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A NOTE ON THE FOLKMAN NUMBER $F(3,3;5)$

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ABSTRACT. For $r > \max\{p,q\}$, let $F(p,q;r)$ denote the minimum number of vertices in a graph G that has the following properties:

(1) G contains no complete subgraph on r vertices,
(2) in any green–red colouring of the edges of G there is a green complete subgraph on p vertices or a red complete subgraph on q vertices.

We show that $F(3,3;5) \leq 16$, which improves a recent result due to Erickson.

For $r > \max\{p,q\}$, let the *Folkman number* $F(p,q;r)$ be the minimum number of vertices in a graph G that has the following properties:

(1) G contains no complete subgraph on r vertices,
(2) in any green–red colouring of the edges of G there is a green complete subgraph on p vertices or a red complete subgraph on q vertices.

The existence of such a non-negative integer was proved by Folkman [2]. If $r > R(p,q)$ ($R(p,q)$ is the *Ramsey number*), then clearly $F(p,q;r) = R(p,q)$.

Very little is known about the Folkman numbers in the case $r \leq R(p,q)$. The only known precise result $F(3,3;6) = 8$ was established by Graham [3]. The corresponding graph is $C_5 + C_3$, the join of a cycle of length 5 and a cycle of length 3. Note that the join $G_1 + G_2$ of two graphs G_1 and G_2 is the graph whose vertex set is the union of the vertex sets of G_1, G_2, and whose edge set is the union of the edge sets of G_1, G_2, together with the set of all possible edges joining a vertex of G_1 to a vertex of G_2.

The only Folkman number that has been bounded reasonably is $F(3,3;5)$. The lower bound $F(3,3;5) \geq 10$ is due to Lin [6]. Graham and Spencer [4] have shown $F(3,3;5) \leq 23$. Later the upper bound was improved to 18 by Irving [5] and recently to 17 by Erickson [1]. Erickson conjectured that $F(3,3;5) = 17$. The aim of this note is to disprove his conjecture by showing that $F(3,3;5) \leq 16$.

As in [1], our proof is based on the following observation.

LEMMA. [1] *If C is a connected graph and $C_5 + C$ has been green–red coloured with no monochromatic triangle, then C is monochromatic.*

Proof. Suppose C is not monochromatic. Then in C there is a green edge vw adjacent to a red edge wx. At least two edges of the same colour (say green) from the vertex w have

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to be joined to some adjacent vertices y and z in C_5. As $C_5 + C$ has no green triangle, each of the edges vy, vz, yz is forced to be red which is a contradiction. □

THEOREM. *We have:*

$$F(3,3;5) \leq 16.$$

Proof. Denote by W the union of three cycles $X = abcde$, $Y = afgdh$, $Z = aijdk$ of length 5 whose vertices are given in cyclic order. Denote by H the graph with vertices $1,2,3,4,5$ and edges 12, 13, 23, 24, 35. Let $H(2,3,4,5)$, $H(1,2,4)$, and $H(1,3,5)$ denote the vertex-induced graph of H for the vertex sets $\{2,3,4,5\}$, $\{1,2,4\}$ and $\{1,3,5\}$, respectively.

We construct a graph G of order 16 to be the union of the graphs $X + H(2,3,4,5)$, $Y + H(1,2,4)$, and $Z + H(1,3,5)$.

As W is triangle free, any complete subgraph K_5 of G must contain the vertices $1,2,3$ which form a triangle in H. But in W there is no edge whose vertices are joined with each of the vertices $1,2,3$. Therefore G contains no K_5.

Suppose G has been coloured with no monochromatic triangle. By applying Lemma we get that:

- the edges 24, 23, 35 are of the same colour (in $X + H(2,3,4,5)$),
- the edges 12, 24 are of the same colour (in $Y + H(1,2,4)$),
- the edges 13, 35 are of the same colour (in $Z + H(1,3,5)$).

These facts yield the existence of the monochromatic triangle 123 which is a contradiction proving the theorem. □

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