Hence, there is a neighbourhood U_1 of x_1 such that $\operatorname{osc} f\leqq\varepsilon$ on $X_1\cap U_1$. If $X\setminus U_1\in\boldsymbol{I}$, then put $\alpha=2$; if not, let $X_2=(X\setminus U_1)^*$. As previously, observe that $f|X_2$ has a point of continuity we denote by x_2. In virtue of Proposition I (a), we have $X_2\subset X\setminus U_1$, so $x_2\notin U_1$. Then there exists a neighbourhood U_2 of x_2, disjoint from U_1, such that $\operatorname{osc} f\leqq\varepsilon$ on $X_2\cap U_2$. If $X\setminus(U_1\cup U_2)\in\boldsymbol{I}$, then put $\alpha=3$; if not, we repeat the construction. In this way, we define, by transfinite induction, a sequence $\{U_n\}_{n<\alpha}$ of nonempty, open, pairwise disjoint sets fulfilling conditions (1), (2), (3). In a separable space such a sequence cannot contain an uncountable number of sets, thus $\alpha<\omega_1$.

Lemma 2. *Let a sequence $\{U_n\}_{n<\alpha}$ fulfil the assertion of Lemma 1. Then there is a set $E\in\boldsymbol{I}$ of type F_σ such that $X\setminus\bigcup_{n<\alpha}(X_n\cap U_n)\subset E$.*

Proof. Let $Y_n=X\setminus\bigcup_{i<n}U_i$, $n<\alpha$. The sets U_n are pairwise disjoint, so $U_n\subset Y_n$, $n<\alpha$. Hence

$$X_n\cap U_n=U_n\setminus(Y_n\setminus X_n),\qquad n<\alpha.$$

Consequently,

$$X\setminus\bigcup_{n<\alpha}(X_n\cap U_n)=X\setminus\bigcup_{n<\alpha}(U_n\setminus(Y_n\setminus X_n))$$

$$=\bigcap_{n<\alpha}((X\setminus U_n)\cup(Y_n\setminus X_n))\subset\bigcap_{n<\alpha}((X\setminus U_n)\cup\bigcup_{m<\alpha}(Y_m\setminus X_m))$$

$$=\left(X\setminus\bigcup_{n<\alpha}U_n\right)\cup\bigcup_{m<\alpha}(Y_m\setminus X_m).$$

From Lemma 1 (3) and Proposition I (b) we deduce that the set

$$E=\left(X\setminus\bigcup_{n<\alpha}U_n\right)\cup\bigcup_{m<\alpha}(Y_m\setminus X_m)$$

is of type F_σ and belongs to $\boldsymbol{I}$.

Proposition 1. *A function f has the property* (K_I) *if and only if it has the property* (K'_I).

Proof. Necessity is evident. To prove sufficiency, consider an arbitrary I-perfect set A. In virtue of the Baire theorem, it is enough to demonstrate that the set D_f of points of discontinuity of the function $f|A$ is of the first category in A. Let $s(x)$ denote the oscillation of the function $f|A$ at a point x. Since $D_f=\bigcup_{n<\omega}\{x\in A:\ s(x)>1/n\}$, we only need to show that, for each $\varepsilon>0$, the set $Z_\varepsilon=\{x\in A:\ s(x)>\varepsilon\}$ is nowhere dense in A. Let $U\neq\emptyset$ be an open set in the subspace A. We shall find a set $V\neq\emptyset$, open in A, such that $V\subset U\setminus Z_\varepsilon$. Consider a sequence $\{U_n\}_{n<\alpha}$ fulfilling the assertion of Lemma 1. We have

$$U=\bigcup_{n<\alpha}(U\cap X_n\cap U_n)\cup\Big(U\setminus\bigcup_{n<\alpha}(X_n\cap U_n)\Big)$$
$$\subset\bigcup_{n<\alpha}(U\cap X_n\cap U_n)\cup(U\cap E)$$

where E fulfils the assertion of Lemma 2. The set U is of the second category in A (by the Baire theorem). The set $U\cap E$ is of the first category in A since it is of type F_σ in A and does not contain any nonempty set, open in A (if it contained such a set, we would have $U\cap E\notin I$ and $E\notin I$). Consequently, the set $\bigcup_{n<\alpha}(U\cap X_n\cap U_n)$ is of the second category in A. Since all the sets $U\cap X_n\cap U_n$, $n<\alpha$, are of type F_σ in A, there exist $n_0<\alpha$ and a set $V\neq\emptyset$, open in A, such that $V\subset U\cap X_{n_0}\cap U_{n_0}$. In virtue of Lemma 1 (2), we have osc $f\leqq\varepsilon$ on V, thus $V\subset U\setminus Z_\varepsilon$. The proof has been completed.

Let K_I denote the family of all functions with the property (K_I), and C_I — the family of all functions whose sets of points of discontinuity belong to I. It is easily seen that $C_I\subset K_I$.

Theorem I ([9], th. 3). *Let* $0<\alpha<\omega_1$. *A function f belongs to* $B_\alpha(C_I)$ *if and only if there exists a function g in* B_α *such that the set* $\{x\in X:\ f(x)\neq g(x)\}$ *is contained in a set of type* F_σ *belonging to* I.

Theorem 1. $K_I\subset B_1(C_I)$.

Proof. Let $f\in K_I$. Since $B_1(C_I)$ is closed with respect to uniform convergence, it suffices to show that, for each $\varepsilon>0$, there exists $f_\varepsilon\in B_1(C_I)$ such that $|f(x)-f_\varepsilon(x)|\leqq\varepsilon$ for all $x\in X$. Let $\{U_n\}_{n<\alpha}$ be the sequence constructed in Lemma 1. For each $n<\alpha$, choose $x_n\in X_n\cap U_n$ and define

$$f_\varepsilon(x)=\begin{cases} f(x_n) & \text{if } x\in X_n\cap U_n, \quad n<\alpha\\ f(x) & \text{if } x\in X\setminus\bigcup_{n<\alpha}(X_n\cap U_n);\end{cases}$$

$$g(x)=\begin{cases} f_\varepsilon(x) & \text{if} \quad x\in X_n\cap U_n, \qquad n<\alpha \\ 0 & \text{if} \quad x\in X\setminus\bigcup_{n<\alpha}(X_n\cap U_n). \end{cases}$$

Observe that condition (2) of lemma 1 gives $|f(x)-f_\varepsilon(x)|\leqq\varepsilon$ for all $x\in X$. We shall prove that $f_\varepsilon\in B_1(C_I)$. By Lemma 2, the set $\{x\in X\colon\ f_\varepsilon(x)\neq g(x)\}$ is contained in a set of type F_σ belonging to $\boldsymbol{I}$. Hence, in virtue of Theorem I, it remains to show that $g\in B_1$. Consider a nonempty closed set A. We shall find a nonempty set, open in A, such that g is constant on it. Consequently, g will belong to B_1. The following cases are possible:

1° $A\subset X\setminus\bigcup_{n<\alpha}U_n$. Then $A\subset X\setminus\bigcup_{n<\alpha}(X_n\cap U_n)$, and so g is constant on A.

2° There exists $n_0<\alpha$ such that $A\cap U_{n_0}\neq\emptyset$. If $A\cap U_{n_0}\subset U_{n_0}\cap X_{n_0}$, then g is constant on $A\cap U_{n_0}$. If $A\cap U_{n_0}\setminus X_{n_0}\neq\emptyset$, we have

$$A\cap U_{n_0}\setminus X_{n_0}\subset\Big(X\setminus\bigcup_{n\neq n_0}U_n\Big)\cap(X\setminus X_{n_0})\subset$$

$$\Big(X\setminus\bigcup_{n\neq n_0}(X_n\cap U_n)\Big)\cup(X\setminus(X_{n_0}\cap U_{n_0}))=X\setminus\bigcup_{n<\alpha}(X_n\cap U_n).$$

Hence g is constant on $A\cap U_{n_0}\setminus X_{n_0}$. This ends the proof.

Remark. We have always $K_I\neq B_1(C_I)$. Indeed, let Y be a countable subset of X^*, dense in X^*, and let f be the characteristic function of the set Y. It is not difficult to check that $f\in B_1(C_I)\setminus K_I$.

Theorem 2. $B_1(K_I)=B_2(C_I)$.

Proof. By Theorem 1, we have $B_1(K_I)\subset B_2(C_I)$. We shall prove that $B_2(C_I)\subset B_1(K_I)$. Let $f\in B_2(C_I)$. In virtue of Theorem I, there exist a function $g\in B_2$ and a set $H\in\boldsymbol{I}$ of type F_σ, such that $\{x\in X\colon\ f(x)\neq g(x)\}\subset H$. Let $H=\bigcup_{n<\omega}H_n$ where all the sets H_n are closed and $H_n\subset H_{n+1}$ for each $n<\omega$. Let $\{g_n\}_{n<\omega}$ be a sequence of functions of the Baire class 1 which tends to g pointwise on X. For each $n<\omega$, let

$$f_n(x)=\begin{cases} g_n(x) & \text{if} \quad x\in X\setminus H_n \\ f(x) & \text{if} \quad x\in H_n. \end{cases}$$

Clearly, the sequence $\{f_n\}_{n<\omega}$ tends to f pointwise on X. Moreover, all the functions f_n belong to K_I. Indeed, let A be an $\boldsymbol{I}$-perfect set and consider an arbitrary set $U\neq\emptyset$, open in the subspace A. Since $g_n\in B_1$, the set of points of continuity of $g_n|A$ is dense in A. We have $U\notin\boldsymbol{I}$, $H_n\in\boldsymbol{I}$; hence the set $V=U\setminus H_n$ is nonempty. Moreover, V is open in A, thus $g_n|A$ has a point of continuity $x_0\in V$.

But $g_n|V=f_n|V$, thus $f_n|A$ is continuous at the point x_0. We have shown that $f\in B_1(K_I)$. This completes the proof.

From Theorem 2 we can deduce relationships between $r(K_I)$ and $r(C_I)$. In particular, we have

Corollary. *If $r(K_I)$ or $r(C_I)$ is infinite, then $r(K_I)=r(C_I)$.*

Remarks. (a) Let I denote the σ-ideal of all countable sets. In this case, $K_I=B_1$ (compare Remark (b) following Definition 1). Thus the inclusion $K_I\subset B_1(C_I)$ is obvious. Moreover, $B_2(C_I)=B_2$ (see [10]), so $B_1(K_I)=B_2(C_I)$ follows immediately, as well.

(b) Let I be the σ-ideal of sets of the first category. Since I does not contain any nonempty open set and $X\setminus X^*$ belonging to I is open (see Proposition I (a), (b)), therefore $X^*=X$. It follows that $K_I\subset C_I$. Indeed, if $f\in K_I$, then $f=f|X^*$ is pointwise discontinuous, and so $f\in C_I$. Hence we have $K_I=C_I$, and thus $r(K_I)=1$ (comp. [6]). We can similarly check that if I is a σ-ideal which contains all sets of the first category and does not contain any nonempty open set, then the equation $K_I=C_I$ holds, as well.

(c) Let $I=I_\mu$ where μ denotes the Lebesgue measure on $\mathbf{R}$. Grande in [3] proved that $C_I\neq K_I$ and $r(K_I)=\omega_1$. Moreover, it was shown in [4] that C_I is a nowhere dense subset of K_I with the metric of uniform convergence.

(d) Let I be the σ-ideal constructed by Mycielski in [11]. It was shown in [1] that $r(C_I)=\omega_1$. Thus, by corollary, $r(K_I)=\omega_1$.

(e) We proved in [1] that if J is a σ-ideal included in I, then $r(C_J)\geqq r(C_I)$. Hence, by Corollary, $J\subset I$ and $r(K_I)=\omega_1$ imply $r(K_J)=\omega_1$.

(f) For any σ-ideal I, the notion of an I-perfect set is identical with the notion of a nonempty perfect set in the topology generated by the operation of the derived set $A\mapsto A^*$ (see [8], [1]). This topology is stronger than the previous one. However, the family of continuous functions in either topology is the same (see [8]).

(g) It seems interesting whether, for every σ-ideal I, there exists a topology τ_I on X such that the family $C(\tau_I)$ of functions $f\colon X\to\mathbf{R}$ continuous with respect to τ_I fulfils the condition $K_I=B_1(C(\tau_I))$ (note that if I denotes the σ-ideal of all countable sets, the answer is affirmative).

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ОБ ОБОБЩЕННОМ СВОЙСТВЕ (K)

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Резюме

Пусть I — σ-идеал множеств в полном, сепарабельном, плотном в себе метрическом пространстве. В статье рассматривается свойство (K_I), которое обобщает свойство (K), определенное в [4]. Теоремы из [4] о точечных пределах последовательностей функций со свойством (K) расширены на случай (K_I). Приведены примеры и замечания.