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PARTIAL ORDER WITH DUALITY AND CONSISTENT CHOICE PROBLEM

Martin Gavalec

(*Communicated by Tibor Katriňák*)

ABSTRACT. The problem of consistent choice in a finite disjoint system of non-empty sets is considered. Any pair of chosen elements has to fulfil a given binary relation of consistency. The width of the system is the maximal cardinality of its members. The consistent choice problem is NP-complete for width > 2.

A connection between the consistent choice of width 2 and partially ordered sets with a unary operation of duality is described. Two $O(n^2)$ algorithms for solving the consistent choice of width 2 are proposed on the base of the above connection. A condition is given under which the algorithms work also for the width > 2.

1. Introduction

In the real life as well as in the theoretical research, we often meet the problem of consistent choice. The consistent choice corresponds to the situation when a simultaneous choice from a disjoint system of non-empty sets is to be performed, and all the chosen elements have to be pairwise compatible with respect to a given binary symmetric relation of consistency.

The notion of consistent choice was considered in [2]. The connection of the consistent choice with the problem of satisfiability was described, and the computational complexity of consistent choice was evaluated. Depending on the width m of the given system of sets, the problem of consistent choice from n sets is solvable in polynomial time $O(n^2)$ for $m = 2$, and it is NP-complete for $m > 2$. The consistent choice was used for solving the problem of balanced location on

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a graph in [3]. An equivalent problem of compatible representatives was studied by Knuth and Ragunathan in [4].

In this paper, a special duality operation on partially ordered sets and directed graphs is introduced. A relation of the partial order with duality to the problem of consistent choice is shown, and efficient algorithms for finding a solution of consistent choice are described. The algorithms work for $m = 2$ and under special conditions also for $m > 2$.

2. Consistent choice problem

We shall formulate the ideas from the previous section in the formal mathematical language. For simpler notation of index sets we shall use the convention by which any natural number n is considered as the set of all smaller natural numbers, i.e., $n = \{0, 1, \dots, n-1\}$.

The problem of consistent choice can be formulated as follows.

DEFINITION 1. Let $\mathcal{M} = (M_i;\ i \in n)$ be a finite pairwise disjoint system of non-empty sets, let $M = \bigcup(M_i;\ i \in n)$, and let P be a symmetric binary relation on M. Then the pair $(\mathcal{M}, P)$ is called a *consistency system.*

DEFINITION 2. Let $(\mathcal{M}, P)$ be a consistency system, let $M = \bigcup \mathcal{M}$. A subset $C \subseteq M$ is called a *consistent choice in* $(\mathcal{M}, P)$ if

$$(\forall i {\in} n)\, |M_i \cap C| = 1\,, \tag{1}$$

$$(\forall x, y {\in} C)\, (x, y) \in P\,. \tag{2}$$

Remark. Equation (2) implies that $(x, x) \in P$ holds true for any $x \in C$. This does not mean that the relation P must necessarily be reflexive. Of course, if $(x, x) \notin P$ for some $x \in M$, then this particular element x must not be contained in any consistent choice C in $(\mathcal{M}, P)$.

DEFINITION 3. *Consistent Choice Problem* (abbr. *CC*):
Given a consistency system $(\mathcal{M}, P)$, is there a consistent choice in $(\mathcal{M}, P)$?

The consistent choice problem is a special case of the problem of compatible representatives formulated by Knuth and Ragunathan in [4].

DEFINITION 4. *Problem of Compatible Representatives* (abbr. *CR*):
Given a system of non-empty sets $\mathcal{M} = (M_i;\ i \in n)$ and a binary relation P on $M = \bigcup(M_i;\ i \in n)$, is there a system of compatible representatives in [illegible]

Clearly, the problem CC is a special case of CR, for disjoint sets M_i, $i \in n$, and for a symmetric relation P. Under these conditions, there is no substantial difference between a system of P-compatible representatives $X = (x_i\,;\ i \in n)$ and a P-consistent choice C. The following theorem shows that, on the other hand, the problem CR can be transformed to CC, and therefore, both problems are equivalent.

THEOREM 1. *Let $\mathcal{M} = (M_i\,;\ i \in n)$ be a system of non-empty sets, and let P be a binary relation on $M = \bigcup(M_i\,;\ i \in n)$. If the system $\overline{\mathcal{M}}$ and the binary relation $\overline{P}$ are defined by the conditions: for any $i, j \in n$ and $x, y \in M$*

$$\overline{\mathcal{M}} = \left(\overline{M}_i\,;\ i \in n\right), \qquad (x, i) \in \overline{M}_i \iff x \in M_i\,,$$
$$\big((x,i),(y,j)\big) \in \overline{P} \iff \big(i < j,\ (x,y) \in P\big) \vee \big(j < i,\ (y,x) \in P\big) \vee \big(i = j,\ x = y\big),$$

then $(\mathcal{M}, P)$ is a yes *instance of CR if and only if $\left(\overline{\mathcal{M}}, \overline{P}\right)$ is a* yes *instance of CC.*

P r o o f. It is easy to verify that any $X = (x_i\,;\ i \in n)$ is a system of compatible representatives in $(\mathcal{M}, P)$ if and only if $\overline{X} = \{(x, i)\,;\ i \in n\}$ is a consistent choice in $\left(\overline{\mathcal{M}}, \overline{P}\right)$. □

The computational complexity of CC depends on the cardinality of sets M_i in $\mathcal{M}$. Let us denote

$$\operatorname{width} \mathcal{M} = \max_{i \in n} |M_i|\,.$$

DEFINITION 5. For natural m, *Consistent m-Choice Problem* (abbr. CC_m) is the problem CC with the additional condition

$$\operatorname{width} \mathcal{M} = m\,.$$

Remark. For any instance $(\mathcal{M}, P)$ of the problem CC_m we may assume, without loss of generality, that $|M_i| = m$ for any $i \in n$. If it is not the case, we can formally extend the set M_i by sufficiently many elements x such that $(x, x) \notin P$. This extension has no effect on the solvability of $(\mathcal{M}, P)$.

It was proved in [2] that the well-known problem of *m-Satisfiability* for conjunctive boolean formulas polynomially transforms to CC_m. By C o o k's theorem [1], *m-Satisfiability* is NP-complete for $m > 2$. Therefore, CC_m is NP-complete for $m > 2$, as well.

On the other hand, the problem CC_2 polynomially $O(n^2)$ transforms to the problem of *2-Satisfiability*, and as a consequence, the problem CC_2 is solvable in polynomial time $O(n^2)$.

Another formulation of the problem can be found in Section 3. The problem CC_2 is presented in the language of directed graphs with a duality operation.

In Section 4, we show that the problem CC_2 is equivalent to the problem of finding a maximal filter in a partial ordered set with a duality operation.

The above equivalency directly induces two algorithms for CC_2, which are described in Section 5. The first algorithm works with the transitive and reflexive closure of the relation P, the second one uses the backtracking method of depth 1. Both algorithms are of complexity $O(n^2)$.

3. Directed graphs with duality

Solving the problem CC_2 in a consistency system $(\mathcal{M}, P)$, we shall assume, in accordance with the remark after Definition 5, that $|M_i| = 2$ holds true for any $i \in n$. This specific situation can be alternatively described by a unary operation $f \colon x \to \overline{x}$ such that any M_i is of the form $M_i = \{x, \overline{x}\}$ for some $x \in M$.

If C is a consistent choice in $(\mathcal{M}, P)$, then condition (1) can be replaced by the following condition (3). For any $x \in M$,

$$x \in C \iff \overline{x} \notin C\,. \tag{3}$$

The consistency relation P induces some implications between the elements of M. Namely, if for some $x, y \in M$,

$$(x, y) \notin P \tag{4}$$

is true, then the elements x, y cannot be contained simultaneously in C. Thus, as a consequence of (3), condition (4) induces the following two implications:

$$\begin{aligned} x \in C &\implies \overline{y} \in C\,, \\ y \in C &\implies \overline{x} \in C\,. \end{aligned} \tag{5}$$

The implications expressed in (5) can be coded by a binary relation A on M. We define the relation A as follows. For $x, y \in M$ we put

$$\begin{aligned} (x, \overline{y}\,) \in A &\iff (x, y) \notin P\,, \\ (y, \overline{x}\,) \in A &\iff (x, y) \notin P\,. \end{aligned} \tag{6}$$

It follows from the symmetry of P that both formulas in (6) are equivalent.

Slightly modifying the notation, we get the duality condition for the relation A: for any $x, y \in M$,

$$(x, y) \in A \iff (\overline{y}, \overline{x}\,) \in A\,. \tag{7}$$

If the relation P is reflexive, then for any $x \in M$,

$$(x, \overline{x}\,) \notin A\,. \tag{8}$$

Finally, if C is a consistent choice in $(\mathcal{M}, P)$, then, by (3) and (5), condition (2) can be replaced by condition (9). For any $x, y \in M$,

$$\big(x \in C\,,\ (x,y) \in A\big) \implies y \in C\,. \tag{9}$$

The above considerations are summarized by the next two definitions.

Definition 6. A *digraph with duality* is a triple (M, A, f) such that

(i) (M, A) is a digraph with the vertex set M and the arrow set A,
(ii) $f\colon x \to \overline{x}$ is a unary operation on M,
(iii) $(\forall x{\in}M)\,\big[\overline{x} \neq x,\ \overline{\overline{x}} = x\big]$,
(iv) $(\forall x, y{\in}M)\,\big[(x,y) \in A \iff (\overline{y}, \overline{x}) \in A\big]$.

Definition 7. A *consistent choice* in (M, A, f) is a subset $C \subseteq M$ such that

(i) $(\forall x{\in}M)\,[x \in C \iff \overline{x} \notin C]$,
(ii) $(\forall x, y{\in}M)\,\big[\big(x \in C,\ (x,y) \in A\big) \implies y \in C\big]$.

Our considerations show that the triple (M, A, f) and the consistent choice C in $(\mathcal{M}, P)$ satisfy the following theorem.

Theorem 2. *Let $(\mathcal{M}, P)$ be a consistency system of width 2, let f be the corresponding unary operation on M, and let the relation A be defined by (6). Then the triple (M, A, f) is a digraph with duality, and any subset $C \subseteq M$ is a consistent choice in $(\mathcal{M}, P)$ if and only if C is a consistent choice in (M, A, f).*

Special cases of the digraphs with duality are analysed in the next section.

4. Partial order with duality

A consistent choice in a digraph with duality (M, A, f) can be found quite easily if the relation A has some special properties. The simplest case is the situation when the relation A is reflexive, transitive and antisymmetric, i.e., when A is a partial order on M.

Definition 8. A *partial order with duality* is a triple (M, A, f), such that

(i) (M, A, f) is a digraph with duality,
(ii) A is a partial order on M.

Definition 9. Let (M, A, f) be a partial order with duality. A subset $F \subseteq M$ is called a *filter* in (M, A, f) if, in the notation $\overline{F} = \{\overline{x}\,;\ x \in F\}$, the following conditions are fulfilled:

(i) $F \cap \overline{F} = \emptyset$,

(ii) F is upward closed, i.e.,

$$(\forall x, y \in M)\left[\left(x \in F,\ (x,y) \in A\right) \implies y \in F\right].$$

The filter F is a *maximal filter* in (M, A, f), if moreover,

(iii) $F \cup \overline{F} = M$.

THEOREM 3. *Let (M, A, f) be a partial order with duality. A subset $C \subseteq M$ is a consistent choice in (M, A, f) if and only if C is a maximal filter in (M, A, f).*

P r o o f . A straightforward verification shows that conditions 9 (i), (iii) are equivalent to condition 7 (i), and condition 9 (ii) is equivalent to 7 (ii). □

The next theorem describes a simple method for constructing a maximal filter in a partial order with duality. The method is based on a subsequent extending of a given filter and does not require any backtracking.

THEOREM 4. *Let (M, A, f) be a partial order with duality, let F be a filter in (M, A, f). If $x \in M - \left(F \cup \overline{F}\right)$ and $(x, \overline{x}) \notin A$, then the set $F_x = F \cup \left\{y \in M : (x, y) \in A\right\}$ is a filter in (M, A, f).*

P r o o f . We have $\overline{F_x} = \overline{F} \cup \left\{\overline{y};\ (\overline{y}, \overline{x}) \in A\right\}$. If there exists an element $z \in F_x \cap \overline{F_x}$, then we shall distinguish four cases: $z \in F \cap \overline{F}$, $z \in F - \overline{F}$, $z \in \overline{F} - F$, $z \in M - \left(F \cup \overline{F}\right)$. The first case is in contradiction with condition 9 (i). In the second case, we have $z \in F$, $(z, \overline{x}) \in A$, which implies $\overline{x} \in F$. Similarly, in the third case, we obtain $x \in \overline{F}$. Thus, both cases are in contradiction with the assumption $x \in M - \left(F \cup \overline{F}\right)$. In the last case, we have $(x, z), (z, \overline{x}) \in A$ and, by the transitivity of A, $(x, \overline{x}) \in A$. This is in contradiction with the assumption $(x, \overline{x}) \notin A$. Therefore, $F_x \cap \overline{F_x} = \emptyset$ holds true. The upward closeness of F_x is evident. □

THEOREM 5. *If (M, A, f) is a partial order with duality, then any filter F in (M, A, f) can be extended to a maximal filter in (M, A, f).*

COROLLARY. *Any partial order with duality is a* yes *instance of* CC_2.

P r o o f o f T h e o r e m 5. Let F be a filter in (M, A, f). If F is not a maximal filter, then there is an element $x \in M - \left(F \cup \overline{F}\right)$. If $(x, \overline{x}) \notin A$, then F can be extended to F_x. If $(x, \overline{x}) \in A$, then, by the antisymmetry of A, the assumption $(\overline{x}, x) \in A$ would imply $x = \overline{x}$, which is in contradiction with Definition 5 (iii). Thus, if $(x, \overline{x}) \in A$, then $(\overline{x}, x) \notin A$ must hold true, and the filter F can be extended to $F_{\overline{x}}$. The process of extending will continue until a maximal filter is obtained.

Definition 9, and Theorems 3 and 4 can be applied also in the situation when the relation A is reflexive and transitive, but not necessarily antisymmetric. In this case, the method described in Theorems 3, 4 and 5 can be used with a small change. Theorem 5 will be modified in the following way.

THEOREM 6. *Let (M, A, f) be a digraph with duality, and let the relation A be reflexive and transitive. If there is an element $x \in M$ such that $(x, \overline{x}), (\overline{x}, x) \in A$, then there is no consistent choice in (M, A, f). If such an element $x \in M$ does not exist, then any filter in (M, A, f) can be extended to a maximal filter.*

P r o o f . Clearly, if $(x, \overline{x}), (\overline{x}, x) \in A$, then no consistent choice exists. Further, the antisymmetry of A was used in the proof of Theorem 5 only at one place, to exclude the situation when $(x, \overline{x}), (\overline{x}, x) \in A$. The rest of the proof remains without change. □

The general situation, when the digraph with duality (M, A, f) has no special properties, can be solved using the transitive and reflexive closure. The transitive and reflexive closure Q of the relation A consists of all loops and of all finite compositions of arrows from A, i.e., for any $x, y \in M$,

$$
\begin{aligned}
(x, y) \in Q \iff \quad & (x = y) \\
& \vee (\exists x_0, \dots, x_k \in M) \left[x = x_0,\ y = x_k,\ (\forall i \in k)\, (x_i, x_{i+1}) \in A \right].
\end{aligned} \tag{10}
$$

It is evident that Q is a reflexive and transitive relation with the property: for any $x, y \in M$,

$$(x, y) \in Q \iff (\overline{y}, \overline{x}) \in Q\,. \tag{11}$$

Moreover, if C is a consistent choice in (M, A, f), then for any $x, y \in M$,

$$\big(x \in C\,,\ (x, y) \in Q \big) \implies y \in C\,. \tag{12}$$

A solution of a general case is described by Theorem 6 and by the following theorem.

THEOREM 7. *Let (M, A, f) be a digraph with duality, let Q be the transitive and reflexive closure of A. A subset $C \subseteq M$ is a consistent choice in (M, A, f) if and only if C is a maximal filter in (M, Q, f).*

P r o o f . The assertion of the theorem immediately follows from (11), (12). □

5. Algorithms for consistent choice of width 2

In this section, we describe two $O(n^2)$ algorithms for solving the problem CC_2. The first algorithm concerns the reflexive and transitive digraphs with duality, the second one is used in a general situation. The algorithms are based on Theorems 6 and 7 of the previous section, with certain modifications in the general case.

ALGORITHM $\mathcal{A}_1$:

Consistent Choice in Reflexive and Transitive Digraphs with Duality

Input: A digraph with duality (M, A, f), where A is a reflexive and transitive relation.

Output:

if $SOL =$ true, then C is a consistent choice (i.e., a maximal filter) in (M, A, f),

if $SOL =$ false, then there is no consistent choice in (M, A, f).

Remark. We assume that the set M is linearly ordered, and there is a procedure $\min R$ which can find the minimal element in any non-empty subset $R \subseteq M$. The set variable R represents the subset $M - (C \cup \overline{C})$, the boolean variable SOL indicates the existence of a solution.

procedure (*input, output*);
begin
    $C := \emptyset;\ SOL := true;\ R := M;$
    while ($R \neq \emptyset$ **and** $SOL = true$) **do**
        $x := \min R;$
        if $(x, \overline{x}) \in A$ **then** $x := \overline{x};$
        if $(x, \overline{x}) \in A$ **then**
            $SOL := false;$
        else
            for all $y \in R$ **do**
                if $(x, y) \in A$ **then** $C := C \cup \{y\};$
            enddo;
            $R := M - (C \cup \overline{C});$
        endif;
    enddo;
end.

The main cycle of the algorithm $\mathcal{A}_1$ will be performed at most n times, because in any run of the main cycle, at least two elements (namely, the elements x, $\overline{x}$) are taken away from $R = M - (C \cup \overline{C})$. The only exception is the case $SOL = false$, when $\mathcal{A}_1$ stops at once. The inner cycle runs at most $2n$ times. Therefore, the algorithm $\mathcal{A}_1$ stops in time $O(n^2)$. The correctness of $\mathcal{A}_1$ was proved in the previous section (Theorems 3–6).

If the relation A is not reflexive and transitive, then the algorithm $\mathcal{A}_1$ can be used in (M, Q, f), where Q is the transitive and reflexive closure of A. This procedure gives the correct result for (M, A, f) in view of Theorem 7. However, this approach is not very advantageous, as the construction of the transitive closure is done, in general, in time $O(n^{2.4})$ (by [5]). We shall apply a special approach to keep the computational complexity at level $O(n^2)$.

The algorithm $\mathcal{A}_1$ is modified to the algorithm $\mathcal{A}_2$, which constructs only the arrows of Q, that are actually necessary in the current computation. $\mathcal{A}_2$ uses a "depth-first search" procedure, *Succ*, which produces the output sequence of all successors of a given element x in the digraph (M, A). Such a procedure is described in [6], and its computational complexity is $O(|M| + |A|)$.

We assume that the work of the procedure *Succ* is organized in such a way that the procedure creates a walk $s = (s(0), s(1), \ldots, s(k))$ in the digraph (M, A), which begins at $x = s(0)$ and which is systematically extended in order to find all successors of x. In accordance with [6], the created walk s uses the arcs from A in both directions and has the following properties:

(TD) any arc is used for the first time in the direction of its orientation,
(T1) any arc is used at most once in each direction,
(T2) any discovering arc (an arc leading to a vertex not yet visited) can be used in the opposite direction only when there is no other possibility,
(T3) if an arc leads to a vertex already visited before, then, in the next step, the arc is used in the opposite direction.

Each time when a new successor y is discovered, i.e., if

$$y = s(k), \qquad (\forall l \in k)\; y \neq s(l),$$

then *Succ* stops and gives the output (s, k). The next call of the procedure *Succ* uses (s, k) as the input and produces an extension $s' \supset s$, of the length $k' > k$, such that $s'(k')$ is the next discovered successor of x. The search is finished when the walk returns to its beginning, i.e., when $s(k) = x$.

Succ uses the set variable $R = M - (C \cup \overline{C})$ as the third parameter and restricts the search to the arrows that are in R. In this way, the total time consumed by all calls of *Succ* is bounded by $O(|M| + |A|) = O(n^2)$.

In $\mathcal{A}_2$, the successors of the considered element x are being compared with the dual element $\overline{x}$ in order to verify the condition

$$(x, \overline{x}) \notin Q. \tag{11}$$

The verification runs parallely in two dual branches, starting in x and in $\overline{x}$. The search stops if one branch successfully verifies condition (11), or, if both branches fail.

In the first case, the successful element x, as well as all its successors, are put into C. If both branches fail, the computation stops with the response *SOL* = *false*.

ALGORITHM $\mathcal{A}_2$:
Consistent Choice of width 2 (general case)

Input: A digraph with duality (M, A, f).
Output:

if SOL = true, then C is a consistent choice in (M, A, f),
if SOL = false, then there is no consistent choice in (M, A, f).

Remark. The algorithm uses the procedure *Succ* described above. The variables R, SOL have the same meaning as in the algorithm $\mathcal{A}_1$. The set variables C_1, C_2 are used in two parallel branches of the computation for temporary keeping the discovered successors of x and $\overline{x}$, respectively. The boolean variables OK_1, OK_2 indicate that condition (11) or the dual condition has not been violated, up to the current step. The variables END_1, END_2 signalize that the "depth-first search" in the corresponding branch has come to the end.

procedure (*input, output*);
begin
  $C, C_1, C_2 := \emptyset;\ SOL := true;\ R := M;$
  while ($R \neq \emptyset$ **and** $SOL = true$) **do**
    $x := \min R;\ OK_1, OK_2 := true;\ END_1, END_2 := false;$
    $k_1, k_2 := 0;\ s_1(0) := x;\ s_2(0) := \overline{x};$
    while (**non**($END_1 = true$ **and** $OK_1 = true$) **and**
        non($END_2 = true$ **and** $OK_2 = true$) **and**
        non($OK_1 = false$ **and** $OK_2 = false$)) **do**
      $(s_1, k_1) := Succ(s_1, k_1, R);\ (s_2, k_2) := Succ(s_2, k_2, R);$
      $y_1 := s_1(k_1);\ y_2 := s_2(k_2);$
      $C_1 := C_1 \cup \{y_1\};\ C_2 := C_2 \cup \{y_2\};$
      if $y_1 = x$ **then** $END_1 := true;$
      if $y_2 = \overline{x}$ **then** $END_2 := true;$
      if $y_1 = \overline{x}$ **then** $OK_1 := false;$
      if $y_2 = x$ **then** $OK_2 := false;$
    enddo;
    if ($OK_1 = false$ **and** $OK_2 = false$) **then**
      $SOL := false;$
    else
      if ($END_1 = true$ **and** $OK_1 = true$) **then** $C, C_2 := C_1;$
      if ($END_2 = true$ **and** $OK_2 = true$) **then** $C, C_1 := C_2;$
    endif;
  enddo;
end.

6. Algorithms for determinate consistent choice

The algorithms proposed in the previous Section 5 are based on the duality operation f introduced in Section 3, for consistent choice of width 2. For their application to the case of width $m > 2$, the algorithms and the notion of duality must be modified. As we have mentioned in the introduction, the consistent choice problem is NP-complete in the case $m > 2$, therefore the modified version will be restricted to a special case of determinate digraphs described in this section. The main results are presented in Theorems 10 and 12.

The notion of duality will be modified to pseudoduality in the following way.

Definition 10. Let $\mathcal{M} = (M_i\,;\ i \in n)$ be a finite disjoint system of non-empty sets, let $M = \bigcup(M_i\,;\ i \in n)$. For any $i \in n,\ x, y \in M_i,\ x \neq y$, we say that the element y is *pseudodual* to x. The set of all elements pseudodual to x will be denoted by $\widetilde{x} = M_i - \{x\}$.

By the definition, the relation of pseudoduality is irreflexive and symmetrical. If width $\mathcal{M} = 2$, then the dual element $\overline{x}$ is the only element in $\widetilde{x}$, i.e., $\widetilde{x} = \{\overline{x}\}$. If width $\mathcal{M} = m > 2$, then any $\widetilde{x}$ consists of $m-1$ elements.

The function $g\colon x \to \widetilde{x}$ represents the pseudoduality as a multivalued unary operation on M, in the sense of the following definition.

Definition 11. A *digraph with pseudoduality* is a triple (M, A, g), such that

(i) (M, A) is a digraph with the vertex set M and the arrow set A,
(ii) $g\colon x \to \widetilde{x}$ is a multivalued unary operation on M,
(iii) $(\forall x \in M)\ x \notin \widetilde{x}$,
(iv) $(\forall x, y \in M)\,[x \in \widetilde{y} \iff y \in \widetilde{x}\,]$.

The relation A is called *determinate* in (M, A, g), if moreover,

(v) $(\forall x, y \in M)(\forall y' \in \widetilde{y})(\exists x' \in \widetilde{x})\ \big[(x, y) \in A \implies (y', x') \in A\big]$.

Definition 12. A *consistent choice* in (M, A, g) is a subset $C \subseteq M$ such that

(i) $(\forall x \in M)\,[x \in C \iff \widetilde{x} \cap C = \emptyset]$,
(ii) $(\forall x, y \in M)\ \big[(x \in C,\ (x, y) \in A) \implies y \in C\big]$.

Definition 13. Let (M, A, g) be a digraph with pseudoduality.

(i) An element $x \in M$ is called *contradictory* if

$$(\exists y, y' \in M)\ \big[y' \in \widetilde{y},\ (x, y), (x, y') \in A\big],$$

(ii) An element $x \in M$ is called *strongly contradictory*, if

$$(\exists x' \in \widetilde{x}\,)\ \big[(x, x') \in A\big].$$

By the notation *Contr, SContr*, respectively, we denote the sets of all contradictory elements or strongly contradictory elements in M. The basic properties of the sets *Contr, SContr* are presented in the following lemma.

LEMMA 1. *Let (M, A, g) be a digraph with pseudoduality. Then the following implications hold true:*

(i) *if the relation A is transitive, then the set Contr is downward closed.*
(ii) *if the relation A is transitive and determinate, then $Contr \subseteq SContr$.*

Proof.
(i) Let $x, z \in M$, $x \in Contr$, $(z, x) \in A$. Then there are $y, y' \in M$ such that $y' \in \widetilde{y}$, $(x, y), (x, y') \in A$. By the transitivity of A, we have $(z, y), (z, y') \in A$, i.e., $z \in Contr$.
(ii) Let $x \in Contr$, then there are $y, y' \in M$, $y' \in \widetilde{y}$, $(x, y), (x, y') \in A$. By Definition 11 (v), there is $x' \in \widetilde{x}$ such that $(y', x') \in A$. By the transitivity of A, we get $(x, x') \in A$, i.e., $x \in SContr$. □

DEFINITION 14. Let (M, A, g) be a digraph with pseudoduality. A subset $F \subseteq M$ is called a *pseudofilter* in (M, A, g) if, in the notation $\widetilde{F} = \bigcup\{\widetilde{x} : x \in F\}$, the following conditions are fulfilled:

(i) $F \cap \widetilde{F} = \emptyset$,
(ii) F is upward closed, i.e.,

$$(\forall x, y \in M) \left[\left(x \in F,\ (x, y) \in A\right) \implies y \in F \right].$$

The pseudofilter F is a *maximal pseudofilter* in (M, A, g) if, moreover,

(iii) $F \cup \widetilde{F} = M$.

LEMMA 2. *Let (M, A, g) be a digraph with pseudoduality, let F be a pseudofilter in (M, A, g). If the relation A is determinate, then $\widetilde{F}$ is downward closed, i.e.,*

$$(\forall z, y \in M) \left[\left(y \in \widetilde{F},\ (z, y) \in A\right) \implies z \in \widetilde{F} \right].$$

Proof. Let $y \in \widetilde{F}$, $(z, y) \in A$. Then, by the definition of $\widetilde{F}$ and by the symmetry of pseudoduality, there is $y' \in F$ such that $y' \in \widetilde{y}$. By the determinateness of A, there is $z' \in \widetilde{z}$ such that $(y', z') \in A$. The upward closeness of F implies that $z' \in F$, therefore $z \in \widetilde{F}$. □

THEOREM 8. *Let (M, A, g) be a digraph with pseudoduality. A subset $C \subseteq M$ is a consistent choice in (M, A, g) if and only if C is a maximal pseudofilter in (M, A, g).*

Proof. We show that the implication

$$(\forall x \in M)\left[x \in C \implies \widetilde{x} \cap C = \emptyset\right]$$

is equivalent to the condition $C \cap \widetilde{C} = \emptyset$.

Let $\widetilde{x} \cap C = \emptyset$ be fulfilled for all $x \in C$, then we have $C \cap \widetilde{C} = C \cap \bigcup\{\widetilde{x};\ x \in C\} = \bigcup\{\widetilde{x} \cap C;\ x \in C\} = \emptyset$.

Let $C \cap \widetilde{C} = \emptyset$, then for any $x \in C$ we have $\widetilde{x} \cap C = \emptyset$ because of the inclusion $\widetilde{x} \subseteq \widetilde{C}$.

Similarly, we show that the converse implication

$$(\forall x \in M)\, [x \notin C \implies \widetilde{x} \cap C \neq \emptyset]$$

is equivalent to the condition $C \cup \widetilde{C} = M$.

Let $\widetilde{x} \cap C \neq \emptyset$ holds true for any $x \notin C$, then there is $y \in C$, $y \in \widetilde{x}$. Using the symmetry property 11 (iv), we have $x \in \widetilde{y}$, therefore $x \in \widetilde{C}$.

Let $C \cup \widetilde{C} = M$. If $x \notin C$, then $x \in \widetilde{C}$, and there is $y \in C$, $x \in \widetilde{y}$, so $y \in \widetilde{x}$, therefore $\widetilde{x} \cap C \neq \emptyset$.

We have proved that condition 12 (i) is equivalent to the conjunction of 14 (i), (iii). Conditions 12 (ii) and 14 (ii) are identical, therefore the proof of the theorem is complete. □

The following theorem is a modification of Theorem 4 and describes a way of constructing a maximal pseudofilter by subsequent extending the given pseudofilter.

THEOREM 9. *Let (M, A, g) be a digraph with pseudoduality, let A be a reflexive, transitive and determinate relation on M, and let F be a pseudofilter in (M, A, g). If $x \in M - \big(F \cup \widetilde{F}\big)$ and $x \notin SContr$, then the set $F_x = F \cup \big\{y \in M;\ (x, y) \in A\big\}$, is a pseudofilter in (M, A, f).*

Proof. By Definition 14, we have $\widetilde{F_x} = \widetilde{F} \cup \bigcup\{\widetilde{y};\ (x, y) \in A\}$. Let us assume that z is an element in $F_x \cap \widetilde{F_x}$. Similarly as in the proof of Theorem 4, we shall distinguish four cases: $z \in F \cap \widetilde{F}$, $z \in F - \widetilde{F}$, $z \in \widetilde{F} - F$, $z \in M - \big(F \cup \widetilde{F}\big)$. The first case is in contradiction with 14 (i).

In the second case, we have $z \in F$ and $z \in \bigcup\{\widetilde{y};\ (x, y) \in A\}$, i.e., there is $y \in M$ such that $(x, y) \in A$, $z \in \widetilde{y}$. By the determinateness of A, there is $x' \in \widetilde{x}$ such that $(z, x') \in A$, and by the upward closeness of F, we get $x' \in F$, which implies $x \in \widetilde{F}$. This is in contradiction with the assumption $x \in M - \big(F \cup \widetilde{F}\big)$.

In the third case, we have $z \in \widetilde{F}$ and $(x, z) \in A$. By Lemma 2, $\widetilde{F}$ is downward closed and therefore $x \in \widetilde{F}$, which leads to the same contradiction.

In the fourth case, $(x, z) \in A$ holds true, and there is an element $y \in M$ such that $z \in \widetilde{y}$, $(x, y) \in A$. By the determinateness of A, there is an element $x' \in \widetilde{x}$ such that $(z, x') \in A$. By the transitivity of A, we get $(x, x') \in A$, i.e., $x \in SContr$.

All the four cases lead to a contradiction, therefore $F_x \cap \widetilde{F_x} = \emptyset$ must be true. The upward closeness of F_x is a consequence of the transitivity of A. □

THEOREM 10. *Let (M, A, g) be a digraph with pseudoduality, let A be a reflexive, transitive and determinate relation on M. Then the problem of consistent choice in (M, A, g) is solvable by the pseudodual version of the algorithm $\mathcal{A}_1$.*

Proof. It is easy to see that, analogously to Theorem 6, if there is an element $x \in M$ such that all elements in $\{x\} \cup \widetilde{x}$ are strongly contradictory, then there is no consistent choice in (M, A, g). If $x \notin SContr$, or if there is $x' \in \widetilde{x}$ such that $x' \notin SContr$, then, by Theorem 9, the following pseudodual version $\mathcal{A}_1'$ will find a solution. □

ALGORITHM $\mathcal{A}_1'$:

Consistent Choice in Reflexive, Transitive and Determinate Digraphs with Pseudoduality

Input: A digraph with pseudoduality (M, A, g), where A is a reflexive, transitive and determinate relation.

Output:

if SOL = true, then C is a consistent choice (i.e., a maximal filter) in (M, A, g),
if SOL = false, then there is no consistent choice in (M, A, g).

Remark. Similarly sa in $\mathcal{A}_1$, we assume that the set M is linearly ordered, and the algorithm can find the minimal element in any non-empty subset of M. The condition $x \in SContr$ used in the algorithm can be verified in $< m$ steps by checking if $(x, x') \in A$ for some element $x' \in \widetilde{x}$.

procedure (*input, output*);
begin
  $C := \emptyset$; $\widetilde{C} := \emptyset$; $SOL := true$; $R := M$;
  while ($R \neq \emptyset$ **and** $SOL = true$) **do**
    $x := \min R$;
    if $x \in SContr$ **then**
      if $\widetilde{x} - SContr = \emptyset$ **then**
        $SOL := false$;
      else
        $x := \min(\widetilde{x} - SContr)$;
      endif;
    endif;

```
      if SOL = true then
         for all y ∈ R do
            if (x,y) ∈ A then C := C ∪ {y}; C̃ := C̃ ∪ ỹ;
         enddo;
         R := M − (C ∪ C̃);
      endif;
   enddo;
end.
```

THEOREM 11. *Let (M,A,g) be a digraph with duality, let the relation A be reflexive and determinate on M. Then the transitive and reflexive closure Q of A is determinate on M.*

P r o o f. Let $x,y \in M$, $(x,y) \in Q$, and let $y' \in \widetilde{y}$. Then there exist elements $x_0, x_1, \dots, x_k \in M$ such that $x = x_0$, $y = x_k$ and $(\forall i{\in}k)\,(x_i, x_{i+1}) \in A$. By repeated use of Definition 11 (v), we get the elements $x'_0, x'_1, \dots, x'_k$ such that $x'_k = y'$, $(\forall i{\in}k)\ x'_i \in \widetilde{x}_i$ and $(\forall i{\in}k)\,(x'_{i+1}, x'_i) \in A$. We denote $x' := x'_0$. Then $x' \in \widetilde{x}$ and $(y', x') \in Q$ hold true. The proof is complete. □

THEOREM 12. *Let (M,A,g) be a digraph with pseudoduality, let A be a reflexive and determinate relation on M. Then the problem of consistent choice in (M,A,g) is solvable by the pseudodual version of the algorithm $\mathcal{A}_2$.*

P r o o f. The pseudodual version of $\mathcal{A}_2$, denoted by $\mathcal{A}'_2$, is similar to the pseudodual version $\mathcal{A}'_1$ of the algorithm $\mathcal{A}_1$. Let us denote $m = \max_{x\in M} |\widetilde{x}| + 1$. Instead of two parallel branches of computation used in $\mathcal{A}_2$, the algorithm $\mathcal{A}'_2$ performs the search of successors in $\le m$ parallel branches beginning in all elements in $\{x\} \cup \widetilde{x}$. Any branch is *OK* until it is discovered that its starting point is strictly contradictory.

When the first *OK* branch comes to end of the search, then all elements of the branch are added to C, and all elements pseudodual to them are added to $\widetilde{C}$. If, in some cycle, all branches are not *OK*, then the consistent choice in (M,A,g) does not exist, and the algorithm stops. □

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