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ON THE LIOUVILLE-TYPE TRANSFORMATION FOR DIFFERENTIAL SYSTEMS

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1. Introduction and preliminary results

Consider a linear differential equation of the second order

$$y'' + p(x)y = 0, \tag{1.1}$$

where $p(x) \neq 0$ is a real function. It is known, see e.g. [illegible] transformation [illegible]

This so-called Liouville transformation is a very useful tool for the investigation of qualitative properties of solutions of (1.1), since many oscillations and asymptotic critera for (1.1) are based on this transformation.

The aim of the present paper is to establish a similar transformation for the differential systems

$$y'' + P(x)y = 0, \tag{1.3}$$

where $P(x)$ is a symmetric $n \times n$ matrix, and also to show some applications of this transformation.

Notation. As usually, $C^m(I)$ denotes the space of real functions having continuous mth derivatives on an interval I. If A is a symmetric $n \times n$ matrix (i.e. $A^T = A$), $A > 0$ ($\geqslant 0$, < 0, $\leqslant 0$) means that A is positive (nonnegative, negative, nonpositive) definite. The inequalities $A > B$ ($\geqslant$, $<$, $\leqslant$) between two symmetric matrices mean that $(A - B) > 0$ ($\geqslant 0$, < 0, $\leqslant 0$). Further, $\|B\| = (\sup_{v^Tv = 1} v^TB^TBv)^{1/2}$ denotes the spectral norm of any square matrix. The symbols E and 0 denote the unit and the zero matrix of any dimension, respectively. Let A be a symmetric matrix. $A^{1/2}$ denotes the (unique) symmetric positive definite matrix for which $A^{1/2}A^{1/2} = A$, the powers $A^{-1/2}$, $A^{\pm 1/4}$ of A have a similar meaning.

Simultaneously with (1.3) we shall consider the associated matrix system

$$Y'' + P(x)Y = 0, \tag{1.4}$$

where $Y(x)$ is an $n \times n$ matrix. A function $y: I \to R^n$ is said to be the solution of (1.3) if $y(x) \in C^2(I)$ and equation (1.3) is identically satisfied on I. Solutions of the matrix system (1.4) are defined analogously.

Now we recall some definitions and concepts which we shall use in the sequel. Two points a, b in an interval I are said to be conjugate relative to a system

$$(F(x)y')' + G(x)y = 0, \tag{1.5}$$

where $F(x)$, $G(x)$ are symmetric $n \times n$ matrices, $F(x) > 0$, if there exists a nontrivial solution $y(x)$ of (1.5) for which $y(a) = 0 = y(b)$. System (1.5) is said to be disconjugate on I if no two distinct points of I are conjugate relative to (1.5). If there exists a number c such that (1.5) is disconjugate on $[c, \infty)$, then this system is said to be nonoscillatory. In the opposite case the system (1.5) is said to be oscillatory.

At the end of this section we recall some properties of the so-called trigonometric matrices. Consider a $2n$-dimensional system of the first order

$$S' = Q(x)C, \quad C' = -Q(x)S, \tag{1.6}$$

where $Q(x)$ is a symmetric $n \times n$ matrix, with the initial condition $S(a) = 0$, $C(a) = E$, $a \in I$. Then $n \times n$ matrices $S(x)$, $C(x)$ are called trigonometric matrices since they have many of the properties of the sine and cosine functions; particularly, they satisfy the following identities

$$\begin{aligned} S^{\mathrm{T}}(x)S(x) + C^{\mathrm{T}}(x)C(x) = E, \quad S^{\mathrm{T}}(x)C(x) = C^{\mathrm{T}}(x)S(x), \\ S(x)S^{\mathrm{T}}(x) + C(x)C^{\mathrm{T}}(x) = E, \quad S(x)C^{\mathrm{T}}(x) = C(x)S^{\mathrm{T}}(x), \end{aligned} \tag{1.7}$$

see e.g. [1]. To emphasize that $\{S(x), C(x)\}$ is a solution of (1.6) with a matrix $Q(x)$ and that the initial condition is given in $x = a$, we shall sometimes denote this solution $\{S(x, Q, a), C(x, Q, a)\}$.

If the matrix $Q(x)$ is nonsingular, system (1.5) can be rewritten as the n-dimensional system of the second order

$$(Q^{-1}(x)Y')' + Q(x)Y = 0 \tag{1.8}$$

and it can be shown, see e.g. [6, p. 264], that the pair of matrices $Y_1(x) = S(x)$, $Y_2(x) = C(x)$ form the base of the solution space of (1.8) in the sense that every solution $Y(x)$ of (1.8) can be expressed in the form $Y(x) = Y_1(x)C_1 + Y_2(x)C_2$, where C_1, C_2 are constant $n \times n$ matrices.

2. The Liouville-type transformation

Theorem 1. *Let $P(x) \in C^2(I)$, $P(x) > 0$, be a symmetric $n \times n$ matrix. There exists a nonsingular $n \times n$ matrix $H(x) \in C^2(I)$ for which*

$$H^{T\prime}(x)H(x) - H^T(x)H'(x) = 0,$$
$$H(x)H^T(x) = P^{-1/2}(x) \tag{2.1}$$

such that the transformation $y = H(x)u$ *transforms the system* (1.3) *into the system*

$$(H^T(x)H(x)u')' + [H^T(x)H''(x) + (H^T(x)H(x))^{-1}]u = 0. \tag{2.2}$$

Proof. Let us denote $K(x) = (P^{-1/4}(x))'P^{-1/4}(x) - P^{-1/4}(x)(P^{-1/4}(x))'$, $L(x) = K(x)P^{1/2}(x) - P^{1/2}(x)K(x)$ and $M(x)$ be a solution of the matrix equation

$$P^{1/2}(x)M(x) + M(x)P^{1/2}(x) = L(x). \tag{2.3}$$

As the matrix $P^{1/2}(x)$ is positive definite, the matrix $M(x)$ is determined by (2.3) uniquely, see [2, p. 205]. From the symmetry of the matrices $L(x)$ and $P^{1/2}(x)$ it follows that $M^T(x)$ also satisfies (2.3), hence $M(x) = M^T(x)$.

Let $G(x)$ be the solution of

$$G' = \tfrac{1}{2}P^{1/2}(x)(K(x) + M(x))G, \quad G(a) = E, \quad a \in I. \tag{2.4}$$

As $P^{1/2}(K + M) + [P^{1/2}(K + M)]^T = P^{1/2}K + P^{1/2}M - KP^{1/2} + MP^{1/2} = P^{1/2}M + MP^{1/2} - L = 0$, the matrix $G(x)$ is orthogonal on I (i.e. $G^{-1} = G^T$). If we set

$$H(x) = P^{-1/4}(x)G(x), \tag{2.5}$$

we have $H^{T\prime}H - H^TH' = [(G^{T\prime}P^{-1/4} + G^T(P^{-1/4})']P^{-1/4}G - G^TP^{-1/4} \cdot [P^{-1/4}G' + (P^{-1/4})'G] = G^{T\prime}P^{-1/2}G + G^T[(P^{-1/4})'P^{-1/4} - P^{-1/4}(P^{-1/4})'] \cdot G - G^TP^{-1/2}G' = \tfrac{1}{2}G^T(M - K)P^{1/2}P^{-1/2}G + G^TKG - \tfrac{1}{2}G^TP^{-1/2}P^{1/2} \cdot (K + M)G = G^T(\tfrac{1}{2}M - \tfrac{1}{2}K + K - \tfrac{1}{2}K - \tfrac{1}{2}M)G = 0$ and $HH^T = P^{-1/4}GG^TP^{-1/4} = P^{-1/4}P^{-1/4} = P^{-1/2}$.

Now, let $y = H(x)u$. Then $H^T(y'' + Py) = H^T(H''u + 2H'u' + Hu'') + H^TPHu = H^T(Hu')' + H^TH'u' + (H^TH'' + H^TP^{1/2}P^{1/2}H)u = (H^THu')' - H^{T\prime}Hu' + H^TH'u' + (H^TH'' + H^T(HH^T)^{-1}(HH^T)^{-1}H)u = (H^THu')' + (H^TH'' + (H^TH)^{-1})u$. As the matrix $H(x)$ is nonsingular, $y(x)$ is a solution of (1.3) if and only if $u(x) = H^{-1}(x)y(x)$ is a solution of (2.2). The proof is complete.

Remark 1. If we replace the assumption "$P(x) > 0$" in Theorem 1 by the weaker condition "$P(x)$ is nonsingular", then we can also use the Liouville-type transformation, but in a modified form, replacing in (2.1) the matrix $P^{-1/2}(x)$ by the matrix $|P(x)|^{-1/2}$, where the matrix $|P(x)|$ is constructed this way: The matrix $P(x)$ can be expressed in the form $T^T(x)\operatorname{diag}\{p_i(x)\}T(x)$, where $p_i(x)$, $i = 1, \ldots, n$, are the eigenvalues of $P(x)$ and $T(x)$ is an orthogonal $n \times n$ matrix. Now we set $|P(x)| = T^T(x)\operatorname{diag}\{|p_i(x)|\}T(x)$, $i = 1, \ldots, n$.

Remark 2. The matrix $H(x)$ is by (2.1) determined uniquely up to a right multiple by some orthogonal $n \times n$ matrix. Indeed, if we replace the initial condition $G(a) = E$ in (2.4) by $G(a) = G_0$, where G_0 is an orthogonal $n \times n$ matrix, the statement of Theorem 1 remains valid.

3. Oscillation criteria

In this section we use the established Liouville-type transformation to derive several oscillation criteria for systems (1.3), which generalize the known oscillation criteria for scalar equations (1.1). In our considerations the following two statements established by Etgen [3] and Morse [5] will play an important role.

Lemma 1 (Etgen). *Let* $F(x) > 0$, $G(x) > 0$ *be symmetric* $n \times n$ *matrices and* $\int^{\infty} \|F^{-1}(x) - G(x)\| \, dx < \infty$. *Then system* (1.5) *is oscillatory if and only if*

Lemma 2 (Morse). *Let* $F_1(x)$, $G_1(x)$ *be symmetric* $n \times n$ *matrices for which* $0 < F_1(x) \leq F(x)$, $G_1(x) \geq G(x)$ $(0 < F(x) \leq F_1(x)$, $G_1(x) \leq G(x))$ *on some interval* $[b, \infty)$. *If the system* (1.5) *is oscillatory* (*nonoscillatory*), *then the system*

$$(F_1(x)y')' + G_1(x)y = 0$$

is also oscillatory (*nonoscillatory*).

In the following statements we suppose that $P(x) > 0$ on some interval $[b, \infty)$.

Corollary 1. *Let the matrix* $H(x)$ *be given by* (2.1), $H^T(x)H''(x) + (H^T(x) \cdot H(x))^{-1} > 0$ *on* $[b, \infty)$ *and* $\int^{\infty} \|H^T(x)H''(x)\| \, dx < \infty$. *Then* (1.3) *is oscillatory if and only if* $\int^{\infty} \operatorname{tr} P^{1/2}(x) \, dx = \infty$.

Proof. As any matrix satisfying (2.1) is given by (2.5) we have

$$\operatorname{tr}(H^T H)^{-1} = \operatorname{tr}(G^T P^{-1/4} P^{-1/4} G)^{-1} = \operatorname{tr}(G^T P^{1/2} G) = \operatorname{tr} P^{1/2}.$$

The statement follows now from Lemma 1 since $n \|H^T H''\| \geqslant \operatorname{tr}(H^T H'') \geqslant -n \|H^T H''\|$.

Corollary 2. *Let the matrix* $H(x)$ *be given by* (2.1). *If* $\int^{\infty} \operatorname{tr} P^{1/2}(x) \, dx = \infty$ *and* $H^T(x)H''(x) \geq 0$ $\left(\int^{\infty} \operatorname{tr} P^{1/2}(x) \, dx < \infty\right.$ *and* $\left.H^T(x)H''(x) \leq 0\right)$ *on some interval* $[b, \infty)$, *then* (1.3) *is oscillatory* (*nonoscillatory*).

Proof. If $\int^{\infty} \operatorname{tr} P^{1/2}(x)\, dx = \infty$, then by Lemma 1 and Corollary 1 the system $(H^T(x)H(x)u')' + (H^T(x)H(x))^{-1}u = 0$ is oscillatory. As $H^T(x)H''(x) \geq \geq 0$, by Lemma 2 the system (2.2) is also oscillatory. The transformation from Theorem 1 preserves the oscillation behaviour of differential systems, hence (1.3) is also oscillatory. Similarly we prove nonoscillation of (1.3) under conditions stated in brackets.

In order to use Corollaries 1 and 2 for detection oscillation or nonoscillation of (1.3), we need to verify assumptions $H^T H'' \geq 0$ (≤ 0) or $\int^{\infty} \|H^T H''\|\, dx < < \infty$. However, we cannot do it directly, since the matrix $G(x)$ in definition of $H(x)$ by (2.5) cannot be, in a general case, computed explicitly. The following statement enables us to avoid this difficulty.

Theorem 2. *Let* $P(x) = D^T(x) \operatorname{diag}\{p_i(x)\} D(x)$, *where* $p_i(x)$, $i = 1, \ldots, n$, *are the eigenvalues of* $P(x)$ *and* $D(x)$ *is an orthogonal* $n \times n$ *matrix. Denote* $R(x) = = \operatorname{diag}\{p_i^{-1/4}(x)\}$, $A(x) = R(x)D(x)D^{T\prime}(x)R^{-1}(x)$, $B(x) = R'(x)D'(x)D^T(x) \cdot \cdot R(x) + R(x)D(x)D^{T\prime}(x)R^{T\prime}(x)$ *and let* $M(x) = (m_{ij}(x))$, *where* $m_{ij}(x) = = -(a_{ij}(x) + a_{ji}(x))(p_i^{1/2}(x) + p_j^{1/2}(x))^{-1}$ *and* $(a_{ij}(x)) = A(x)$.

i) *If* $\int^{\infty} \operatorname{tr} P^{1/2}(x)\, dx = \infty$ *and the symmetric matrix* $M' + MR^{-2}M + + MA^T + AM + B + RR''$ *is nonnegative definite on some interval* $[b, \infty)$, *then* (1.3) *is oscillatory.*

ii) *If* $\int^{\infty} \operatorname{tr} P^{1/2}(x)\, dx < \infty$ *and* $M' + MR^{-2}M + MA^T + AM + B + + RR'' \leq 0$ *on* $[b, \infty)$, *then* (1.3) *is nonoscillatory.*

Proof. We shall prove only part i), the proof of ii) is similar. Denote $H(x) = D^T(x)R(x)T(x)$, where $T(x)$ is the solution of

$$T' = (A^T(x) + R^{-2}(x)M(x))T, \quad T(a) = E, \quad a \in [b, \infty).$$

$A^T + R^{-2}M + (A^T + R^{-2}M)^T = \operatorname{diag}\{p_i^{1/2}\}M + M\operatorname{diag}\{p_i^{1/2}\} + A^T + A = = 0$ (see the definition of M). It implies that $T(x)$ is orthogonal. Further $H^{T\prime}H - H^T H' = (T^{T\prime}RD + T^T R'D + T^T RD')D^T RT - T^T RD(D^{T\prime}RT + + D^T R'T + D^T RT') = T^T[(MR^{-2} + A)R^2 - R^2(A^T + R^{-2}M)]T + T^T \cdot \cdot (R'R - RR')T + T^T R(D'D^T - DD^{T\prime})RT = T^T(M + RDD^{T\prime}R - RD' \cdot \cdot D^T R - M + RD'D^T R - RDD^{T\prime}R)T = 0$ and $HH^T = D^T RTT^T RD = = D^T R^2 D = D^T \operatorname{diag}\{p_i^{-1/2}\}D = (D^T \operatorname{diag}\{p_i^{-1/2}\}D)^{1/2} = P^{-1/2}(x)$. Hence the transformation $y = H(x)u$ transforms (1.3) into (2.2). $H^T H'' = (T^T RD) \cdot \cdot (D^T RT)'' = T^T RD(D^{T\prime\prime}RT + D^T R''T + D^T RT'' + 2D^{T\prime}R'T + 2D^{T\prime}RT' + + 2D^T R'T') = T^T RDD^{T\prime\prime}RT + T^T RR''T + T^T R^2[(A^T + R^{-2}M)T]' +$

$+ 2T^{\mathrm{T}}RDD^{\mathrm{T}\prime}R'T + 2T^{\mathrm{T}}RDD^{\mathrm{T}\prime}R(A^{\mathrm{T}} + R^{-2}M)T + 2T^{\mathrm{T}}RR'(A^{\mathrm{T}} + R^{-2}M)T =$
$= T^{\mathrm{T}}[RDD^{\mathrm{T}\prime\prime}R + RR'' + R^2(R^{-1}D'D^{\mathrm{T}}R)' - 2R^2R^{-3}R'M + R^2R^{-2}M' +$
$+ R^2(A^{\mathrm{T}} + R^{-2}M)(A^{\mathrm{T}} + R^{-2}M) + 2RDD^{\mathrm{T}\prime}R' + 2RDD^{\mathrm{T}\prime}RR^{-1}D'D^{\mathrm{T}}R +$
$+ 2RDD^{\mathrm{T}\prime}R^{-1}M + 2RR'R^{-1}D'D^{\mathrm{T}}R + 2RR'R^{-2}M]T = T^{\mathrm{T}}[RDD^{\mathrm{T}\prime\prime}R +$
$+ RR'' - R'D'D^{\mathrm{T}}R + RD''D^{\mathrm{T}}R + RD'D^{\mathrm{T}\prime}R + RD'D^{\mathrm{T}}R' - 2R^{-1}R'M +$
$+ M' + R^2(R^{-1}D'D^{\mathrm{T}}R)(R^{-1}D'D^{\mathrm{T}}R) + R^2R^{-1}D'D^{\mathrm{T}}RR^{-2}M + MR^{-1}D'D^{\mathrm{T}}R +$
$+ MR^{-2}M + 2RDD^{\mathrm{T}\prime}R' + 2RD'D^{\mathrm{T}}DD^{\mathrm{T}\prime}R + 2RDD^{\mathrm{T}\prime}R^{-1}M + 2R'D'D^{\mathrm{T}}R +$
$+ 2R^{-1}R'M]T = T^{\mathrm{T}}[RDD^{\mathrm{T}\prime\prime}R + RR'' - R'D'D^{\mathrm{T}}R + RD''D^{\mathrm{T}}R + RD'D^{\mathrm{T}\prime}\cdot$
$\cdot R + RD'D^{\mathrm{T}}R' + M' - RD'D^{\mathrm{T}\prime}R + RD'D^{\mathrm{T}}R^{-1}M + MR^{-1}D'D^{\mathrm{T}}R +$
$+ MR^{-2}M + 2RDD^{\mathrm{T}\prime}R' + 2RD'D^{\mathrm{T}\prime}R - 2RD'D^{\mathrm{T}}R^{-1}M + 2R'D'D^{\mathrm{T}}R]T =$
$= T^{\mathrm{T}}[R(DD^{\mathrm{T}})''R + RR'' + MR^2M + M' - RDD^{\mathrm{T}\prime}R^{-1}M + 2RDD^{\mathrm{T}\prime}\cdot$
$\cdot R^{-1}M + MR^{-1}D'D^{\mathrm{T}}R + R'D'D^{\mathrm{T}}R + RDD^{\mathrm{T}\prime}R']T = T^{\mathrm{T}}(M' + MR^{-2}M +$
$+ AM + MA^{\mathrm{T}} + B + RR'')T$, where the fact that D is orthogonal, i.e. $D'D^{\mathrm{T}} + DD^{\mathrm{T}\prime} = 0$, has been used. Hence, we can write system (2.2) in the form

$$(T^{\mathrm{T}}R^2Tu')' + T^{\mathrm{T}}(RR'' + R^{-2} + M' + MR^{-2}M + AM + \\ + MA^{\mathrm{T}} + B)Tu = 0. \tag{3.1}$$

Now, consider the system

$$(T^{\mathrm{T}}R^2Tu')' + T^{\mathrm{T}}R^{-2}Tu = 0.$$

As $\int^{\infty} \operatorname{tr}(T^{\mathrm{T}}R^{-2}T)\,\mathrm{d}x = \int^{\infty} \operatorname{tr} R^{-2}\,\mathrm{d}x = \int^{\infty} \operatorname{tr} P^{1/2}\,\mathrm{d}x = \infty$, this system is oscillatory, see Lemma 1. Since $M' + MR^{-2}M + AM + MA^{\mathrm{T}} + B + RR'' \geqslant 0$, the system (3.1) is oscillatory, see Lemma 2, and hence (1.3) is also oscillatory.

Remark 3. Consider a diagonal system $y'' + \operatorname{diag}\{p_i(x)\}y = 0$, $i = 1, \ldots, n$, and let this system be, e.g., oscillatory (i.e. at least one of the scalar equations $y_i'' + p_i(x)y = 0$, $i = 1, \ldots, n$, is oscillatory). The set of all orthogonal $n \times n$ matrices $G(x) \in C^2\,[b, \infty)$ is by this system decomposed into two classes. The first consists of the matrices $G(x)$ for which the system $y'' + G^{\mathrm{T}}(x)\cdot \cdot\operatorname{diag}\{p_i(x)\}G(x)y = 0$ is oscillatory, the second consists of $G(x)$ for which this system is nonoscillatory. Theorem 2 gives a sufficient condition for $G(x)$ to be in one of these classes. It would be interesting to describe these classes in greater detail, e.g., in the case when $n = 2$, or to give some nontrivial condition on $p_1(x), \ldots, p_n(x)$ for one of these classes to be empty.

4. Asymptotic formulae

In this section we shall use the Liouville-type transformation to derive certain asymptotic formulae for solutions of (1.4) involving trigonometric matrices.

Theorem 3. *Let* $P(x) > 0$; $P(x) \in C^2\,[b, \infty)$, $b \in R$, $H(x)$ *be given by* (2.1) *and* $\int^{\infty} \| H^{\mathrm{T}}(x)H''(x)\| \,\mathrm{d}x < \infty$. *Then a pair of linearly independent solutions* $Y_1(x)$, $Y_2(x)$ *of* (1.4) *can be expressed in the form*

$$Y(x) = H(x)[S(x, (H^{\mathrm{T}}(x)H(x))^{-1}, a) + o(1)]$$
$$Y_2(x) = H(x)[C(x, (H^{\mathrm{T}}(x)H(x))^{-1}, a) + o(1)],$$

where the symbol $o(1)$ *denotes an* $n \times n$ *matrix tending to the zero matrix as* $x \to \infty$.

Proof. The transformation $Y = H(x)U$ transforms (1.4) into the matrix system

$$(H^{\mathrm{T}}(x)H(x)U')' + [H^{\mathrm{T}}(x)H''(x) + (H^{\mathrm{T}}(x)H(x))^{-1}]U = 0. \tag{4.1}$$

Denote in this system $Q(x) = (H^{\mathrm{T}}(x)H(x))^{-1}$, $W(x) = H^{\mathrm{T}}(x)H''(x)$, i.e. $U(x)$ is a solution of

$$(Q^{-1}(x)U')' + Q(x)U = -W(x)U.$$

As $S(x, Q, a)$, $C(x, Q, a)$ form the base of the solution space of the homogeneous system $(Q^{-1}U')' + QU = 0$, using the standard method of variation of constants a particular solution of (4.1) can be written in the form

$$U(x) = S(x, Q, a)D_1(x) + C(x, Q, a)D_2(x),$$

where $D_1(x)$, $D_2(x)$ are $n \times n$ matrices for which

$$\begin{bmatrix} S(x, Q, a) & C(x, Q, a) \\ C(x, Q, a) & -S(x, Q, a) \end{bmatrix} \begin{bmatrix} D_1'(x) \\ D_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -W(x)U \end{bmatrix},$$

and according to (1.7) we have

$$\begin{bmatrix} D_1'(x) \\ D_2'(x) \end{bmatrix} = \begin{bmatrix} S^{\mathrm{T}}(x, Q, a) & C^{\mathrm{T}}(x, Q, a) \\ C^{\mathrm{T}}(x, Q, a) & -S^{\mathrm{T}}(x, Q, a) \end{bmatrix} \begin{bmatrix} 0 \\ -W(x)U \end{bmatrix},$$

i.e.

$$D_1(x) = \int_x^{\infty} C^{\mathrm{T}}(t, Q, a)W(t)U(t)\,\mathrm{d}t$$
$$D_2(x) = -\int_x^{\infty} S^{\mathrm{T}}(t, Q, a)W(t)U(t)\,\mathrm{d}t,$$

(convergence of the last integrals will be proved later) and thus

$$U(x) = \int_x^{\infty} S(x, Q, t)W(t)U(t)\,\mathrm{d}t, \tag{4.2}$$

where the identity $S(x, Q, a)C^{\mathrm{T}}(t, Q, a) - C(x, Q, a)S^{\mathrm{T}}(t, Q, a) = S(x, Q, t)$, see

[6, p. 265], has been used. Using (4.2) we can now verify that the matrices $U_1(x) = S(x, Q, a) + V(x)$, $U_2(x) = C(x, Q, a) + V(x)$ are solutions of (4.1) if and only if $V(x)$ is a fixed point of the integral operator

$$T(V)(x) = \int_x^\infty S(x, Q, t) W(t) (S(t, Q, a) + V(t)) \, dt$$

and it can also be proved that these solutions form a base of the solution space of (4.1). Denote $\mathscr{U} = \{U(x),\ U\colon [b, \infty) \to R^{n^2},\ U(x) = O(1) \text{ for } x \to \infty\}$ and define a norm on $\mathscr{U}$, $\|U(x)\|_{\mathscr{U}} = \sup_{x \in [b, \infty)} \|U(x)\|$. Then $\|\ \|_{\mathscr{U}}$ is the Banache norm on $\mathscr{U}$ and

$$\lim_{x\to\infty} |T(U)(x)| = \lim \left| \int_x^\infty S(x, Q, a) W(t) (S(t, Q, a + U(t)) \, dt \right| \le$$

$$\le \lim_{x\to\infty} \int_x^\infty \|S(x, Q, t)\| (\|S(t, Q, a)\| + \|U(t)\|) \|W(t)\| \, dt \le$$

$$\le k \lim_{x\to\infty} \int_x^\infty \|W(t)\| \, dt = 0,$$

k being a real constant, for $U(x) \in \mathscr{U}$, since $\|S(x, Q, t)\| \le 1$. Thus T maps $\mathscr{U}$ into itself. Further,

$$\|T(U_1)(x) - T(U_2)(x)\|_{\mathscr{U}} = \sup_{x \in [b, \infty)} \left\| \int_x^\infty (S(x, Q, t) W(t) (U_1(t) - \right.$$

$$\left. - U_2(t)) \, dt \right\| \le \int_x^\infty \|S(x, Q, t)\| \, \|W(t)\|.$$

$$\sup_{t \in [b, \infty)} \|U_1(t) - U_2(t)\| \, dt \le \|U_1(x) - U_2(x)\|_{\mathscr{U}} \int_x^\infty \|W(t)\| \, dt,$$

hence T is a contraction mapping for sufficiently large x. By the Banach contraction principle there exists $V(x) \in \mathscr{U}$ such that $V(x) = T(V)(x)$, hence $U_1(x) = S(x, Q, a) + V(x)$ is a solution of (4.1) and $Y_1(x) = H(x)(S(x, Q, a) + V(x))$ is a solution of (1.7). Similarly we obtain the asymptotic formula $Y_2(x) = H(x)(C(x, Q, a) + V(x))$. The proof is complete.

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ПРЕОБРАЗОВАНИЕ ЛИУВИЛЛЯ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ СИСТЕМ

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Резюме

В работе установливается преобразование Лиувилля для линейных дифференциальных систем второго порядка

$$y'' + P(x)y = 0,$$

где $P(x)$ симметрическая матрица размерности $n \times n$. С помощью этого преобразования обобщаются известные асимптотические и колебательные критерия для скалярных уравнений на случай дифференциальных систем.