Mathematica Slovaca

Ivan Dobrakov

Multilinear integration of bounded scalar valued functions

Mathematica Slovaca, Vol. 49 (1999), No. 3, 295--304

Persistent URL: http://dml.cz/dmlcz/136753

Math. Slovaca, **49** (1999), No. 3, 295–304

MULTILINEAR INTEGRATION OF BOUNDED SCALAR VALUED FUNCTIONS

IVAN DOBRAKOV†

(*Communicated by Miloslav Duchoň*)

ABSTRACT. In this paper we prove a general convergence theorem which extends our Multilinear Lebesgue bounded convergence theorem. The extension is that we have not only convergent sequences of functions, but also a convergent sequence of multimeasures (polymeasures). As an application we prove a particular Fubini theorem.

1. The multilinear integral

In the following, d will be a fixed positive integer denoting the dimension of multilinearity, and Y will be a Banach space over the scalar field K of real or complex numbers. We have d measurable spaces $(T_1, \mathcal{S}_1), \dots, (T_d, \mathcal{S}_d)$, where $\mathcal{S}_i$ is a σ-ring of subsets of T_i for $i = 1, \dots, d$, and there is given a separately σ-additive vector d-multimeasure (d-polymeasure) $\gamma\colon \mathcal{S}_1 \times \dots \times \mathcal{S}_d \to Y$. Note that the separate σ-additivity of γ means that:

1) $\gamma(\cdot, A_2, \dots, A_d)\colon \mathcal{S}_1 \to Y$ is a σ-additive vector measure for each $(A_2, \dots, A_d) \in \mathcal{S}_2 \times \dots \times \mathcal{S}_d$,

$\vdots \qquad\qquad \vdots$

d) $\gamma(A_1, \dots, A_{d-1}, \cdot)\colon \mathcal{S}_d \to Y$ is a σ-additive vector measure for each $(A_1, \dots, A_{d-1}) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_{d-1}$

(see [1; Definition 1]). We will use the abbreviations: $\times C_i = C_1 \times \dots \times C_d$, and $(C_i) = (C_1, \dots, C_d)$.

We obtain the multilinear integral of d-tuples of bounded scalar valued measurable functions in an elementary way, in contrast with the general case

AMS Subject Classification (1991): Primary 28B99.
Key words: vector d-multimeasure, multilinear integral, Multilinear Lebesgue bounded convergence theorem, Fubini theorem.

of vector valued functions, see [2] and [4]. First, let us recall the definition of the supremation $\overline{\gamma}$ of γ (see [1; Definition 2])

$$\overline{\gamma}(A_i) = \sup\{|\gamma(B_i)|;\ B_i \in A_i \cap \mathcal{S}_i,\ i = 1,\dots,d\}.$$

Then

$$\text{(N)} \qquad \overline{\gamma}(T_i) = \sup_{(A_i)\in\times\mathcal{S}_i} \overline{\gamma}(A_i) < +\infty$$

by Nikodým uniform boundedness theorem for multimeasures (see [1; p. 490]).

For $i = 1,\dots,d$, $S(\mathcal{S}_i)$ denotes the linear normed space of all $\mathcal{S}_i$-simple functions $f_i\colon T_i \to K$ with the norm $\|f_i\|_{T_i} = \sup_{t_i\in T_i} |f_i(t_i)|$. For $(A_i) \in \times\mathcal{S}_i$ and for $(f_i) \in \times S(\mathcal{S}_i)$ with $f_i = \sum_{j=1}^{r_i} a_{ij}\cdot\chi_{A_{ij}}$, $a_{ij} \in K$, $i = 1,\dots,d$, $j = 1,\dots,r_i$, and $A_{ij} \in \mathcal{S}_i$, $j = 1,\dots,r_i$, pairwise disjoint for each $i = 1,\dots,d$, we define the multilinear integral naturally as

$$\gamma\big[(f_i),(A_i)\big] = \int_{(A_i)} (f_i)\,\mathrm{d}\gamma = \sum_{j_1=1}^{r_1}\cdots\sum_{j_d=1}^{r_d} a_{1j_1}\cdot\ldots\cdot a_{1d_d}\cdot\gamma(A_i \cap A_{ij_i}).$$

Let us note first that the finite iterated sum on the right hand side neither depends on the order of summation nor on any grouping of the summation. We call this property the Inner Fubini property of the multilinear integral, and we will state it explicitly later when the integral is extended to d-tuples of bounded scalar valued measurable functions.

For $i = 1,\dots,d$ let $\overline{S(\mathcal{S}_i)}$ be the closure of $S(\mathcal{S}_i)$ in the norm $\|\cdot\|_{T_i}$ of uniform convergence on T_i in the Banach space of the bounded scalar valued functions on T_i. It is well known that $f_i \in \overline{S(\mathcal{S}_i)}$ if and only if $f_i\colon T_i \to K$ is bounded and $\mathcal{S}_i$-measurable. Our extension of the integral is based on this density of the space of simple functions.

Our integral mapping $\gamma\big[(\cdot),(\cdot)\big]\colon \times S(\mathcal{S}_i) \times (\times\mathcal{S}_i) \to Y$ has the properties:

1) $\gamma\big[(f_i),(A_i)\big] = \gamma\big[(f_i),(A_i \cap \{t_i \in T_i;\ f_i(t_i) \neq 0\})\big]$ for each $(f_i) \in \times S(\mathcal{S}_i)$ and each $(A_i) \in \times\mathcal{S}_i$. We use this equality to define $\gamma\big[(f_i),(T_i)\big]$.
2) $\gamma\big[(f_i),(\cdot)\big]\colon \times\mathcal{S}_i \to Y$ is separately σ-additive for each d-tuple of functions $(f_i) \in \times S(\mathcal{S}_i)$.
3) $\gamma\big[(\cdot),(A_i)\big]\colon \times S(\mathcal{S}_i) \to Y$ is separately linear for each d-tuple of sets $(A_i) \in \times\mathcal{S}_i$. Its norm $\|\gamma\|(A_i)$, called the semivariation of γ on (A_i), satisfies the inequality

$$\|\gamma\|(A_i) \le 4^d\cdot\overline{\gamma}(A_i) \le 4^d\cdot\overline{\gamma}(T_i) < +\infty,$$

see [1; Theorem 3. 4)] and (N) above.

Hence, by continuity it has a unique separately linear and bounded extension $g[(\cdot),(A_i)]: \times\overline{S(\mathcal{S}_i)} \to Y$ with the same norm $\|\gamma\|(A_i)$. More precisely, we have the following extension of our multilinear integral:

DEFINITION. Let $(f_i) \in \times\overline{S(\mathcal{S}_i)}$ and $(A_i) \in \times\mathcal{S}_i$, and let for each $i = 1,\dots,d$, $f_{i,n_i} \in S(\mathcal{S}_i)$, $n_i = 1,2,\dots$ be such that $\lim\limits_{n_i\to\infty} \|f_i - f_{i,n_i}\|_{T_i} = 0$. Then

$$\gamma[(f_i),(A_i)] = \int\limits_{(A_i)} (f_i)\,\mathrm{d}\gamma = \lim_{n_1,\dots,n_d\to\infty} \int\limits_{(A_i)} (f_{i,n_i})\,\mathrm{d}\gamma \in Y$$

exists uniformly with respect to $(A_i) \in \times\mathcal{S}_i$. This limit is independent of the converging sequences (f_{i,n_i}).

Obviously the analogs of properties 1) and 2) above hold for this extension of integral. Note also that if $\mu\colon \mathcal{S}_i\otimes\mathcal{S}_d \to Y$ is a countably additive vector measure, if $\gamma_\mu = \mu\colon \times\mathcal{S}_i \to Y$, and if $(f_i) \in \times S(\mathcal{S}_i)$, then $f_1\cdot\ldots\cdot f_d$ is integrable with respect to μ, and we have the equality

$$\int\limits_{(A_i)} (f_i)\,\mathrm{d}\gamma_\mu = \int\limits_{A_1\times\ldots\times A_d} f_1\cdot\ldots\cdot f_d\,\mathrm{d}\mu \qquad \text{for each} \quad (A_i) \in \times\mathcal{S}_i\,.$$

Note that there are many polymeasures which are not restrictions of measures (see [5; Remarks 2 and 3]).

Since finite iterated sums depend neither on the order nor on the grouping of summation, and since, by our extension of the integral, we have uniform limits, the following important property holds:

THE INNER FUBINI PROPERTY. *Suppose d_1 is a positive integer such that $1 \le d_1 < d$, and let $f_i \in \overline{S(\mathcal{S}_i)}$ and $A_i \in \mathcal{S}_i$ for $i = 1,\dots,d$. Then obviously $(\chi_{A_1},\dots,\chi_{A_{d_1}},f_{d_1+1},\dots,f_d) \in \times\overline{S(\mathcal{S}_i)}$, the mapping $(A_1,\dots,A_{d_1}) \mapsto \int\limits_{(A_i)} (\chi_{A_1},\dots,\chi_{A_{d_1}},f_{d_1+1},\dots,f_d)\,\mathrm{d}\gamma$, $(A_1,\dots,A_{d_1}) \in \mathcal{S}_1\times\ldots\times\mathcal{S}_{d_1}$, is a d_1-polymeasure, and*

$$\int\limits_{(A_i)} (f_i)\,\mathrm{d}\gamma = \int\limits_{(A_1,\dots,A_{d_1})} (f_1,\dots,f_{d_1})\,\mathrm{d}\left(\int\limits_{(\dots,A_{d_1+1},\dots,A_d)} \cdot\ (\dots,f_{d_1+1},\dots,f_d)\,\mathrm{d}\gamma\right).$$

Hence

$$\int\limits_{(A_i)} (f_i)\, \mathrm{d}\gamma$$

$$= \int\limits_{A_1} f_1\, \mathrm{d}\Bigg(\int\limits_{(\dots, A_2)} (\dots, f_2)\, \mathrm{d}\Bigg(\dots$$

$$\dots \Bigg(\int\limits_{(\dots, A_{d-1})} (\dots, f_{d-1})\, \mathrm{d}\Bigg(\int\limits_{(\dots, A_d)} (\dots, f_d)\, \mathrm{d}\gamma \Bigg)\Bigg) \dots \Bigg)\Bigg)$$

$$= \int\limits_{A_d} f_d\, \mathrm{d}\Bigg(\int\limits_{(A_{d-1},\cdot)} (f_{d-1},\cdot)\, \mathrm{d}\Bigg(\int\limits_{(A_2,\dots)} (f_2,\dots)\, \mathrm{d}\Bigg(\int\limits_{(A_1,\dots)} (f_1,\dots)\, \mathrm{d}\gamma \Bigg)\Bigg) \dots \Bigg)$$

= *the analog for any permutation of* $\{1,\dots,d\}$

= *the analog for any decomposition of* $\{1,\dots,d\}$ *into finite groups, and any order of it.*

2. The Basic convergence theorem

First we prove the 1-dimensional version. In the following two theorems, $\mathcal{S}$ will be a σ-ring of subsets of a non empty set T.

THEOREM 1. *Suppose that* $\mu_n \colon \mathcal{S} \to Y$, $n = 1,2,\dots$, *are countably additive, and* $\mu_n(A) \to \mu(A) \in Y$ *for each* $A \in \mathcal{S}$. *Further, let* $f, f_k \in \overline{S(\mathcal{S})}$, $k = 1,2,\dots$, *let* $f_k \to f$ *pointwise and let* $|f_k| \le C < +\infty$ *for each* $k = 1,2,\dots$. *Then*

$$\lim_{k,n\to\infty} \int\limits_A f_k\, \mathrm{d}\mu_n = \int\limits_A f\, \mathrm{d}\mu \qquad \textit{for each} \quad A \in \mathcal{S}.$$

This limit is uniform with respect to $A \in \mathcal{S}$, *provided that* $\mu_n(A) \to \mu(A)$ *uniformly with respect to* $A \in \mathcal{S}$.

Proof. By the Nikodým uniform boundedness theorem, $a = \sup_n \overline{\mu}_n(T) < +\infty$ (see [1; p. 490]). Further, by the Vitali-Hahn-Saks-Nikodým theorem, the supremations $\overline{\mu}_n, \mu \colon \mathcal{S} \to [0,+\infty)$, $n = 1,2,\dots$, are uniformly continuous from above at the empty set $\emptyset$ (see [1; p. 490]). Hence, if we put

$$\lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\overline{\mu}_n(A)}{1 + \overline{\mu}_n(T)} \qquad \text{for} \quad A \in \mathcal{S},$$

then clearly $\overline{\mu}_n$, μ, $n = 1, 2, \ldots$, are uniformly λ-absolutely continuous.

Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $E \in \mathcal{S}$ and $\lambda(E) < \delta$ imply $\left|\int_E f_k \, \mathrm{d}\mu_n\right| \le C \cdot \overline{\mu}_n(E) < \frac{\varepsilon}{4}$ for each $k, n = 0, 1, 2, \ldots$, where $f_0 = f$ and $\mu_0 = \mu$. By Egoroff's theorem there exists $E_\varepsilon \in \mathcal{S}$ such that $\lambda(E_\varepsilon) < \delta$, and the sequence f_k, $k = 1, 2, \ldots$, converges uniformly to the function f on $T - E_\varepsilon$. Take $g_j \in S(\mathcal{S})$, $j = 1, 2, \ldots$, so that $\lim_{j\to\infty} \|f - g_j\|_T = 0$, and let $A \in \mathcal{S}$. Then

$$
\begin{aligned}
&\left|\int_A f_k \, \mathrm{d}\mu_n - \int_A f \, \mathrm{d}\mu\right| \\
&\le \frac{\varepsilon}{2} + \left|\int_{A-E_\varepsilon} f_k \, \mathrm{d}\mu_n - \int_{A-E_\varepsilon} f \, \mathrm{d}\mu\right| \\
&\le \frac{\varepsilon}{2} + \left|\int_{A-E_\varepsilon} [(f_k - f) + (f - g_j)] \, \mathrm{d}\mu_n\right| \\
&\qquad + \left|\int_{A-E_\varepsilon} g_j \, \mathrm{d}\mu_n - \int_{A-E_\varepsilon} g_j \, \mathrm{d}\mu\right| + \left|\int_{A-E_\varepsilon} (g_j - f) \, \mathrm{d}\mu\right| \\
&\le \frac{\varepsilon}{2} + [\|f_k - f\|]_{T-E_\varepsilon} + \|f - g_j\|_T] \cdot a \\
&\qquad + \left|\int_{A-E_\varepsilon} g_j \, \mathrm{d}\mu_n - \int_{A-E_\varepsilon} g_j \, \mathrm{d}\mu\right| + \|g_j - f\|_T \cdot a
\end{aligned}
$$

for each $j, k, n = 1, 2, \ldots$.

Due to uniform limits, and finiteness of a, there exists j_0 such that

$$
\left|\int_A f_k \, \mathrm{d}\mu_n - \int_A f \, \mathrm{d}\mu\right| \le \frac{3}{4}\varepsilon + \left|\int_{A-E_\varepsilon} g_{j_0} \, \mathrm{d}\mu_n - \int_{A-E_\varepsilon} g_{j_0} \, \mathrm{d}\mu\right|
$$
$$
\text{for} \quad k \ge j_0, \quad n = 1, 2, \ldots.
$$

Since g_{j_0} is a $\mathcal{S}$-simple function, the convergence $\mu_n(B) \to \mu(B)$ for each $B \in \mathcal{S}$ implies the existence of $n_0 \ge j_0$ such that the estimated difference is less than ε for $n \ge n_0$. This estimation does not depend on A, provided that $\mu_n(B) \to \mu(B)$ uniformly with respect to $B \in \mathcal{S}$. The theorem is proved. □

We will need the following extension of the previous Theorem 1.

THEOREM 2. *Suppose* $\mu_{n_1,\ldots,n_d} \colon \mathcal{S} \to Y$, $n_1, \ldots, n_d = 1, 2, \ldots$ *are countably additive, and* $\lim_{n_1,\ldots,n_d\to\infty} \mu_{n_1,\ldots,n_d}(A) = \mu(A) \in Y$ *exists for each* $A \in \mathcal{S}$. *Further* *let* $f, f_k \in \overline{S(\mathcal{S})}$*,* $k = 1, 2, \dots$ *, let* $f_k \to f$ *pointwise and let* $|f_k| \le C < +\infty$ *for each* $k = 1, 2, \dots$ *. Then*

$$\lim_{k, n_1, \dots, n_d \to \infty} \int_A f_k \, \mathrm{d}\mu_{n_1, \dots, n_d} = \int_A f \, \mathrm{d}\mu$$

for each $A \in \mathcal{S}$*. This limit is uniform with respect to* $A \in \mathcal{S}$ *provided* $\lim\limits_{n_1, \dots, n_d \to \infty} \mu_{n_1, \dots, n_d}(A) = \mu(A) \in Y$ *uniformly with respect to* $A \in \mathcal{S}$*.*

P r o o f . If the assertion of the theorem does not hold, then there exist $A \in \mathcal{S}$, $a > 0$, and subsequences k_j, $n_{1,j}$, ..., $n_{d,j}$, $j = 1, 2, \dots$, such that

$$\left| \int_A f_{k_j} \, \mathrm{d}\mu_{n_{1,j}, \dots, n_{d,j}} - \int_A f \, \mathrm{d}\mu \right| > a \qquad \text{for each} \quad j = 1, 2, \dots .$$

Hence, if $f'_j = f_{k_j}$ and $\mu'_j = \mu_{n_{1,j}, \dots, n_{d,j}}$ for $j = 1, 2, \dots$, we have a contradiction with Theorem 1. We obtain the assertion concerning uniform limit similarly. □

Theorem 3. Basic convergence theorem. *Suppose* $\gamma_n \colon \times \mathcal{S}_i \to Y$*,* $n = 1, 2, \dots$ *, are separately countably additive and let* $\gamma_n(A_i) \to \gamma(A_i) \in Y$ *for each* $(A_i) \in \times \mathcal{S}_i$*. Let* f_i*,* $f_{i,k}$*,* $k = 1, 2, \dots$ *, be bounded* $\mathcal{S}_i$*-measurable functions such that* $f_{i,k} \to f_i$ *pointwise, for each* $i = 1, \dots, d$*. Suppose finally that* $|f_{i,k}| \le C < +\infty$ *for each* $i = 1, \dots, d$ *and each* $k = 1, 2, \dots$ *. Then*

$$\lim_{k_1, \dots, k_d, n \to \infty} \int_{(A_i)} (f_{i,k_i}) \, \mathrm{d}\gamma_n = \int_{(A_i)} (f_i) \, \mathrm{d}\gamma \qquad \textit{for each} \quad (A_i) \in \times \mathcal{S}_i .$$

P r o o f . We proceed by induction on the dimension d. For $d = 1$ the assertion follows from Theorem 1. Suppose the theorem is valid for $d - 1$ with $d \ge 2$. Let $A_2 \in \mathcal{S}_2$, ..., $A_d \in \mathcal{S}_d$ be fixed. Put $\mu_{k_1, \dots, k_d, n}(A_1) = \int_{(A_i)} (\chi_{A_1}, f_{2,k_2}, \dots, f_{d,k_d}) \, \mathrm{d}\gamma_n$ for $A_1 \in \mathcal{S}_1$, and for $k_2, \dots, k_d, n = 1, 2, \dots$.
Then, by the inductive assumption,

$$\lim_{k_2, \dots, k_d, n \to \infty} \mu_{k_2, \dots, k_d, n}(A_1) = \int_{(A_i)} (\chi_{A_1}, f_2, \dots, f_d) \, \mathrm{d}\gamma = \mu(A_1) \in Y$$

$$\text{for each} \quad A_1 \in \mathcal{S}_1 .$$

Now

$$\lim_{k_1, \dots, k_d, n \to \infty} \int_{(A_i)} (f_{i,k_i}) \, \mathrm{d}\gamma_n = \lim_{k_1, \dots, k_d, n \to \infty} \int_{A_1} (f_{1,k_1}) \, \mathrm{d}\mu_{k_2, \dots, k_d, n}$$

$$= \int_{A_1} f_1 \, \mathrm{d}\mu = \int_{(A_i)} (f_i) \, \mathrm{d}\gamma$$

by Theorem 2 and the Inner Fubini property. □

COROLLARY 1. *Let $\gamma_n = \gamma$ for each $n = 1, 2, \dots$ in Theorem 3. Then we have the Multilinear Lebesgue bounded convergence theorem.*

Let us note that this Corollary 1 suffices for the multilinear part of Theorem 1 in [5]. Note also the difference with respect to the proof of Theorem 3 in [3] what is essentially the same assertions.

COROLLARY 2. *Let $\gamma_n \colon \times \mathcal{S}_i \to Y$, $n = 1, 2, \dots$, be separately countably additive and let $\gamma_n(A_i) \to \gamma(A_i) \in Y$ for each $(A_i) \in \times \mathcal{S}_i$. Further let $(A_{i,k}) \in \times \mathcal{S}_i$, $k = 1, 2, \dots$, and let $A_{i,k} \to A_i$ for each $i = 1, \dots, d$. Then*

$$\lim_{n, k_1, \dots, k_d \to \infty} \gamma_n(A_{i,k_i}) = \gamma(A_i).$$

COROLLARY 3. THE JOINT CONTINUITY OF A d-POLYMEASURE. *Let $\gamma \colon \times \mathcal{S}_i \to Y$ be separately countably additive, let $(A_{i,k}) \in \times \mathcal{S}_i$, $k = 1, 2, \dots$, and let $A_{i,k} \to A_i$ for each $i = 1, \dots, d$. Then*

$$\lim_{k_1, \dots, k_d \to \infty} \gamma(A_{i,k_i}) = \gamma(A_i).$$

COROLLARY 4. *Let $\gamma \colon \times \mathcal{S}_i \to Y$ be separately countably additive and let $A_{i,n} \in \mathcal{S}_i$, $n = 1, 2, \dots$, be pairwise disjoint for each $i = 1, \dots, d$. Then*

$$\gamma\left(\bigcup_{n_i=1}^{\infty} A_{i,n_i} \right) = \lim_{N_1, \dots, N_d \to \infty} \sum_{n_1=1}^{N_1} \cdots \sum_{n_d=1}^{N_d} \gamma(A_{i,n_i}).$$

3. A Particular Fubini theorem

We give a Fubini theorem for n-tuples of bounded measurable functions with respect to the product, more precisely the indirect product, of our vector d-multimeasure with a family of scalar multimeasure, which in general depend on the coordinate variable.

As before, let $\gamma \colon \times \mathcal{S}_i \to Y$ be the given vector d-multimeasure. Suppose there are $i \in \{1, \dots, d\}$ for which there are measurable spaces $(T_{i,j}, \mathcal{S}_{i,j})$, $j = 1, \dots, n_i$, $\mathcal{S}_{i,j}$ being a σ-ring, and a mapping $\gamma_i(\cdot, \dots) \colon T_i \times \mathcal{S}_{i,j} \times \dots \times \mathcal{S}_{i,n_i} \to K$ such that:

a) $\gamma_i(t_i, \dots) \colon \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_{i,n_i} \to K$ is separately countably additive for each $t_i \in T_i$,

b) $\gamma_i(\cdot, A_{i,1}, \dots, A_{i,n_i}) \colon T_i \to K$ is $\mathcal{S}_i$-measurable for each $(A_{i,1}, \dots, A_{i,n_i}) \in \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_{i,n_i}$, and this family of functions is uniformly bounded on T_i.

First we consider all such i's, and for the remaining $i \in \{1, \dots, d\}$ for notational simplicity, we put $n_i = 1$, $T_{i,1} = \{1\}$, $\mathcal{S}_{i,1} = \{\{1\}, \emptyset\}$, $\gamma_i(t_i, \{1\}) = 1$ for each $t_i \in T_i$, and $\gamma_i(t_i, \emptyset) = 0$ for each $t_i \in T_i$. Hence the assumptions in each coordinate $i = 1, \dots, d$ are the same, i.e., a) and b) above.

For each $i = 1, \dots, d$ and $j = 1, \dots, n_i$ we denote by $(T_i \times T_{i,j}, \mathcal{S}_i \otimes \mathcal{S}_{i,j})$ the usual product of the given measurable spaces, i.e., $\mathcal{S}_i \otimes \mathcal{S}_{i,j}$ is the σ-ring over $\mathcal{S}_i \times \mathcal{S}_{i,j}$. For $t_i \in T_i$ the t_i-section of a set $E_{i,j} \in \mathcal{S}_i \otimes \mathcal{S}_{i,j}$ is given by the equality

$$E_{i,j}^{t_i} = \{t_{i,j} \in T_{i,j}\,;\ (t_i, t_{i,j}) \in E_{i,j}\} \in \mathcal{S}_{i,j}\,.$$

THEOREM 4. *Let $i \in \{1, \dots, d\}$ be fixed, let $f_{i,j} \colon T_i \times T_{i,j} \to K$, $j = 1, \dots, n_i$, be bounded $\mathcal{S}_i \otimes \mathcal{S}_{i,j}$-measurable functions, and let $f_{i,j,k} \colon T_i \times T_{i,j} \to K$, $k = 1, 2, \dots$, be $\mathcal{S}_i \otimes \mathcal{S}_{i,j}$-simple function such that $f_{i,j,k} \to f_{i,j}$ and $|f_{i,j,k}| \uparrow |f_{i,j}|$ pointwise for each $j = 1, \dots, n_i$. Then:*

1) *the functions $t_i \mapsto \gamma_i(t_i, E_{i,1}^{t_i}, \dots, E_{i,n_i}^{t_i})$, $t_i \in T_i$, $(E_{i,1}, \dots, E_{i,n_i}) \in \mathcal{S}_i \otimes \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_i \otimes \mathcal{S}_{i,n_i}$, are $\mathcal{S}_i$-measurable, and are uniformly bounded on T_i,*
2)

$$\lim_{k\to\infty} \int\limits_{(E_{i,1}^{t_i}, \dots, E_{i,n_i}^{t_i})} (f_{i,1,k}(t_i, \cdot), \dots, f_{i,n_i,k}(t_i, \cdot))\, \mathrm{d}\gamma_i(t_i, \dots) = \int\limits_{(E_{i,1}^{t_i}, \dots, E_{i,n_i}^{t_i})} (f_{i,1}(t_i, \cdot), \dots, f_{i,n_i}(t_i, \cdot))\, \mathrm{d}\gamma_i(t_i, \dots)$$

for each $(E_{i,1}, \dots, E_{i,n_i}) \in \mathcal{S}_i \otimes \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_i \otimes \mathcal{S}_{i,n_i}$ and each $t_i \in T_i$,

3) *the functions $t_i \mapsto \int_{(E_{i,1}^{t_i}, \dots, E_{i,n_i}^{t_i})} (f_{i,1}(t_i, \cdot), \dots, f_{i,n_i}(t_i, \cdot))\, \mathrm{d}\gamma_i(t_i, \dots)$, $t_i \in T_i$ and $(E_{i,1}, \dots, E_{i,n_i}) \in \mathcal{S}_i \otimes \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_i \otimes \mathcal{S}_{i,n_i}$, are $\mathcal{S}_i$-measurable, and are uniformly bounded on T_i.*

Proof.

1) If $E_{i,j} \in \varrho(\mathcal{S}_i \times \mathcal{S}_{i,j})$ — the ring over the rectangles $\mathcal{S}_i \times \mathcal{S}_{i,j}$, for $j = 1, \dots, n_i$, then 1) holds by assumption b) and the separate additivity of $\gamma_i(t_i, \dots)$. Denote by M_1 the class of all $E_{i,1} \in \mathcal{S}_i \otimes \mathcal{S}_{i,j}$ for which 1) holds provided $E_{i,j} \in \mathcal{S}_i \times \mathcal{S}_{i,j}$ for $j > 1$. Then M_1 is a monotone class over the ring $\varrho(\mathcal{S}_i \times \mathcal{S}_{i,1})$ owing to the separate countable additivity of $\gamma_i(t_i, \dots)$, and assumption b). Hence $M_1 = \mathcal{S}_i \otimes \mathcal{S}_{i,1}$. If $n_i > 1$, denote by M_2 all these sets $E_{i,2} \in \mathcal{S}_i \otimes \mathcal{S}_{i,2}$ for which 1) holds provided $E_{i,1} \in \mathcal{S}_i \otimes \mathcal{S}_{i,1}$ and $E_{i,j} \in \mathcal{S}_i \otimes \mathcal{S}_{i,j}$ for $j > 2$. Then $M_2 = \mathcal{S}_i \otimes \mathcal{S}_{i,2}$, similarly as in the case of M_1. Continuing in this way we obtain the validity of 1).

2) follows directly from the Multilinear Lebesgue Bounded convergence theorem; see Corollary 1 of Theorem 3.

3) Since a pointwise limit of a sequence of measurable functions is measurable, the asserted $\mathcal{S}_i$-measurability follows from 1) and 2). Assumption b) and the boundedness of $f_{i,j}$ for each $j = 1, \dots, n_i$ imply the uniform boundedness assertions. □

Using assertion 1) of Theorem 4, assumption a), and the Multilinear Lebesgue bounded convergence theorem (see Corollary 1 of Theorem 3), we immediately obtain the following theorem on the existence of the indirect product of the multimeasure γ with the family of multimeasures $\{\gamma_i(t_i, \dots)\}$.

THEOREM 5. *For* $((E_{i,1}, \dots, E_{i,n})) \in \times(\mathcal{S}_i \otimes \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_i \otimes \mathcal{S}_{i,n_i})$ *put*

$$\Big(\gamma \otimes \{\gamma_i(t_i, \dots)\}\Big)(E_{i,1}, \dots, E_{i,n_i}) = \int\limits_{(T_i)} (\gamma_i(t_i E^{t_i}_{i,1}, \dots, E^{t_i}_{i,n_i}))\, \mathrm{d}\gamma .$$

Then $\gamma \otimes \{\gamma_i(t_i, \dots)\}\colon \times (\mathcal{S}_i \otimes \mathcal{S}_{i,1} \times \dots \times \mathcal{S}_i \otimes \mathcal{S}_{i,n_i}) \to Y$ *is a separately countably additive* $\sum_{i=1}^{d} n_i$*-multimeasure which we call the indirect product of the multimeasure* γ *with the family of multimeasures* $\{\gamma_i(t_i, \dots)\}$.

We are now ready to give a short proof of the following theorem.

THEOREM 6. PARTICULAR FUBINI THEOREM. *Suppose* $f_{i,j}\colon T_i \times T_{i,j} \to K$, $i = 1, \dots, d$, $j = 1, \dots, n_i$, *is a bounded* $\mathcal{S}_i \otimes \mathcal{S}_{i,j}$*-measurable function. Then*

$$\int\limits_{((E_{i,1}, \dots, E_{i,n_i}))} ((f_{i,1}, \dots, f_{i,n_i}))\, \mathrm{d}\big(\gamma \otimes \{\gamma_i(t_i, \dots)\}\big)$$

$$= \int\limits_{(T_i)} \Bigg(\int\limits_{(E^{t_i}_{i,1}, \dots, E^{t_i}_{i,n_i})} (f_{i,1}(t_i, \cdot), \dots, f_{i,n_i}(t_i, \cdot))\, \mathrm{d}\gamma_i(t_i, \dots) \Bigg)\, \mathrm{d}\gamma$$

for each $E_{i,j} \in \mathcal{S}_i \otimes \mathcal{S}_{i,j}$, $i = 1, \dots, d$, *and* $j = 1, \dots, n_i$.

P r o o f. The integrands in the iterated integral are measurable and uniformly bounded, hence integrable with respect to γ, by assertion 3) of Theorem 4. For $i = 1, \dots, d$ and $j = 1, \dots, n_i$ take a sequence of $\mathcal{S}_i \otimes \mathcal{S}_{i,j}$-simple functions $f_{i,j,k}\colon T_i \times T_{i,j} \to K$, $k = 1, 2, \dots$, such that $f_{i,j,k} \to f_{i,j}$ and $|f_{i,j,k}| \uparrow |f_{i,j}|$ pointwise. By Theorem 5 the Fubini equality holds for $(f_{i,j,k})$, $i = 1, \dots, d$ and $j = 1, \dots, n_i$ for each $k = 1, 2, \dots$. Now by the Multilinear Lebesgue bounded convergence theorem and assertion 2) of Theorem 4 we obtain the equality of the theorem. □

REFERENCES

[1] DOBRAKOV, I.: *On integration in Banach spaces, VIII* (*Polymeasures*), Czechoslovak Math. J. **37(112)** (1987), 487-506.

[2] DOBRAKOV, I.: *On integration in Banach spaces, IX; X* (*Integration with respect to polymeasures*), Czechoslovak Math. J. **38(113)** (1988), 589–601; 713–725.

[3] DOBRAKOV, I.: *Representation of multilinear operators on* $\times C_0(T_i)$, Czechoslovak Math. J. **39(114)** (1989), 288–302.

[4] DOBRAKOV, I.: *On integration in Banach spaces, XI; XII; XIII* (*Integration with respect to polymeasures*), Czechoslovak Math. J. **40(115)** (1990), 8–24; 424–440; 566–582.

[5] DOBRAKOV, I.: *Feynman type integrals as multilinear integrals, I*, Rend. Circ. Mat. Palermo (2), Suppl. **28** (1992), 169–180.

[6] HALMOS, P. R.: *Measure Theory*, D. Van Nostrand, Toronto, 1950.

Received January 18, 1996
Revised September 26, 1996

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA
E-mail: dobrakov@savba.sk