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RESULTS ON THE RATIOS OF THE TERMS OF SECOND ORDER LINEAR RECURRENCES

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ABSTRACT. We survey the results concerning the diophantine approximative property of second order linear recurrences and discuss the dependence of the estimate on the discriminant of a characteristic polynomial.

Let $R = \{R_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where R_0, R_1 and A, B are fixed integers with $AB \neq 0$, $R_0^2 + R_1^2 \neq 0$ and $\mathbf{D} = A^2 + 4B \neq 0$. Let α and β be the roots of the equation $x^2 - Ax - B = 0$, where $\alpha \neq \beta$ since $\mathbf{D} \neq 0$. It is known that the terms of R can be expressed in the form

$$R_n = a\alpha^n + b\beta^n \tag{1}$$

for any $n \geq 0$, where

$$a = \frac{R_1 - \beta R_0}{\alpha - \beta} \qquad \text{and} \qquad b = \frac{R_1 - \alpha R_0}{\beta - \alpha}$$

(see e.g. [1], pp. 106–108).

Throughout this paper we assume $|\alpha| \geq |\beta|$, $ab \neq 0$ and the sequence is non-degenerate, i.e. α/β is not a root of unity. We may also suppose that $R_n \neq 0$ for $n > 0$ since in [2] it was proved that a non-degenerate sequence R has at most one zero term and after a change of indices this condition will be fulfilled.

If $\mathbf{D} = A^2 + 4B > 0$, i.e. if α and β are real numbers, then $(\beta/\alpha)^n \to 0$ as $n \to \infty$, so by (1) we have

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \lim_{n\to\infty} \frac{1 + (b/a)(\beta/\alpha)^{n+1}}{1 + (b/a)(\beta/\alpha)^n} \cdot \alpha = \alpha . \tag{2}$$

This raises the following interesting problem: what is the quality of approximation of α by rationals of the form R_{n+1}/R_n? In [3], using the continued fraction expansion of α, we proved that this approximation is "good" only if $|B| = 1$.

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Theorem 1. *If* $\mathbf{D} > 0$, $R_0 = 0$, $R_1 = 1$ *and* α *is irrational, then the inequality*

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{cR_n^2}$$

holds with a positive real number c *for infinitely many integer* n *if and only if* $|B| = 1$ *and* $c \leq \sqrt{\mathbf{D}}$. *Furthermore, if* p/q *is a rational number such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{\mathbf{D}}q^2},$$

then p/q *has the form* $p/q = R_{n+1}/R_n$.

In some other special cases similar results follow from [4] and [7].

In a joint paper with Z. Sinka [5] the complete answer to the problem was obtained by determining the measure of approximation for general second order recurrences.

Theorem 2. *Let* R *be a non-degenerate linear recurrence with* $\mathbf{D} > 0$. *Define the numbers* k_0 *and* c_0 *by*

$$k_0 = 2 - \frac{\log|B|}{\log|\alpha|} \qquad \text{and} \qquad c_0 = \frac{(\sqrt{\mathbf{D}})^{-1}}{|a^{k_0-1}b|}$$

and let k *and* c *be positive real numbers. Then*

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{cR_n^k}$$

holds for infinitely many integers n *if and only if*

(i) $k < k_0$ *and* c *is arbitrary, or*
(ii) $k = k_0$ *and* $c < c_0$, *or*
(iii) $k = k_0$, $c = c_0$ *and* $B > 0$, *or*
(iv) $k = k_0$, $c = c_0$, $B < 0$ *and* $b/a > 0$.

Since $1 \leq |B| = |\alpha\beta| < |\alpha|^2$, $k_0 > 0$ always holds. Furthermore, Theorem 1 is a special case of Theorem 2 since $a = -b = 1/(\alpha - \beta) = 1/\sqrt{\mathbf{D}}$ if $R_0 = 0$, $R_1 = 1$, and $k_0 = 2$ if $|B| = 1$.

The case $\mathbf{D} < 0$ is far more complicated. In this case α and β are non-real complex numbers with $|\alpha| = |\beta|$ and (2) does not hold even if we consider the absolute values of the numbers. Since α $(= \overline{\beta})$ and a $(= \overline{b})$ are complex conjugates of β and b respectively, we can write

$$\beta = re^{\pi\theta i}, \qquad \alpha = re^{-\pi\theta i}, \qquad \beta/\alpha = e^{2\pi\theta i}$$

and

$$b = r_1 e^{2\pi\gamma i}, \qquad a = r_1 e^{-2\pi\gamma i}, \qquad b/a = e^{2\pi\omega i},$$

where r, r_1, θ, γ and ω are positive real numbers with $0 < \theta, \gamma, \omega < 1$. Using these notations, we get from (1):

$$\begin{aligned}\left|\frac{R_{n+1}}{R_n}\right| &= |\alpha| \cdot \left|\frac{1+(b/a)(\beta/\alpha)^{n+1}}{1+(b/a)(\beta/\alpha)^n}\right| = \\ &= |\alpha| \cdot \left|\frac{1+e^{2\pi(n+1)\theta i+2\pi\omega i}}{1+e^{2\pi n\theta i+2\pi\omega i}}\right|.\end{aligned} \tag{3}$$

Under our conditions β/α is not a root of unity and so θ is an irrational number. This implies that the sequence $(n\theta+\omega)$, $n = 1, 2, \dots$, is uniformly distributed modulo 1 and then $n\theta+\omega$ can come arbitrarily "close" to the real number $\frac{1}{2}-\frac{\theta}{2}$ for infinitely many n. Hence

$$e^{2\pi i(n\theta+\omega)} \approx e^{2\pi i(1/2-\theta/2)} = z$$

and

$$e^{2\pi i((n+1)\theta+\omega)} \approx e^{2\pi i(1/2+\theta/2)} = \overline{z}$$

for these n's (z and $\overline{z}$ are conjugate complex numbers). From this and from (3) it follows that

$$|\alpha| \cdot \left(1-\frac{\varepsilon}{|\alpha|}\right) < \left|\frac{R_{n+1}}{R_n}\right| < |\alpha| \cdot \left(1+\frac{\varepsilon}{|\alpha|}\right)$$

and

$$\left||\alpha| - \left|\frac{R_{n+1}}{R_n}\right|\right| < \varepsilon \tag{4}$$

hold for infinitely many n with any $\varepsilon > 0$.

(4) raises the question: what can we say about ε? In a joint paper with R. Tichy [6], using a result for the discrepancy of the sequence $(n\theta+\omega)$, the following result was proved:

Theorem 3. *For any non-degenerate second order linear recurrence R, for which $\mathbf{D} < 0$, there is a constant $c > 0$ such that*

$$\left||\alpha| - \left|\frac{R_{n+1}}{R_n}\right|\right| < \frac{1}{n^c} \tag{5}$$

for infinitely many n.

This approximation of $|\alpha|$ by $|R_{n+1}/R_n|$ is very "bad", since $n \approx \log|R_n|$ by (1). However, surprisingly, this approximation, apart from the constant c, is the best possible one.

Also in [6], using results about the estimation of linear forms of logarithms of algebraic numbers, we have shown:

Theorem 4. *For any non-degenerate second order linear recurrence* R, *for which* $\mathbf{D} < 0$, *there is a constant* c' *such that*

$$\left| |\alpha| - \left| \frac{R_{n+1}}{R_n} \right| \right| > \frac{1}{n^{c'}} \tag{6}$$

for all sufficiently large n.

The only remaining open problem consists of determining the best values of the constants c and c' in Theorems 3 and 4. In [5] we obtained the following improvement of Theorem 3 for a special case:

Theorem 5. *Let* R *be a non-degenerate second order linear recurrence with* $\mathbf{D} < 0$ *and initial values* $R_0 = 0$, $R_1 = 1$. *Then there is a constant* c_1 (> 0) *such that*

$$\left| |\alpha| - \left| \frac{R_{n+1}}{R_n} \right| \right| < \frac{c_1}{n}$$

for infinitely many n.

I conjecture that there exists an absolute constant C not depending on the parameters of the sequence R such that (5) holds with any $c < C$ for infinitely many n and (6) holds with any $c' > C$ for all sufficiently large n.

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