Abolghassem Bozorgnia; M. Bhaskara Rao

Some comments on a result of Hanš on strong convergence of sequences of random elements in separable Banach spaces

Mathematica Slovaca, Vol. 39 (1989), No. 2, 191--197

Persistent URL: http://dml.cz/dmlcz/136489

Math. Slovaca 39, 1989, No. 2, 191—197

SOME COMMENTS ON A RESULT OF HANŠ ON STRONG CONVERGENCE OF SEQUENCES OF RANDOM ELEMENTS IN SEPARABLE BANACH SPACES

A. BOZORGNIA—M. BHASKARA RAO

1. Introduction

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space and X_n, $n \geq 1$ a sequence of random elements defined on Ω taking values in a separable Banach space B. The starting point of this paper is a result of Hanš on convergence almost everywhere of the sequence X_n, $n \geq 1$. Before we state this result, we need the following set.

$$A = \{w \in \Omega;\ \{X_n(w),\ n \geq 1\} \text{ is relatively strongly compact in } B\}.$$

One can show that $A \in \mathscr{A}$ and that A is a tail set. See Hanš [4, p. 88].

Theorem 1 (Hanš [4, Theorem 19, p. 89]. *Let T be a total subset of B^*, the dual of B. Then X_n, $n \geq 1$ converges a.e. $[P]$ if and only if the following hold.*

Hanš conditions. (α) $P(A) = 1$

(β) $f(X_n)$, $n \geq 1$ *converges a.e.* $[P]$ *for every* f *in* T.

This paper has three objectives to achieve. The first objective is to examine Hanš conditions (α) and (β) in relationship to the following condition.

(γ) X_n, $n \geq 1$ *converges in probability.*

Using Theorem 1, Hanš characterized almost sure convergence of sequences of random elements taking values in some special Banach spaces c_0, l_1 etc. Some of these characterizations are not right. The second objective is to provide the relevant counter examples. Section 2 of this paper focuses on these two objectives.

The third objective is to establish a result in the spirit of the Hanš Theorem above for convergence in probability and this is covered in Section 3.

2. Hanš conditions, convergence in probability, and convergence in some special Banach spaces

Suppose X_n, $n \geq 1$ converges in probability. Then every subsequence of X_n, $n \geq 1$ admits a further subsequence which converges almost surely $[P]$. (See Lemma 1 below.) Then it might be tempting to conclude that the Hanš condition (α) and convergence in probability are equivalent. We give examples to show that these two concepts are different.

The following is an example to show that the validity of Hanš condition (α) does not imply convergence in probability.

Example 1. Let $\Omega = \{0, 1\}^N$, $\mathscr{A}$ its Borel σ-field and P the product measure $\lambda_1 \times \lambda_2 \times \ldots$, where each $\lambda_i(\{0\}) = 1/2 = \lambda_i(\{1\})$ and N is the set of all natural numbers. Let $B = R$, the real line. Let for each $n \geq 1$,

$$X_n(x_1, x_2, \ldots) = x_n$$

for every $(x_1, x_2, \ldots) \in \Omega$. X_n, $n \geq 1$ is a sequence of independent 0-1 valued random variables. It is obvious that $A = \{\omega \in \Omega;\ \{X_n(\omega), n \geq 1\}$ is relatively strongly compact in $B\} = \Omega$ and consequently $P(A) = 1$. But X_n, $n \geq 1$ does not converge in probability.

Suppose X_n, $n \geq 1$ converges in probability. Does this imply that $P(A) = 1$? The answer is no. The answer is still no even when the Hanš condition (β) holds for the sequence X_n, $n \geq 1$ additionally. The relevant counter example is given below.

Example 2. Beck and Warren [2, p. 922] exhibited a sequence Y_n, $n \geq 1$ of random elements taking values in c_0 with the following properties. (i) Y_n, $n \geq 1$ is uniformly bounded in norm. (ii) Y_n, $n \geq 1$ is identically distributed. (iii) $\mathrm{E} Y_n = 0$ for every $n \geq 1$. (iv) Y_n, $n \geq 1$ is weakly orthogonal, i.e., $f(Y_n)$, $n \geq 1$ is a sequence of pairwise uncorrelated random variables for every f in B^*. (v) $X_n = (Y_1 + Y_2 + \ldots + Y_n)/n$, $n \geq 1$ does not converge almost surely $[P]$.

This sequence X_n, $n \geq 1$ is the sequence of interest. By Theorem 5.1.2 of Chung [3, p. 103], it follows that $f(X_n)$, $n \geq 1$ converges a.e. $[P]$ for every f in B^*. Thus the Hanš condition (β) holds for X_n, $n \geq 1$. By Theorem 2.3 of Wang and Bhaskara Rao [5, p. 128], it follows that X_n, $n \geq 1$ converges in probability. But the Hanš condition (α) evidently is not valid by (v) above.

Hanš [4] gave several applications of Theorem 1 to l^p spaces, c_0 (the space of sequences of real numbers converging to zero), c (the space of all convergent sequences of real numbers), $L^p[0, 1]$ spaces and $C[0, 1]$. See Theorems 25 to 37 of Hanš [4]. Now we look at some of these applications.

First, consider the Banach spaces l^p, $1 \leq p < \infty$, c and c_0. Consider the functionals

$$f_n(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = x_n, \qquad n \geq 1.$$

Each of these functionals can be defined on every one of the spaces mentioned above and they constitute a total subset of the corresponding dual spaces.

The following is Theorem 25 of Hanš [4, p. 92]. "*Let* $(\Omega, \mathscr{A}, P)$ *be a probability space and* $X_0, X_1, \ldots$ *a sequence of random elements with values in* l^1. *Then* X_n, $n \geq 1$ *converges to* X_0 *almost surely* $[P]$ *if and only if the following two conditions* (a) *and* (b) *hold.*

(a) *For every* $\varepsilon > 0$ *there exists a positive integer* k_ε *(dependent on* ε *only) such that for every* $n = 1, 2, \ldots$

$$P\left\{\omega \in \Omega;\ \sum_{i \geq k_\varepsilon} |f_i(X_n(\omega))| < \varepsilon\right\} = 1.$$

(b) $f_i(X_n)$, $n \geq 1$ *converges to* $f_i(X_0)$ *almost surely* $[P]$ *for every* $i = 1, 2, \ldots$"

The conclusion of this theorem as it stands is false. We have checked the sufficiency part of this theorem to be correct. The necessity part of this theorem is false. We give the following counter example to substantiate our claim.

Example 3. Let $\delta_n = (x_1, x_2, \ldots)$ be defined by $x_i = 1$ for $i = n$, and $= 0$ for $i \neq n$, $n \geq 1$. Let $\Omega = \{w_1, w_2, \ldots\}$ be countable and $P(\{w_k\}) > 0$ for every $k \geq 1$. For each $n \geq 1$, define

$$\begin{aligned} V_n(w_i) &= \delta_i && \text{for } 1 \leq i \leq n, \\ &= n\delta_{n+1} && \text{for } i \geq n+1. \end{aligned}$$

Also define $V_0(w_i) = \delta_i$, $i \geq 1$. It is easy to check that $\lim_{n \to \infty} V_n(w) = V_0(w)$ for every $\omega \in \Omega$. Further, it can be checked that (a) does not hold for this sequence.

Hanš has a similar theorem for l^p spaces for $1 < p < \infty$. See Theorem 31 [4, p. 93]. One can give an example to show that the necessary part of this theorem is not true.

The following theorem is stated by Hanš [4, Theorem 27 and 28, p. 92]. "*Let* $(\Omega, \mathscr{A}, P)$ *be a probability space and* V_n, $n \geq 0$ *a sequence of random elements taking values either in* c *or* c_0. *Then* V_n, $n \geq 1$ *converges to* V_0 *almost surely if and only if the following two conditions* (a) *and* (b) *hold.*

(a) *For every* $\varepsilon > 0$ *there exists a positive integer* k_ε *(dependent on* ε *only) such that for every* $i \geq k_\varepsilon$ *and* $n = 1, 2, \ldots,$

$$P\{\omega \in \Omega;\ |f_i(V_n(\omega))| < \varepsilon\} = 1.$$

(b) $f_i(V_n)$, $n \geq 1$ *converges to* $f_i(V_0)$ *almost surely* $[P]$, *for every* $i = 1, 2, 3, \ldots$"

We claim that the necessary part of this theorem is false. The example given above also serves here.

3. Convergence in probability

In this section, we characterize convergence in probability in the same spirit as Theorem 1 above. We also give some applications of this result in some special Banach spaces.

Lemma 1. *Let V_n, $n \geq 0$ be a sequence of random elements defined on a probability space $(\Omega, \mathscr{A}, P)$ taking values in a separable Banach space B. Then V_n, $n \geq 1$ converges to V_0 in probability if and only if every subsequence of V_n, $n \geq 1$ admits a further subsequence V_{n_k}, $k \geq 1$ converging to V_0 almost surely* $[P]$.

Proof. This result is well known for the case when B is the real line and the same proof goes through for the general case. See Chung [3, Theorem 4.2.3, p. 73].

Theorem 2. *Let X_n, $n \geq 0$ be a sequence of random elements defined on a complete probability space $(\Omega, \mathscr{A}, P)$ taking values in a separable Banach space B. Suppose the following two conditions are satisfied*:

(i) $f(X_n)$, $n \geq 1$ *converges to* $f(X_0)$ *in probability for each* f *in* A, *where* A *is some total subset of* B^*.

(ii) *Every subsequence of* X_n, $n \geq 1$ *admits a further subsequence* X_{n_k}, $k \geq 1$ *such that* $P\{\omega \in \Omega; \{X_{n_k}(\omega); k \geq 1\}$ *is relatively strongly compact*$\} = 1$.

Then X_n, $n \geq 1$ *converges to* X_0 *in probability. Further, conditions* (i) *and* (ii) *are also necessary for convergence in probability.*

Proof. Sufficiency. In view of Lemma 1, it suffices to show that given any subsequence of X_n, $n \geq 1$, it admits a further subsequence which converges to X_0 almost surely.

Since B is separable, we can find a countable subset C of A which is total. See Banach [1, Theorem 4, p. 124]. Let $C = \{f_1, f_2, \ldots\}$. For the given subsequence of X_n, $n \geq 1$ choose a further subsequence X_{n_k}, $k \geq 1$ such that

$$P\{\omega \in \Omega; \{X_{n_k}(\omega); k \geq 1\} \text{ is relatively strongly compact}\} = 1.$$

Since $f_1(X_{n_k})$, $k \geq 1$ converges to $f_1(X_0)$ in probability, we can find a subsequence $f_1(X_{n_{1r}})$, $r \geq 1$ converges to $f_1(X_0)$ almost everywhere. By a similar argument, we can find a subsequence $X_{n_{2r}}$, $r \geq 1$ of $X_{n_{1r}}$, $r \geq 1$ such that $f_2(X_{n_{2r}})$, $r \geq 1$ converges to $f_2(X_0)$ almost everywhere. Continuing this process for each f_k, we get a double array

$$\begin{array}{l} f_1(X_{n_{11}}),\ f_1(X_{n_{12}}).\ f_1(X_{n_{13}}),\ \ldots \\ f_2(X_{n_{21}}),\ f_2(X_{n_{22}}),\ f_2(X_{n_{23}}),\ \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_p(X_{n_{p1}}),\ f_p(X_{n_{p2}}),\ f_p(X_{n_{p3}}),\ \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

where each row sequence is a subsequence of the row sequence just ahead of it. It has the following properties:

(i) $X_{n_{kk}}$, $k \geq 1$ is a subsequence of X_{n_k}, $k \geq 1$ and hence a subsequence of the given subsequence with which we originally started.

(ii) $f_p(X_{n_{kk}})$, $k \geq 1$ converges to $f_p(X_0)$ almost surely for each $p \geq 1$.

(iii) $P\{\omega \in \Omega; \{X_{n_{kk}}(\omega), k \geq 1\}$ is relatively strongly compact$\} = 1$, since the set involved with in the flower brackets contains $\{\omega \in \Omega; \{X_{n_k}(\omega), k \geq 1\}$ is relatively strongly compact$\}$.

By Theorem 1, $X_{n_{kk}}$, $k \geq 1$ converges to X_0 almost surely $[P]$.

Necessity. Necessity of the conditions (i) and (ii) follows from Lemma 1 and from the necessity part of Theorem 1.

We now apply Theorem 2 to some special spaces.

Theorem 3. *Let V_n, $n \geq 0$ be a sequence of random elements defined on a complete probability space $(\Omega, \mathscr{A}, P)$, taking values in l^p, $1 \leq p < \infty$. Then V_n, $n \geq 1$ converges to V_0 in probability if the following two conditions are satisfied.*

(a) *Every subsequence of V_n, $n \geq 1$ admits a further subsequence V_{n_k}, $k \geq 1$ such that for every $\varepsilon > 0$ there exists a positive integer k_ε satisfying*

$$P\left\{\omega \in \Omega; \sum_{i \geq k_\varepsilon} |f_i(V_{n_k}(\omega))|^p < \varepsilon\right\} = 1 \quad \text{for each } k \geq 1.$$

(b) *$f_i(V_n)$, $n \geq 1$ converges to $f_i(V_0)$ in probability for every $i \geq 1$.*

Proof. We want to apply Theorem 2 to prove this result. We will show that (a) and (b) together imply (i) and (ii) of Theorem 2. Suppose (a) and (b) hold. (b), obviously, implies (i) if we take $A = \{f_n, n \geq 1\}$. Now, we will show that (a) + (b) implies (ii). Let V_{n_k}, $k \geq 1$ be any arbitrary subsequence of V_n, $n \geq 1$. Since $f_i(V_{n_k})$, $k \geq 1$ converges in probability to $f_i(V_0)$ for each $i \geq 1$, by Cantor's diagonal technique we can find a subsequence $\tilde{V}_p$, $p \geq 1$ of V_{n_k}, $k \geq 1$ such that $f_i(\tilde{V}_p)$, $p \geq 1$ converges to $f_i(V_0)$ almost surely $[P]$ for each $i \geq 1$.

There exists a set $D \in \mathscr{A}$ such that $P(D) = 1$ and for each $\omega \in D$, $f_i(\tilde{V}_p(\omega))$, $p \geq 1$ converges to $f_i(V_0(\omega))$. Assume, without loss of generality, that for the subsequence $\tilde{V}_p$, $p \geq 1$, (a) is satisfied. If not, we can take a further subsequence of $\tilde{V}_p$, $p \geq 1$ satisfying (a).

For $\varepsilon = 1/n$, denote $k_{1/n}$ by k_n. Let

$$\mathrm{E} = \bigcap_{r \geq 1} \bigcap_{n \geq 1} \left\{\omega \in \Omega; \sum_{i \geq k_n} |f_i(\tilde{V}_r(\omega))|^p < 1/n\right\}.$$

Then condition (a) implies $P(E) = 1$. Let $F = D \cap E$. We have $P(F) = 1$. Now, we claim that for each $\omega \in F$, $\tilde{V}_r(\omega)$, $r \geq 1$ is a Cauchy sequence.

Let $\omega \in F$. Let $\varepsilon > 0$. Then we can find $n \geq 1$ such that $(1/n) < \varepsilon$.

$$\| \tilde{V}_r(\omega) - \tilde{V}_m(\omega) \|^p = \sum_{i \geq 1} |f_i(\tilde{V}_r(\omega)) - f_i(\tilde{V}_m(\omega))|^p =$$

$$\sum_{i=1}^{k_n - 1} |f_i(\tilde{V}_r(\omega)) - f_i(\tilde{V}_m(\omega))|^p +$$

$$\sum_{i \geq k_n} |f_i(\tilde{V}_r(\omega)) - f_i(\tilde{V}_m(\omega))|^p \leq$$

$$\sum_{i=1}^{k_n - 1} |f_i(\tilde{V}_r(\omega)) - f_i(\tilde{V}_m(\omega))|^p + 2/n.$$

Since $f_i(\tilde{V}_r(\omega))$, $r \geq 1$ converges for each $i = 1$ to $k_n - 1$, we can find $M \geq 1$ such that

$$|f_i(\tilde{V}_r(\omega)) - f_i(\tilde{V}_m(\omega))|^p \leq \varepsilon/(k_n - 1)$$

for every $i = 1$ to $k_n - 1$ and r, $m \geq M$.

Consequently, if r, $m \geq M$,

$$\| \tilde{V}_r(\omega) - \tilde{V}_m(\omega) \|^p \leq 3\varepsilon.$$

From this, it follows that

$$P\{\omega \in \Omega ; \{\tilde{V}_r(\omega), r \geq 1\} \text{ is relatively strongly compact}\} = 1.$$

This proves that (a) and (b) imply (ii) of Theorem 2.

Theorem 4. *Let V_n, $n \geq 0$ be a sequence of random elements defined on a complete probability space $(\Omega, \mathscr{A}, P)$ taking values in c_0. Then V_n, $n \geq 1$ converges to V_0 in probability if the following two conditions are satisfied.*

(a) *Every subsequence of V_n, $n \geq 1$ admits a further subsequence V_{n_k}, $k \geq 1$ such that for every $\varepsilon > 0$, there exists a positive integer k_ε satisfying*

$$P\{\omega \in \Omega : |f_i(V_{n_k}(\omega))| < \varepsilon\} = 1$$

for every $i \geq k_\varepsilon$ and $k \geq 1$.

(b) *$f_i(V_n)$, $n \geq 1$ converges to $f_i(V_0)$ in probability for every $i \geq 1$.*

Proof. A proof can be supplied in the same way as that of Theorem 3.

Hanš [4, Theorem 34, p. 94] characterized convergence almost surely in $L^p[0, 1]$ for $p \geq 1$. Here the measure on $[0, 1]$ is the Lebesgue measure. We give a characterization of convergence in probability for general L^p spaces.

Theorem 5. *Let $(Y, \mathscr{C}, \mu)$ be a probability space, where $\mathscr{C}$ is a separable σ-field on Y. Let E_n, $n \geq 1$ be a generator of $\mathscr{C}$ closed under finite intersections and containing Y. Let V_n, $n \geq 0$ be a sequence of random elements defined on a complete probability space $(\Omega, \mathscr{A}, P)$ taking values in $L^p(Y, \mathscr{C}, \mu)$ for some $p \geq 1$. Then V_n, $n \geq 1$ converges to V_0 in probability if the following two conditions are satisfied.*

(a) *Every subsequence of V_n, $n \geq 1$ admits a further subsequence V_{n_k}, $k \geq 1$ such that*

$$\int_Y |V_{n_k}(\omega)(y)|^p \mu(\mathrm{d}y), \; k \geq 1 \textit{ converges to}$$

$$\int_Y |V_0(\omega)(y)|^p \mu(\mathrm{d}y), \textit{ for almost all } \omega \in \Omega.$$

(b) $$\int_{E_i} V_n(\cdot)(y)\,\mu(\mathrm{d}y),\ n \geq 1 \textit{ converges to}$$

$$\int_{E_i} V_0(\cdot)(y)\,\mu(\mathrm{d}y) \textit{ in probability for every } i = 1, 2, \ldots$$

Proof. A proof of this result can be patterned along the lines of the proof of Theorem 3.

REFERENCES

[1] BANACH, S.: Theorie des Opérations Linéaires. Warszawa 1932.

[2] BECK, A.—WARREN, P.: Strong laws of large numbers for weakly orthogonal sequences of Banach space-valued random variables. Ann. Probability, Vol. 2, 1974, 918—925.

[3] CHUNG, K. L.: A Course in Probability Theory. Second edition, Academic Press, London 1974.

[4] HANŠ, O.: Generalized random variables. In: Trans. First Prague Conf. on Information Theory, Statistics, Decision Functions, and Random Processes, 1957, 61—103.

[5] WANG, X. C.—RAO, M. B.: Convergence in the pth — mean and some weak laws of large numbers for weighted sums of random elements in separable normed linear spaces. J. Multivariate Analysis, Vol. 15, No. 1, 124—134.

Received July 4, 1985

Department of Probability and Statistic
The University Sheffield S10 2TN
England

Division of Mathematical Sciences
North Dakota State University
Fargo, ND 58105
U.S.A.

НЕКОТОРЫЕ ЗАМЕЧАНИЯ ПО ПОВОДУ ВЫВОДОВ ГАНША О СИЛЬНОЙ СХОДИМОСТИ ПОСЛЕДОВАТЕЛЬНОСТЕЙ СЛУЧАЙНЫХ ЭЛЕМЕНТОВ В РАЗДЕЛЬНЫХ БАНАХОВЫХ ПРОСТРАНСТВАХ

A. Bozorgnia—M. Bhaskara Rao

Резюме

В работе рассматриваются выводы Ганша о сильной сходимости последовательности случайных элементов в раздельном банаховом пространстве в сопоставлении с подобными результатами, имеющимися в литературе.

Рассматривается также использование этих выводов в некоторых специальный банаховых пространствах.