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MULTIPLE SOLUTIONS OF A THIRD ORDER BOUNDARY VALUE PROBLEM

MILAN GERA *) — FELIX SADYRBAEV **)

ABSTRACT. An estimate from below of the number of solutions to the boundary value problem $x''' = f(t,x,x',x'') = 0$, $x(a) = A$, $x'(a) = A_1$, $x(b) = B$ is given provided that f along with first partial derivatives f_x, $f_{x'}$, $f_{x''}$ are continuous functions and $f_x \leqq 0$, $f_{x'} \geqq 0$.

We shall consider the two-point boundary value problem

$$x''' = f(t,x,x',x''), \tag{1}$$

$$x(a) = A, \quad x'(a) = A_1, \tag{2}$$

$$x(b) = B, \tag{3}$$

where A, A_1, B, a, b ($a < b$) are given real numbers, provided that $f\colon [a,b]\times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ along with the first partial derivatives f_x, $f_{x'}$, $f_{x''}$ are continuous functions.

The aim of this paper is to give an estimate from below of the number of solutions of BVP (1–3).

The proof of the main result is based on a certain technique developed by one of the authors in [1] and on some results from the theory of third-order linear differential equations.

The following linear problem will be very important in our considerations:

$$\begin{aligned} y''' = f_{x''}\big(t,\xi(t),\xi'(t),\xi''(t)\big)y'' + f_{x'}\big(t,\xi(t),\xi'(t),\xi''(t)\big)y' \\ +f_x\big(t,\xi(t),\xi'(t),\xi''(t)\big)y\,, \end{aligned} \tag{4}$$

$$y(a) = y'(a) = 0, \quad y''(a) = 1. \tag{5}$$

where ξ is the solution of BVP (1–3). (The differential equation (4) is a linear variational equation for the solution ξ.)

We shall distinguish solutions of (1–3) by a certain property. The precise definition is the following:

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DEFINITION. *Let ξ be a solution of the* BVP (1–3) *and h the solution of the corresponding initial value problem* (4), (5). *Then the number of zeros of h in the interval (a,b) is called the index of ξ.*

Now we are ready to present the main result of the paper.

THEOREM. *Assume that*

(A1) *there exists a solution ξ of* BVP (1–3) *with nonzero index m;*
(A2) *there exists a solution η of* (1), (2) *on $[a,b]$ such that $\eta(t) > \xi(t)$ for all $t \in (a,b]$;*
(A3) *all solutions u of* (1), (2) *for which $\xi''(a) \leqq u''(a) \leqq \eta''(a)$ extend to $[a,b]$;*
(A4) *the function f fulfils the following condition:*

$$\forall (t,x,x',x'') \in [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \colon f_x(t,x,x',x'') \leqq 0\,,$$
$$f_{x'}(t,x,x',x'') \geqq 0\,.$$

Then the boundary value problem (1–3) *has at least $m+1$ solutions.*

We first state some lemmas needed for the proof of the Theorem.

LEMMA 1. *Let p, q, r be continuous functions on $[a,b]$ and $r(t) \leqq 0$, $q(t) \geqq 0$ for all $t \in [a,b]$. Then a solution of the initial value problem*

$$x''' = p(t)x'' + q(t)x' + r(t)x\,, \tag{6}$$

$$x(a) = x'(a) = 0, \quad x''(a) \neq 0 \tag{7}$$

cannot have multiple zeros on $(a,b]$. All its zeros on $(a,b]$ (if they exists) are simple.

Proof. Let y be a solution of (6), (7). Suppose that there exists $t_0 \in [a,b]$ such that $y(t_0) = y'(t_0) = 0$, $y''(t_0) \neq 0$. Then, by Theorem 1 [2], the solution y has no zero on $[a,t_0)$, i.e. $y(t) \neq 0$ for all $t \in [a,t_0)$. This contradicts the fact that $y(a) = 0$. Thus, when y has a zero in $(a,b]$, it is simple. The proof is complete.

Now denote by S the set of all solutions x of (1), (2) satisfying the condition

$$\xi''(a) \leqq x''(a) \leqq \eta''(a)\,,$$

where ξ and η are the solutions of (1) which fulfil the assumptions of the theorem.

Note that $\xi''(a) < \eta''(a)$, otherwise the solutions ξ, η must coincide since solutions of initial value problems for (1) are uniquely determined.

In what follows we establish some properties of the set S.

LEMMA 2. ([3] *or* [4]). *S is a compact set in $C^2([a,b],\mathbb{R})$.*

LEMMA 3. *Let M_0, M_1, M_2 be real positive numbers. Denote by G_i ($i = 0,1,2$) a family of elements $g_i \in C([a,b],\mathbb{R})$ such that $|g_i(t)| \leqq M_i$ for all $t \in [a,b]$. Then there exists a number $\delta > 0$ depending on M_0, M_1, M_2 only and such that for each $g_i \in G_i$, $i = 0,1,2$, the length of any subinterval which contains three zeros counting their multiplicity of an arbitrary nontrivial solution of*

$$x''' = g_2(t)x'' + g_1(t)x' + g_0(t)x \tag{8}$$

is larger than δ.

P r o o f. Let δ be a positive number such that the following inequality holds

$$\frac{2}{81}M_0\delta^3 + \frac{1}{6}M_1\delta + \frac{2}{3}M_2 < 1 .$$

Then the differential equation (8) is disconjugate on each interval $[a_1, b_1] \subset [a,b]$ for which $b_1 - a_1 \leqq \delta$ is true [5]. (The differential equation (8) is disconjugate on $[a_1, b_1]$ if each of its nontrivial solutions has at most two zeros, counting their multiplicity, on $[a_1, b_1]$.) Thus, if a nontrivial solution of (8) has three zeros on any interval $[a_2, b_2] \subset [a,b]$, it must be $b_2 - a_2 > \delta$. The lemma is proved.

P r o o f o f t h e T h e o r e m . The set S is compact in $C^2([a,b],\mathbb{R})$ by Lemma 2. Consequently, there exist positive constants K_0, K_1, K_2 such that $|x(t)| \leqq K_0$, $|x'(t)| \leqq K_1$, $|x''(t)| \leqq K_2$ for all $t \in [a,b]$ and $x \in S$.

We may assume that $|f|$ with $|f_x|$, $|f_{x'}|$ and $|f_{x''}|$ are bounded on $[a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. This means no loss in generality because the values of f outside the finite region $D = \{(t,x,y,z);\ a \leqq t \leqq b,\ |x| \leqq K_0,\ |y| \leqq K_1,\ |z| \leqq K_2\}$ are not involved in our considerations. So we can always make $|f_x|$, $|f_{x'}|$, $|f_{x''}|$ bounded by changing the functions outside the region D [6]. Thus, let M_0, M_1, M_2, be positive numbers such that $|f_x| \leqq M_0$, $|f_{x'}| \leqq M_1$, $|f_{x''}| \leqq M_2$ on $[a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Let $x_\lambda = x(t;\lambda)$ be a solution of the initial value problem

$$x''' = f(t,x,x',x''),$$
$$x(a) = A,\ x'(a) = A_1,\ x''(a) = \lambda ,$$

where $\lambda_0 := \xi''(a) \leqq \lambda \leqq \eta''(a) =: \Lambda_1$. Then $\dfrac{\partial^i x(t;\lambda)}{\partial t^i}$ are continuous functions of $t \in [a,b]$, $\lambda \in [\lambda_0, \Lambda_1]$, $x_\lambda \in S$,

$$\lim_{\lambda \to \lambda_0^+} \frac{\partial^i x(t;\lambda)}{\partial t^i} = \xi^{(i)}(t) \quad \left(= \frac{\partial^i x(t;\lambda_0)}{\partial t^i}\right),$$

$$\lim_{\lambda \to \Lambda_1^-} \frac{\partial^i x(t;\lambda)}{\partial t^i} = \eta^{(i)}(t) \quad \left(= \frac{\partial^i x(t;\Lambda_1)}{\partial t^i}\right)$$

are uniform for $t \in [a,b]$ ($i \in \{0,1,2\}$) and the function $\dfrac{\partial x(t;\lambda_0)}{\partial \lambda}$ $\left(= \lim\limits_{\lambda \to \lambda_0^+} \dfrac{x(t;\lambda) - \xi(t)}{\lambda - \lambda_0}\right)$ is the solution of (4), (5), i.e. $\dfrac{\partial x(t;\lambda_0)}{\partial \lambda} = h(t)$ for all $t \in [a,b]$ [7].

Setting $y(t;\lambda) = \bigl(x(t;\lambda) - \xi(t)\bigr)/(\lambda - \lambda_0)$ ($\lambda_0 < \lambda \leqq \Lambda_1$), we have that the function $y_\lambda \colon t \mapsto y(t;\lambda)$, $t \in [a,b]$, solves the initial value problem

$$\begin{aligned} &y''' = g_2(t,\lambda)y'' + g_1(t;\lambda)y' + g_0(t;\lambda)y\,,\\ &y(a) = y'(a) = 0, \quad y''(a) = 1\,, \end{aligned}$$

where

$$g_2(t;\lambda) = \int_0^1 f_{x''}\bigl(t, x_0(t,\tau,\lambda), x_1(t,\tau,\lambda), x_2(t,\tau,\lambda)\bigr)\,\mathrm{d}\tau\,,$$

$$g_1(t;\lambda) = \int_0^1 f_{x'}\bigl(t, x_0(t,\tau,\lambda), x_1(t,\tau,\lambda), x_2(t,\tau,\lambda)\bigr)\,\mathrm{d}\tau \geqq 0\,,$$

$$g_0(t;\lambda) = \int_0^1 f_{x}\bigl(t, x_0(t,\tau,\lambda), x_1(t,\tau,\lambda), x_2(t,\tau,\lambda)\bigr)\,\mathrm{d}\tau \leqq 0\,,$$

$$x_i(t,\tau;\lambda) = \xi^{(i)}(t) + \tau\left[\frac{\partial^i x(t;\lambda)}{\partial t^i} - \xi^{(i)}(t)\right], \qquad 0 \leqq \tau \leqq 1\,, \quad i = 0,1,2\,,$$

and

$$\begin{aligned} f\left(t, x(t;\lambda), \frac{\partial x(t;\lambda)}{\partial t}, \frac{\partial^2 x(t;\lambda)}{\partial t^2}\right) &- f\bigl(t, \xi(t), \xi'(t), \xi''(t)\bigr)\\ &= \sum_{i=0}^{2} g_i(t;\lambda)\Bigl[\frac{\partial^i x(t;\lambda)}{\partial t^i} - \xi^{(i)}(t)\Bigr] \end{aligned}$$

for $t \in [a,b]$, $\lambda \in [\lambda_0, \Lambda_1]$. It is obvious that $|g_i| \leqq M_i$ on $[a,b] \times [\lambda_0, \Lambda_1]$ and also g_i, $i = 0,1,2$, are continuous functions on $[a,b] \times [\lambda_0, \Lambda_1]$. Further it holds that

$$\lim_{\lambda \to \lambda_0^+} \frac{\partial^i y(t;\lambda)}{\partial t^i} = h^{(i)}(t), \qquad \lim_{\lambda \to \Lambda_1^-} \frac{\partial^i y(t;\lambda)}{\partial t^i} = \frac{\eta^{(i)}t - \xi^{(i)}t}{\Lambda_1 - \lambda_0} \tag{9}$$

are uniform for $t \in [a,b]$ ($i = 0,1,2$) [8].

Let $t_1(\lambda), t_2(\lambda) < \dots$ ($t_1(\lambda) < t_2(\lambda), \dots$) denote zeros of y_λ in $(a,b]$. By Lemma 1, with regard to the assumption $(A4)$, all zeros of y_λ and h in $(a,b]$ are simple. Since the solution h has exactly m zeros in (a,b), and y_λ tends to h uniformly on $[a,b]$ as $\lambda \to \lambda_0^+$, the functions $t_j(\lambda)$, $j = 1,2,\dots,m$, are defined for sufficiently small $\lambda - \lambda_0 > 0$. However, this implies also that $\frac{\partial y}{\partial t}(t_j(\lambda);\lambda) \neq 0$. Using the Implicit Function Theorem to solve $y(t;\lambda) = 0$ we get that $t_j(\lambda)$ are continuous functions of the parameter λ in their domains of definition. Besides, on the basis of Lemma 3, there is a positive number δ such that $t_j(\lambda) > a + \delta$ for $j = 1,2,\dots$, independently of λ.

Now we show that there exists a number $\delta_1 = \delta_1(S) > 0$ such that $t_{j+1}(\lambda) - t_j(\lambda) > \delta_1$ for all $j = 1,2,\dots$, independently of λ. Suppose, for contradiction, that, e.g., $t_2(\lambda) - t_1(\lambda) > \delta_1$ is not true. We have two possibilities: either we can find a sequence $(\bar\lambda_n)_1^\infty$ in (λ_0, Λ_1), $t_2(\bar\lambda_n) - t_1(\bar\lambda_n) \to 0$ such that $p_{\lambda_n} = \max\Big\{ \big|x(t;\bar\lambda_n) - \xi(t)\big|;\quad t_1(\bar\lambda_n) \leqq t \leqq t_2(\lambda_n) \Big\}$, $n = 1,2,\dots$, is bounded from below by a positive number, or p_{λ_n} is not separated in this way from zero. In the first case we obtain that $\dfrac{\partial x(t;\lambda)}{\partial t}$ is not bounded for some $\lambda \in [\lambda_0, \Lambda_1]$, but this is a contradiction with the compactness of the set S in $C^2([a,b],\mathbb{R})$. In the second case both $t_1(\lambda)$ and $t_2(\lambda)$ are separated from $t = a$. Since S is compact in $C^2([a,b],\mathbb{R})$, there is a uniformly convergent subsequence of $(x_{\lambda_n})_1^\infty$ which converges to $\bar x \in S$, such that $\bar x - \xi$ has a double zero in $(a,b]$. This is a contradiction with the assertion of Lemma 1. Therefore, the differences $t_{j+1}(\lambda) - t_j(\lambda)$ ($j = 1,2,\dots$) are separated from zero. From these facts along with $\eta(t) > \xi(t)$ for every $t \in (a,b]$ it follows that there exist $\lambda_j = \sup\{\lambda;\ \lambda \in (\lambda_0, \Lambda_1),\ t_j(\lambda) < b\}$ and at the same time $t_j(\lambda_j) = b$, $j = 1,2,\dots,m$, $\lambda_1 > \lambda_2 > \dots > \lambda_m$. It is clear that the functions x_{λ_j}, $j = 1,2,\dots,m$, are solutions of BVP (1–3) . The proof is complete.

COROLLARY. *Suppose that the function f is bounded on $[a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and such that $f_x(t,x,x',x'') \leqq 0$, $f_{x'}(t,x,x',x'') \geqq 0$ for all $(t,x,x',x'') \in [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $f(t,0,0,0) = 0$, $r(t) := f_x(t,0,0,0)$, $q(t) := f_{x'}(t,0,0,0)$,*

$p(t) := f_{x''}(t,0,0,0)$ for $t \in [a,b]$ and let the solution of the initial value problem

$$x''' = p(t)x'' + q(t)x' + r(t)x \,, \tag{10}$$

$$x(a) = x'(a) = 0, \quad x''(a) = 1 \tag{11}$$

have m zeros in (a,b). Then the boundary value problem

$$x''' = f(t,x,x',x'') \,, \tag{12}$$

$$x(a) = x'(a) = x(b) = 0 \tag{13}$$

has at least $m+1$ solutions. In particular, the boundary value problem

$$x''' = f(x) \,,$$
$$x(a) = x'(a) = x(b) = 0 \,,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is bounded, $f(0) = 0$, $f'(0) < 0$, $f'(x) \leqq 0$ for $x \in \mathbb{R}$ and $b - a \geqq 2(m+1)\pi/\sqrt{3}\,\sqrt[3]{-f'(0)}$, has at least $m+1$ solutions.

Proof. Since the function f is bounded, all the solutions of (12) extend to $[a,b]$. Let M be a positive constant such that $|f| \leqq M$ and let $\xi = 0$. Let η be the solution of the initial value problem

$$x''' = f(t,x,x',x'') \,,$$
$$x(a) = 0 \,, \quad x'(a) = 0 \,, \quad x''(a) = \beta \,,$$

where β is a real number, $\beta > (b-a)M/3$. Then for η we have

$$\eta(t) = \beta\frac{(t-a)^2}{2} + \int_a^t \frac{(t-\tau)^2}{2} f\big(\tau,\eta(\tau),\eta'(\tau),\eta''(\tau)\big)\,\mathrm{d}\tau$$
$$\geqq \beta\frac{(t-a)^2}{2} - M\frac{(t-a)^3}{6} \geqq \frac{(t-a)^2}{2}\Big[\beta - \frac{M}{3}(b-a)\Big] > 0$$

for $t \in (a,b]$, i.e. $\eta(t) > \xi(t) = 0$ for all $t \in (a,b]$. Since the solution of (10), (11) has m zeros in (a,b), the solution ξ of BVP (12), (13) is of the index m. We see that all the hypotheses of the Theorem are satisfied. Thus BVP (12), (13) has at least $m+1$ solutions and the proof is complete.

For example, the boundary value problem

$$x''' = -\frac{8}{3\sqrt{3}} \operatorname{arctg} x \,,$$
$$x(a) = x'(a) = 0, \; x(b) = 0$$

has at least $m+1$ solutions when $b-a \geqq (m+1)\pi$.

Remark. We still note that in the paper [9] there is investigated the existence and uniqueness of solutions of multipoint boundary value problems for an nth order nonlinear differential equation under the assumption of the existence and uniqueness of solutions of certain boundary value problems for the corresponding variational equations (in particular, when the variational equations are disconjugate on a given interval). For example, from Corollary 2 [9], provided that the assumptions (A), (B), (C) from [9] hold, we obtain the following result.

If the differential equation (4) is disconjugate on $[a,b]$ for all solutions ξ of (1), then BVP (1–3) has a unique solution. Consequently, if BVP (1–3) has several solutions, then the variational equation (4) for some solution ξ of (1) is not disconjugate on $[a,b]$.

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