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A REMARK ON CONJUGACY OF HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation properties of the second order half-linear differential equation

$$\big(r(t)\Phi(y')\big)' + c(t)\Phi(y) = 0\,, \qquad \Phi(s) = |s|^{p-2}s\,, \quad p > 1\,, \tag{$*$}$$

are investigated. Equation $(*)$ is viewed as a perturbation of the disconjugate equation

$$\big(r(t)\Phi(y')\big)' + \tilde{c}(t)\Phi(y) = 0$$

and an integral condition for the difference $c(t) - \tilde{c}(t)$ is given which guaranties that $(*)$ possesses a nontrivial solution having at least two different zeros in a given interval.

I. Introduction

Consider the half-linear second order differential equation

$$\big(r(t)\Phi(y')\big)' + c(t)\Phi(y) = 0\,, \qquad t \in I := (a,b) \subseteq \mathbb{R}, \tag{1.1}$$

where $\Phi(s) = |s|^{p-2}s$, r, c are continuous real-valued functions with $r(t) > 0$. The aim of this paper is to derive conditions under which (1.1) possesses a nontrivial solution having at least two different zeros in I. Conditions of this kind are usually referred as conjugacy criteria. If $p = 2$, then (1.1) reduces to the Sturm-Liouville second order differential equation

$$\big(r(t)y'\big)' + c(t)y = 0 \tag{1.2}$$

and conjugacy of this equation has been investigated in several papers, see [2] [4], [15], [16] and the references given therein. In particular, the following (linear) perturbation principle has been established in [2] and [3].

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PROPOSITION. *Suppose that $\tilde{c}(t)$ is a continuous real-valued function such that the equation*

$$\big(r(t)y'\big)' + \tilde{c}(t)y = 0$$

is disconjugate in $I = (a,b)$ and that the principal solutions of this equation at end points a, b coincide. If

$$\liminf_{t_1 \downarrow a,\, t_2 \uparrow b} \int_{t_1}^{t_2} \big(c(t) - \tilde{c}(t)\big)u_0^2(t)\, dt \ge 0\,, \qquad c(t) - \tilde{c}(t) \not\equiv 0\,, \quad t \in I\,,$$

where u_0 is the principal solution of (1.2) *at a and b, then* (1.2) *is conjugate in I, i.e., there exists a nontrivial solution of this equation having at least two zeros in I.*

In this paper we derive a similar perturbation principle for half-linear equations (1.1). The main difficulty in extending Proposition to (1.1) consists in the fact that the solution space of half-linear equation (1.1) is only "half-linear", i.e., it is homogeneous but not generally additive (what is also the characterization of half-linear equations). A more detailed explanation of the difference between linear and half-linear equations is given in the next section.

Conjugacy criteria for half-linear equations have been investigated in [5], [6], [17]. In the papers [6], [17], equation (1.1) is viewed as a perturbation of the one-term equation

$$\big(r(t)\Phi(y')\big)' = 0\,. \tag{1.3}$$

In this case one can take the advantage of the fact that the solution space of (1.3) has the linear structure, so a suitable modification of the linear method applies also to (1.1). In [5] the case $r(t) \equiv 1$ and $I = (0,\infty)$ is investigated. There the equation

$$\big(\Phi(y')\big)' + c(t)\Phi(y) = 0$$

is viewed as a perturbation of the generalized Euler equation with the critical constant

$$\big(\Phi(y')\big)' + \frac{\gamma_0}{t^p}\Phi(y) = 0\,, \qquad \gamma_0 = \left(\frac{p-1}{p}\right)^p. \tag{1.4}$$

The solution space of (1.4) is no longer linear, however, a certain special solution of (1.4), namely $y = t^{\frac{p-1}{p}}$, can be computed explicitly and this again enables the application of the modified linear method.

In this paper we show that this idea, when suitable modified, extends also to the general situation when (1.1) is viewed as a perturbation of a disconjugate equation of the same form. This extension is possible due to the new construction of the principal solution of (1.1) which is presented in the next section.

The paper is organized as follows. In the next section we recall basic properties of solutions of (1.1) including the recently established concept of principal solution of this equation. The main result of the paper — a conjugacy criterion for (1.1) — is given in Section 3. The last section is devoted to remarks and comments concerning the results of the paper and their possible extension.

II. Auxiliary results

Suppose that y is a solution of (1.1) such that $y \neq 0$ in a subinterval $I_0 \subseteq I$. Then the function

$$w = \frac{r\Phi(y')}{\Phi(y)} \tag{2.1}$$

verifies in I_0 the generalized Riccati equation

$$w' + (p-1)r^{1-q}(t)|w|^q + c(t) = 0\,, \qquad \frac{1}{p} + \frac{1}{q} = 1\,. \tag{2.2}$$

Since solutions of this equation behave essentially in the same way as those of the "linear" Riccati equation

$$w' + r^{-1}(t)w^2 + c(t) = 0\,,$$

Sturmian type theorems extends to (1.1). In particular, any solution of (1.1) is either oscillatory or nonoscillatory. Indeed, if y is a nontrivial solution of this equation with consecutive zeros $t_1 < t_2$, then the corresponding solution w of (2.2) satisfies $w(t_1+) = \infty$, $w(t_2-) = -\infty$. Consequently, no solution of (2.2) may exist on the whole interval $[t_1, t_2]$ (since through any point in the plane passes exactly one solution of (2.2)) and hence any solution of (1.1) independent of y has a zero in (t_1, t_2). The relationship between (1.1) and (2.2), usually referred as Riccati technique, has a broad application in oscillation theory of (1.1), see [7], [8], [11], [13] and the references given therein.

Another useful tool when investigating oscillatory properties of half-linear equations is the variational principle consisting in the relation between disconjugacy of (1.1) and positivity of the functional

$$\mathcal{F}(y) = \int_a^b \left[r(t)|y'|^p - c(t)|y|^p\right]\,\mathrm{d}t\,. \tag{2.3}$$

More precisely, (1.1) is disconjugate in $[a,b]$ if and only if $\mathcal{F}(y) > 0$ for every nontrivial function piecewise of the class C^1 satisfying $y(a) = 0 = y(b)$. This statement follows from the half-linear version of Picone's identity (see [12])

$$\mathcal{F}(y) = w(t)|y|^p\Big|_a^b + p\int_a^b \left[\frac{r(t)|y'|^p}{p} - \Phi(y)w(t)y' + r^{1-q}(t)\frac{|\Phi(y)w(t)|^q}{q}\right]\mathrm{d}t\,, \tag{2.4}$$

where w is any solution of (2.2), since the integrand in (2.4) is nonnegative by the Young inequality $\frac{|\alpha|^p}{p} + \frac{|\beta|^q}{q} \geq \alpha\beta$ setting

$$\alpha = r^{\frac{1}{p}}|y'|\,, \qquad \beta = \Phi(y)wr^{\frac{1}{p}}\,.$$

For an alternative approach to the investigation of the relationship between positivity of the functional $\mathcal{F}$ and disconjugacy of (1.1) see [14].

The crucial role in our investigation is played by the principal solution of nonoscillatory solution of (1.1). Basic properties of such solution have been established in [10] using the generalized Prüfer transformation. Principal solution y_b of (1.1) at the end point b of the interval $I = (a, b)$ is a solution for which

$$\frac{y_b'(t)}{y_b(t)} < \frac{y'(t)}{y(t)} \qquad \text{and} \qquad \frac{y(t)}{y_b(t)} \quad \text{is increasing near } b$$

for any solution y of (1.1) which is linearly independent of y_b. Consequently, similarly as in the linear case, the principal solution at b generates the minimal solution of (2.2) near b given by

$$w_b = \frac{r(t)\Phi(y_b')}{\Phi(y_b)}\,. \tag{2.5}$$

Here we present a slightly different approach to the construction of this solution based on the minimality of the solution given by (2.5).

Suppose that (1.1) is nonoscillatory at b, i.e., there exists $c \in (a, b)$ such that the solution y_c of (1.1) given by $y_c(c) = 0$, $r(c)\Phi(y')(c) = 1$ is positive in (c, b). Denote $w_c(t) = \frac{r(t)\Phi(y_c')}{\Phi(y_c)}$ and for $d \in (c, b)$ let w_d be the solution of (2.2) determined by the solution y_d of (1.1) satisfying the initial condition $y(d) = 0$, $r(d)\Phi(y')(d) = -1$. Then $w_d(d-) = -\infty$ and $w_d(t) < w_c(t)$ for $t \in (c, d)$. Moreover, if $c < d_1 < d_2 < b$ then $w_{d_1}(t) < w_{d_2}(t) < w_c(t)$ for $t \in (c, d_1)$. This implies that for $t \in (c, b)$ there exists the limit $w_b(t) := \lim\limits_{c \to b-} w_c(t)$ and monotonicity of this convergence (with respect to the "subscript" variable) implies that this convergence is uniform on every compact subinterval of (c, b). Consequently, the limit function w_b solves (2.2) too and any solution w of this equation which is extensible up to b satisfies the inequality $w(t) > w_b(t)$ near b. Indeed, if a solution $\bar{w}$ would satisfy the inequality $w(t) < w_b(t)$ on some interval $(\bar{c}, b)$, then for $\bar{t} \in (\bar{c}, b)$ and d sufficiently close to b we have $w(t) < w_d(\bar{t}) < w_b(\bar{t})$. But this would contradict to the fact that $w_d(d-) \quad -\infty$ and that graphs of solutions of (2.2) cannot intersect.

Now, having defined the minimal solution w_b of (2.2), we define the principal solution of (1.1) at b as the solution of the first order equation

$$y' = |r(t)w_b(t)|^{\frac{1}{p-1}} \operatorname{sgn} w_b(t)y\,. \qquad 2.6$$

Concerning the principal solution at the left-end point, we proceed in the same way. If (1.1) is nonoscillatory at a, then for t sufficiently close to a there exists the limit

$$w_a(t) := \lim_{d \to a+} w_d(t),$$

where

$$w_d(t) = \frac{r(t)\Phi(y_d')}{\Phi(y_d)},$$

y_d being given by the initial condition $y_d(d) = 0$, $r(d)\Phi(y')(d) = 1$. If w is any solution of (2.2) different from w_a which is extensible to the left up to $t = a$, then this solution satisfies the inequality $w_a(t) > w(t)$ near a. The principal solution of (2.2) at $t = a$ is now defined by (2.6) with w_b replaced by w_a.

III. Conjugacy criterion

In this section we formulate our main result — a conjugacy criterion for (1.1). We will use the following terminology.

DEFINITION. Equation (1.1) is said to be 1-*special* in an interval $I = (a,b)$ if there exists exactly one (up to the multiplication by a real constant) solution y_0 of this equation which has no zero in I.

Note that in the linear case $p = 2$ the classification of nonoscillatory equations as being n-special or n-general, $n \in \mathbb{N}$, was introduced by B o r ů v k a [1] in connection with transformation theory of second order linear differential equations. In the half-linear case we have no analogue of the linear transformation theory, however, as we will show in this section, when investigating conjugacy of (1.1) in a given interval, 1-special equations play here the same role as for linear equations. Observe that if (1.1) is 1-special in $I = (a,b)$ then the principal solutions y_a, y_b of this equation at end points $t = a$ and $t = b$ coincide. This follows from the fact that these solutions have no zero point in I. Indeed, if e.g. y_b would have a zero at some $\bar{t} \in I$, i.e., the solution $w_b = \frac{r\Phi(y_b')}{\Phi(y_b)}$ satisfies $w_b(\bar{t}+) = \infty$, then for any $\tilde{t} \in (a, \bar{t})$ the solution $\tilde{w}$ of (2.2) satisfying $\tilde{w}(\tilde{t}+) = \infty$ cannot be extensible up to b. Indeed, if this solution would be extensible, then according to the unique solvability of (2.2), this solution satisfies the inequality $\tilde{w}(t) < w_b(t)$ for $t \in (\bar{t}, b)$ which contradicts to minimality of w_b near b. Consequently, $\tilde{w}$ has to blow down to $-\infty$ at some $\hat{t} \in (\tilde{t}, b)$ and hence the solution $\tilde{y}$ of (1.1) which generates $\tilde{w}$ by the formula $\tilde{w} = \frac{r\Phi(\tilde{y}')}{\Phi(\tilde{y})}$ has zeros at $\tilde{t}$ and $\hat{t}$ which contradicts to our assumption that (1.1) is disconjugate in I.

Concerning the interval $[t_1, t_2]$, we have (again by identity (2.4))

$$\begin{aligned}
&\int_{t_1}^{t_2} \left[r(t)|y'|^p - \tilde{c}(t)|y|^p\right] \,\mathrm{d}t \\
&= w_h|h|^p\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[r(t)|y'|^p - pw_h(t)y'\Phi(y) + (p-1)r^{1-q}(t)|w_h(t)|^q|y|^p\right] \,\mathrm{d}t \\
&= w_h|h|^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \Big\{ r(t)|h' + \delta(\Delta h)'|^p - pr(t)\frac{\Phi(h')}{\Phi(h)} y' h^{p-1}(1+\delta\Delta)^{p-1} + \\
&\qquad\qquad + (p-1)r^{1-q}(t)\left|\frac{r(t)\Phi(h')}{\Phi(h)}\right|^q h^p(1+\delta\Delta)^p \Big\} \,\mathrm{d}t \\
&= w_h|h|^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} r(t)\big\{|h'|^p + p\delta(\Delta h)'\Phi(h') + o(\delta) \\
&\qquad\qquad - p\big(h' + \delta(\Delta h)'\big)\Phi(h')(1 + (p-1)\delta\Delta + o(\delta)) \\
&\qquad\qquad + (p-1)|h'|^p\big(1 + p\delta\Delta + o(\delta)\big)\big\} \,\mathrm{d}t \\
&= w_h|h|^p\Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} r(t)\big\{|h'|^p + p\delta(\Delta h)'\Phi(h') + p|h'|^p - p\delta\Phi(h')(\Delta h)' \\
&\qquad\qquad - p(p-1)\delta\Delta|h'|^p + (p-1)|h'|^p + (p-1)p\delta\Delta|h'|^p + o(\delta)\big\} \,\mathrm{d}t \\
&= w_h|h|^p\Big|_{t_1}^{t_2} + o(\delta)\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{t_0}^{t_3} \left[r(t)|y'|^p - \tilde{c}(t)|y|^p\right] \,\mathrm{d}t \\
&= w_f|f|^p\Big|_{t_0}^{t_1} + w_h|h|^p\Big|_{t_1}^{t_2} + w_g|g|^p\Big|_{t_2}^{t_3} + o(\delta) \\
&= |h(t_1)|^p\big(w_f(t_1) - w_h(t_1)\big) + |h(t_2)|^p\big(w_h(t_2) - w_g(t_2)\big) + o(\delta) \\
&\qquad\qquad \text{as} \quad \delta \to 0+\,.
\end{aligned}$$

Further, observe that the function $\frac{f}{h}$ is increasing in (t_0, t_1) since $\frac{f}{h}(t_0) = 0$, $\frac{f}{h}(t_1) = 1$ and $\left(\frac{f}{h}\right)' = \frac{f'h - fh'}{h^2} \neq 0$ in (t_0, t_1). Indeed, if $f'h - fh' = 0$ at some point $\tilde{t} \in (t_0, t_1)$, i.e. $\frac{f'}{f}(\tilde{t}) = \frac{h'}{h}(\tilde{t})$ then $w_f(\tilde{t}) = w_h(\tilde{t})$ which contradicts the unique solvability of (3.3). By the second mean value theorem of integral calculus there exists $\xi_1 \in (t_0, t_1)$ such that

$$\int_{t_0}^{t_1} \big(c(t) - \tilde{c}(t)\big)|f|^p \,\mathrm{d}t = \int_{t_0}^{t_1} \big(c(t) - \tilde{c}(t)\big)|h|^p \frac{|f|^p}{|h|^p} \,\mathrm{d}t$$

$$= \int_{\xi_1}^{t_1} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t\,.$$

By the same argument $\frac{g}{h}$ is decreasing in (t_2, t_3) and

$$\int_{t_2}^{t_3} \big(c(t) - \tilde{c}(t)\big)|g|^p \,\mathrm{d}t = \int_{t_2}^{\xi_2} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t$$

for some $\xi_2 \in (t_2, t_3)$.

Concerning the interval (t_1, t_2) we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \big(c(t) - \tilde{c}(t)\big)|y|^p \,\mathrm{d}t \\
&= \int_{t_1}^{\bar{t}-\varrho} \big(c(t) - \tilde{c}(t)\big)|h|^p + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \big(c(t) - \tilde{c}(t)\big)|h|^p (1 + \delta\Delta)^p \,\mathrm{d}t \\
&\qquad\qquad + \int_{\bar{t}+\varrho}^{t_2} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t + \delta \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \big(c(t) - \tilde{c}(t)\big)|h|^p \Delta(t) \,\mathrm{d}t + o(\delta) \\
&\geq \int_{t_1}^{t_2} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t + \delta K + o(\delta)\,,
\end{aligned}$$

where $K = d \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \Delta(t) \,\mathrm{d}t > 0$. Therefore

$$\int_{t_0}^{t_3} \big(c(t) - \tilde{c}(t)\big)|y|^p \,\mathrm{d}t \geq \int_{\xi_1}^{\xi_2} \big(c(t) - \tilde{c}(t)\big)|h|^p \,\mathrm{d}t + K\delta + o(\delta)\,.$$

Summarizing our computations, we have

$$\begin{aligned}\mathcal{F}(y;t_0,t_3) \le |h(t_1)|^p\big(w_f(t_1)-w_h(t_1)\big) + |h(t_2)|^p\big(w_h(t_2)-w_g(t_2)\big) \\ -\int_{\xi_1}^{\xi_2}\big(c(t)-\tilde{c}(t)\big)|h|^p\,\mathrm{d}t - \big(K\delta + o(\delta)\big)\end{aligned}$$

with a positive constant K.

Now, let $\delta > 0$ (sufficiently small) be such that $K\delta + o(\delta) =: \varepsilon > 0$. According to (3.2) the points t_1, t_2 can be chosen in such a way that

$$\int_{s_1}^{s_2}\big(c(t)-\tilde{c}(t)\big)|h|^p\,\mathrm{d}t > -\frac{\varepsilon}{4}$$

whenever $s_1 \in (a,t_1)$, $s_2 \in (t_2,b)$. Further, since w_h is generated by the solution h of (3.1) which is principal both at $t=a$ and $t=b$, according to the "Riccati equation" construction of principal solution mentioned in the previous section, we have (for t_1, t_2 fixed for a moment)

$$\lim_{t_0\to a+}\big[w_f(t_1)-w_h(t_1)\big] = 0\,, \qquad \lim_{t_3\to b-}\big[w_g(t_2)-w_h(t_2)\big] = 0\,.$$

Hence

$$|h(t_1)|^p\big[w_f(t_1)-w_h(t_1)\big] < \frac{\varepsilon}{4}\,, \qquad |h(t_2)|^p\big[w_h(t_2)-w_g(t_2)\big] < \frac{\varepsilon}{4}$$

if $t_0 < t_1$, $t_2 < t_3$ are sufficiently close to a and b, respectively.

Consequently, for the above specified choice of $t_0 < t_1 < t_2 < t_3$ we have

$$\begin{aligned}\mathcal{F}(y;t_0,t_3) &= \int_{t_0}^{t_3}\big[r(t)|y'|^p - \tilde{c}(t)|y|^p\big]\,\mathrm{d}t - \int_{t_0}^{t_3}\big(c(t)-\tilde{c}(t)\big)|y|^p\,\mathrm{d}t \\ &\le |h(t_1)|^p\big[w_f(t_1)-w_h(t_1)\big] + |h(t_2)|^p\big[w_h(t_2)-w_g(t_2)\big] \\ &\qquad -\int_{\xi_1}^{\xi_2}\big(c(t)-\tilde{c}(t)\big)|h|^p\,\mathrm{d}t - \big(K\delta+o(\delta)\big) \\ &< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}-\varepsilon < 0\,.\end{aligned}$$

The proof is now complete. □

IV. Remarks

(i) If the function r in (1.1) satisfies

$$\int_a r^{1-q}(t)\ \mathrm{d}t = \infty = \int^b r^{1-q}(t)\ \mathrm{d}t \tag{4.1}$$

then one can take $\tilde{c} \equiv 0$ in Theorem, i.e., to consider (1.1) as a perturbation of the one term equation

$$\big(r(t)\Phi(y')\big)' = 0\,. \tag{4.2}$$

Indeed, if (4.1) holds, then $y \equiv 1$ is the only (up to multiplication) of (4.2) without zero point in (a,b). In [6] we have shown that under (4.1) equation (1.1) is conjugate in (a,b) provided there exist $t_0 \in (a,b)$, $a < T_1 < t_0 < T_2 < b$ and $\alpha_1, \alpha_2 \in \big(-\frac{1}{p}, p-2\big]$ such that $c(t) \not\equiv 0$ both in (a,t_0), (t_0,b) and

$$\begin{aligned} &\int_t^{t_0}\left[\int_s^{t_0} r^{1-q}(\tau)\ \mathrm{d}\tau\right]^{\alpha_1} r^{1-q}(s)\left(\int_s^{t_0} c(\tau)\ \mathrm{d}\tau\right)\mathrm{d}s \ge 0\,, \qquad t \in (a,T_1)\,,\\ &\int_{t_0}^{t}\left[\int_{t_0}^{s} r^{1-q}(\tau)\ \mathrm{d}\tau\right]^{\alpha_2} r^{1-q}(s)\left(\int_{t_0}^{s} c(\tau)\ \mathrm{d}\tau\right)\mathrm{d}s \ge 0\,, \qquad t \in (T_2,b)\,. \end{aligned} \tag{4.3}$$

It seems that this criterion and that given in Theorem are not generally comparable in the sense that no one is a consequence of the other one. However, if we replace (3.2) in Theorem by the stronger requirement (observe that $\tilde{c} \equiv 0$ implies $h \equiv 1$ in our situation)

$$\liminf_{s_1\downarrow a,\, s_2\uparrow b} \int_{s_1}^{s_2} c(s)\ \mathrm{d}s > 0\,, \tag{4.4}$$

then this criterion is already a consequence of (4.3). In fact, (4.4) implies that there exists $t_0 \in (a,b)$ such that

$$\liminf_{s_1\downarrow a} \int_{s_1}^{t_0} c(s)\ \mathrm{d}s > 0\,, \qquad \liminf_{s_2\uparrow b} \int_{t_0}^{s_1} c(s)\ \mathrm{d}s > 0\,,$$

hence $\int_t^{t_0} c(s)\ \mathrm{d}s$, $\int_{t_0}^{t} c(s)\ \mathrm{d}s$ are positive for t sufficiently close to a and b, respectively, which means that (4.3) holds for any $\alpha_1, \alpha_2 \in \big(-\frac{1}{p}, p-2\big]$.

(ii) In [2] we have found conditions which guarantee that the linear second order equation (1.2) possesses a nontrivial solution with at least $(n+1)$ zeros in (a,b), $n \geq 1$. This statement is based on the relationship between positivity of the functional

$$\mathcal{J}(y) = \int_a^b \left[r(t)(y')^2 - c(t)y^2\right] \,\mathrm{d}t \tag{4.5}$$

and oscillation properties of (1.2) given by the following variational lemma.

LEMMA. *Suppose that there exist linearly independent functions $y_1, \dots, y_n$ which are piecewise of the class $C^1(a,b)$, $y_k(a) = 0 = y_k(b)$, y_k has exactly $(k-1)$ zeros in (a,b), and $\mathcal{J}(y_k) < 0$, $k = 1, \dots, n$. Then there exists a nontrivial solution of* (1.2) *having at least $(n+1)$ zeros in (a,b).*

We conjecture that a similar statement holds also for half-linear equations and corresponding functionals and we hope to use this statement in order to investigate "$(n+1)$ conjugacy" of half-linear equations in a subsequent paper.

(iii) If we suppose that $c - \tilde{c} \geq 0$ near a and b, a closer examination of the proof of Theorem reveals that $\liminf$ in (3.2) may be replaced by $\limsup$.

(iv) Consider the Schrödinger partial differential equation

$$\Delta u + Q(x)u\,, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n \tag{4.6}$$

where $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and $Q\colon \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. Oscillation properties, in particular, the sufficient conditions for the existence of a *nodal domain* of a solution of (4.6), have been investigated in [18]. Recall that a bounded set $\Omega \subset \mathbb{R}^n$ is said to be the nodal domain of a nontrivial solution u of (4.6) if $u(x) = 0$ for $x \in \delta\Omega$. It seems that some of the results of [18] extend to the p-Laplace equation

$$\operatorname{div}\bigl(R(x)|\nabla u|^{p-2}\nabla u\bigr) + Q(x)\Phi(u) = 0\,,$$

where $R\colon \mathbb{R}^n \to \mathbb{R}$ is a positive differentiable function. This idea is the subject of the present investigation.

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Now we are in a position to formulate the main result of the paper. Here the equation (1.1) is viewed as a perturbation of a certain 1-special equation of the same form.

THEOREM. *Suppose that $\tilde{c}$ is a continuous function such that the equation*

$$\big(r(t)\Phi(y')\big)' + \tilde{c}(t)\Phi(y) = 0 \tag{3.1}$$

is 1*-special in $I = (a,b)$ and let h be its only nonzero solution in I. If*

$$\liminf_{s_1 \downarrow a,\, s_2 \uparrow b} \int_{s_1}^{s_2} \big(c(t) - \tilde{c}(t)\big)|h(t)|^p \,\mathrm{d}t \geq 0\,, \qquad c(t) \not\equiv \tilde{c}(t) \quad \text{in } (a,b)\,, \tag{3.2}$$

then (1.1) *is conjugate in I, i.e., there exists a nontrivial solution of this equation having at least two zeros in I.*

P r o o f . Our proof is based on the relation between positivity of the functional $\mathcal{F}$ given by (2.3) and disconjugacy of (1.1) mentioned in the previous section. We construct a nontrivial function piecewise of the class C^1, with $\operatorname{supp} y \subset I$, such that $\mathcal{F}(y) < 0$.

Continuity of the functions c, $\tilde{c}$ and (3.2) imply the existence of $\bar{t} \in I$ and $d, \varrho > 0$ such that $\big(c(t) - \tilde{c}(t)\big)|h(t)|^p > d$ for $(\bar{t} - \varrho, \bar{t} + \varrho)$ and let Δ be any positive differentiable function with the compact support in $(\bar{t}-\varrho, \bar{t}+\varrho)$. Further, let $a < t_1 < \bar{t} - \varrho < \bar{t} + \varrho < t_2 < t_3 < b$ and let f, g be solutions of (3.1) satisfying the boundary conditions

$$f(t_0) = 0\,, \quad f(t_1) = h(t_1)\,, \qquad g(t_2) = h(t_2)\,, \quad g(t_3) = 0\,.$$

Note that such solutions exist for any $t_0, t_1, t_2, t_3 \in I$ due to disconjugacy of (3.1) and the fact that the solution space of this equation is homogeneous. Define the function y as follows

$$y(t) = \begin{cases} 0 & t \in (a, t_0]\,, \\ f(t) & t \in [t_0, t_1]\,, \\ h(t) & t \in [t_1, t_2] \setminus [\bar{t} - \varrho, \bar{t} + \varrho]\,, \\ h(t)\big(1 + \delta\Delta(t)\big) & t \in [\bar{t} - \varrho, \bar{t} + \varrho]\,, \\ g(t) & t \in [t_2, t_3]\,, \\ 0 & t \in [t_3, b)\,, \end{cases}$$

where δ is a real parameter. Then we have

$$\begin{aligned}
\mathcal{F}(y;t_0,t_3) &= \int_{t_0}^{t_3} \left[r(t)|y'|^p - c(t)|y|^p\right] \,\mathrm{d}t \\
&= \int_{t_0}^{t_3} \left[r(t)|y'|^p - \tilde{c}(t)|y|^p\right] \,\mathrm{d}t - \int_{t_0}^{t_3} \left[c(t) - \tilde{c}(t)\right]|y|^p \,\mathrm{d}t \\
&= \int_{t_0}^{t_1} \left[r(t)|f'|^p - \tilde{c}(t)|f|^p\right] \,\mathrm{d}t - \int_{t_0}^{t_1} \left[c(t) - \tilde{c}(t)\right]|f|^p \,\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \left[r(t)|y'|^p - \tilde{c}(t)|y|^p\right] \,\mathrm{d}t - \int_{t_1}^{t_2} \left[c(t) - \tilde{c}(t)\right]|y|^p \,\mathrm{d}t \\
&\quad + \int_{t_2}^{t_3} \left[r(t)|g'|^p - \tilde{c}(t)|g|^p\right] \,\mathrm{d}t - \int_{t_2}^{t_3} \left[c(t) - \tilde{c}(t)\right]|g|^p \,\mathrm{d}t .
\end{aligned}$$

Denote by w_f, w_g, w_h the solutions of the Riccati equation associated with (3.1)

$$w' + \tilde{c}(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{3.3}$$

generated by f, g and h, respectively, i.e.,

$$w_f = \frac{r\Phi(f')}{\Phi(f)}, \qquad w_g = \frac{r\Phi(g')}{\Phi(g)}, \qquad w_h = \frac{r\Phi(h')}{\Phi(h)} .$$

Then using Picone's identity (2.4)

$$\begin{aligned}
&\int_{t_0}^{t_1} \left[r(t)|f'|^p - \tilde{c}(t)|f|^p\right] \,\mathrm{d}t \\
&= w_f|f|^p\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[r(t)|f'|^p - p w_f(t) f'\Phi(f) + (p-1)r^{1-q}(t)|w_f(t)|^q|f|^p\right] \,\mathrm{d}t \\
&= w_f|f|^p\Big|_{t_0}^{t_1} ,
\end{aligned}$$

similarly,

$$\int_{t_2}^{t_3} \left[r(t)|g'|^p - \tilde{c}(t)|g|^p\right] \,\mathrm{d}t = w_g|g|^p\Big|_{t_2}^{t_3} .$$