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# ALMOST MIDCONVEX AND ALMOST CONVEX SET-VALUED FUNCTIONS

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ABSTRACT. Let $X$ be a topological vector space and $D$ be a convex and open subset of $X$. Let $Y$ be a separable normed space. Denote by $\Im_1$ and $\Im_2$ proper linearly invariant $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying certain conditions (a) and (b). Given a set-valued function $F\colon D \to cc(Y)$ such that

$$\frac{F(x)+F(y)}{2} \subset F\left(\frac{x+y}{2}\right) \qquad \Im_2\text{-a.e. in } D \times D,$$

we claim that there exists a unique midconvex set-valued function $G\colon D \to cc(Y)$ such that

$$F(x) = G(x) \qquad \Im_1\text{-a.e. in } D.$$

A similar result is obtained for almost convex set-valued functions.

This paper is devoted to $\Im$-almost midconvex and $\Im$-almost convex set-valued functions. We will answer the question if every almost midconvex (convex) set-valued function is equivalent to a midconvex (convex) one. This problem was considered for single-valued functions by M. Kuczma and R. Ger (cf. [4], [3] and [2]). Earlier such a question was raised for additive functions by P. Erdös in 1960. A positive answer was given by N. G. de Bruijn and W. B. Jurkat; some generalization was also obtained by J. L. Denny and R. Ger (cf. [4], [2] and the references given there).

We shall use definitions and notations introduced by M. Kuczma in [4]. Nevertheless we will recall some definition and lemmas.

Let $X$ be an arbitrary set and let $2^X$ denote the set of all subsets of $X$. A non-empty family $\Im \subset 2^X$ is said to be *$\sigma$-ideal*, if it satisfies conditions

(i) $A \in \Im \wedge B \subset A \implies B \in \Im$,

(ii) $A_n \in \Im,\ n \in \mathbb{N}, \implies \bigcup_{n=1}^{\infty} A_n \in \Im$.

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If a $\sigma$-ideal $\Im$ additionally satisfies the condition

(iii) $X \notin \Im$,

then it is called *proper*.

Let $\Im$ be a proper $\sigma$-ideal. We say that a condition depending on $x \in X$ is satisfied $\Im$-*almost everywhere in* $X$ ($\Im$-a.e. in $X$), if there exists a set $A \in \Im$ such that the given condition is satisfied for all $x \in X \setminus A$.

We say that a $\sigma$-ideal defined on a (not necessarily commutative) group $(X, +)$ is *linearly invariant*, if it satisfies the condition

(iv) $\forall x \in X \quad \forall A \in \Im \qquad x - A \in \Im$.

In the sequel the phrase "a proper linearly invariant $\sigma$-ideal" will be abbreviated to "a p.l.i. $\sigma$-ideal".

**Lemma A.** ([4]) *If* $(X, +)$ *is a group and* $\Im \subset 2^X$ *is a linearly invariant* $\sigma$*-ideal, then for all* $x \in X$ *and* $A \in \Im$ *we have:*

$$-A \in \Im, \qquad x + A \in \Im, \qquad A + x \in \Im.$$

Let us define, for an arbitrary set $A \subset X \times X$ and $x \in X$, the set

$$A[x] := \{y \in X : \ (x, y) \in A\}.$$

Let $\Im_1$ be a $\sigma$-ideal in $X$ and let $\Im_2$ be a $\sigma$-ideal in $X \times X$. We say that $\sigma$-ideals $\Im_1$ and $\Im_2$ are *conjugate*, if for any set $A \in \Im_2$

$$A[x] \in \Im_1 \qquad \Im_1\text{-a.e. in } X.$$

Now let $X$ be a topological vector space. In the sequel we will be considering $\sigma$-ideals $\Im_1$ and $\Im_2$ in $X$ and $X \times X$, respectively, satisfying conditions

(a) $A \in \Im_1 \wedge a \in \mathbb{R} \implies aA \in \Im_1$,

(b) $A \in \Im_2 \implies T(A) \in \Im_2$,

where $T \colon X \times X \to X \times X$ is given by the formula $T(x, y) := \frac{1}{2}(x + y, x - y)$, $x, y \in X$.

**Lemma B.** ([4]) *Let* $\Im$ *be a p.l.i.* $\sigma$*-ideal in* $X$ *satisfying condition* (a). *If* $D \subset X$ *is a non-empty open set, then* $D \notin \Im$.

**Lemma C.** ([4]) *Let* $X$ *be a topological vector space,* $D \subset X$ *be a convex and open set,* $\Im$ *be a p.l.i.* $\sigma$*-ideal in* $X$ *satisfying condition* (a) *and let* $f \colon D \to [-\infty, \infty)$ *and* $g \colon D \to [-\infty, \infty)$ *be midconvex functions equal* $\Im$*-a.e. in* $D$. *Then* $f = g$ *in* $D$.

Now let us recall a result of M. K u c z m a.

**THEOREM A.** *Let $D$ be a convex and open subset of a topological vector space $X$. Let us denote by $\Im_1$ and $\Im_2$ p.l.i. conjugate $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying conditions* (a) *and* (b).

*If $f\colon D \to [-\infty, \infty)$ is $\Im_2$-almost midconvex, then there exists a unique midconvex function $g\colon D \to [-\infty, \infty)$ such that $f(x) = g(x)$ $\Im_1$-a.e. in $D$.*

In fact Kuczma's theorem is formulated for $X = \mathbb{R}^n$ and for real-valued functions, but it can be proved for any topological vector space $X$ and functions with values in $[-\infty, \infty)$, in the same way (see also R. Ger [2; Theorem 3] and its proof for $\varepsilon = 0$).

Let $Y$ be a topological vector space. We will denote by $\mathrm{n}(Y)$ the family of all non-empty subsets of the set $Y$, by $\mathrm{ccl}(Y)$ the family of all convex and closed elements of $\mathrm{n}(Y)$ and by $\mathrm{cc}(Y)$ the family of all convex and compact elements of $\mathrm{n}(Y)$.

A set-valued function $F\colon D \to \mathrm{n}(Y)$ is called *midconvex* if

$$\frac{F(x) + F(y)}{2} \subset F\left(\frac{x+y}{2}\right), \qquad x, y \in D.$$

If $\Im_2$ is a $\sigma$-ideal in $X \times X$ and the above condition holds $\Im_2$-a.e. in $D \times D$, then $F$ is called *$\Im_2$-almost midconvex.*

Using the quoted above theorem we can prove the following result.

**THEOREM 1.** *Let $X$ be a topological vector space and $D$ be a convex and open subset of $X$. Denote by $\Im_1$ and $\Im_2$ two conjugate p.l.i. $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying conditions* (a) *and* (b).

*If a set-valued function $F\colon D \to \mathrm{ccl}(\mathbb{R})$ is $\Im_2$-almost midconvex then there exists a unique midconvex set-valued function $G\colon D \to \mathrm{ccl}(\mathbb{R})$ such that*

$$F(x) = G(x) \qquad \Im_1\text{-a.e. in } D. \tag{1}$$

Proof. Let us consider two functions $f\colon D \to [-\infty, \infty)$ and $g\colon D \to (-\infty, \infty]$ defined by the formulas

$$f(x) := \begin{cases} \inf F(x), & \text{if } F(x) \text{ is bounded below,} \\ -\infty, & \text{if } F(x) \text{ is unbounded below,} \end{cases}$$

and

$$g(x) := \begin{cases} \sup F(x), & \text{if } F(x) \text{ is bounded above,} \\ +\infty, & \text{if } F(x) \text{ is unbounded above.} \end{cases}$$

Since $F$ is $\Im_2$-almost midconvex, the functions $f$ and $g$ are such that

$$\frac{f(x) + f(y)}{2} \geqslant f\left(\frac{x+y}{2}\right) \qquad \text{and} \qquad \frac{g(x) + g(y)}{2} \leqslant g\left(\frac{x+y}{2}\right)$$

$$\Im_2\text{-a.e. in } D \times D.$$

Thus by K u c z m a 's theorem there exist a midconvex function $r\colon D \to [-\infty, \infty)$ equal to $f$ $\Im_1$-a.e. in $D$ and a midconcave function $s\colon D \to (-\infty, \infty]$ equal to $g$ $\Im_1$-a.e. in $D$. Let $A \subset X$ be such that $f(x) = r(x)$ and $g(x) = s(x)$ for all $x \in D \setminus A$.

We will show that $r(x) \leqslant s(x)$ for all $x \in D$. Notice that if $r$ is equal to $-\infty$ at some point $x \in D$, then it is equal to $-\infty$ everywhere in $D$, and if $s(x) = +\infty$ for some $x \in D$, then $s$ equals $+\infty$ for every $x \in D$. So, when $r \equiv -\infty$ or $s \equiv +\infty$, then, certainly, $r(x) \leqslant s(x)$, $x \in D$.

Now assume that both of these functions are real-valued. If $x$ belongs to $D \setminus A$, then $r(x) = f(x) \leqslant g(x) = s(x)$. Suppose now that there exists $x_0 \in D$ such that $r(x_0) > s(x_0)$. Certainly $x_0 \in A$. Since $D$ is an open set, there exists a neighbourhood $U$ of $x_0$ contained in $D$. Without loss of generality we may assume that $U = x_0 + V_0$, where $V_0$ is a symmetric neighbourhood of zero $(V_0 = -V_0)$.

Fix an arbitrary $x \in (x_0 + V_0) \setminus A$ and take $y := 2x_0 - x \in (x_0 + V_0)$. Since $x \notin A$, $r(x) \leqslant s(x)$ we have

$$\begin{aligned}\frac{s(x)+s(y)}{2} &\leqslant s\left(\frac{x+y}{2}\right) = s(x_0)\\ &< r(x_0) = r\left(\frac{x+y}{2}\right) \leqslant \frac{r(x)+r(y)}{2}\\ &\leqslant \frac{s(x)+r(y)}{2}.\end{aligned}$$

Thus $s(y) < r(y)$. This means that $y \in A$ and consequently $x = 2x_0 - y \in 2x_0 - A$.

So $(x_0 + V_0) \setminus A \subset 2x_0 - A \in \Im_1$ and hence $(x_0 + V_0) \setminus A \in \Im_1$. Since

$$(x_0 + V_0) \subset \big[(x_0 + V_0) \setminus A\big] \cup A \in \Im_1 ,$$

then $(x_0 + V_0) \in \Im_1$, and this is a contradiction with Lemma B. Hence the set-valued function $G(x) := \{y \in \mathbb{R} :\ r(x) \leqslant y \leqslant s(x)\}$ is well-defined and it is midconvex.

Now let us take another midconvex set-valued function $H\colon D \to \operatorname{ccl}(\mathbb{R})$ equal to $F$ $\Im_1$-a.e. in $D$. Certainly it can be represented in the following way $H(x) = \{y \in \mathbb{R} :\ u(x) \leqslant y \leqslant v(x)\}$, $x \in D$, where $u\colon D \to [-\infty, \infty)$ is midconvex and $v\colon D \to (-\infty, \infty]$ is midconcave. Because $u = r$ $\Im_1$-a.e. in $D$ and both of these functions are midconvex, then, according to Lemma C, they are equal. Similarly $v$ is equal to $s$, so in fact $H = G$ in $D$, which means that $G$ is unique $\square$

To prove the next theorem we need the following result (cf. [1]) which forces us to make an additional assumption that the set-valued functions under consideration have values in a separable normed space.

**THEOREM B.** *Let $Y$ be a separable normed space. Then there is a countable set $W$ of continuous linear functionals on $Y$ such that if $K$ is a non-empty, compact, convex subset of $Y$ and $B$ is a closed ball in $Y$ disjoint from $K$, then there exists $f \in W$ for which* $\max f(K) < \inf f(B)$.

The main result of this paper reads as follows:

**THEOREM 2.** *Let $X$ be a topological vector space, $D$ be a convex and open subset of $X$ and $Y$ be a separable normed space. Denote by $\Im_1$ and $\Im_2$ p.l.i. conjugate $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying conditions* (a) *and* (b).

*If a set-valued function $F\colon D \to \mathrm{cc}(Y)$ is $\Im_2$-almost midconvex, then there exists a unique midconvex set-valued function $G\colon D \to \mathrm{cc}(Y)$ such that*

$$F(x) = G(x) \qquad \Im_1\text{-}a.e.\ in\ D. \tag{2}$$

P r o o f. It is easy to verify that for each $p \in Y^*$ the set-valued function $p \circ F\colon D \to 2^{\mathbb{R}}$ is $\Im_2$-almost midconvex. Notice that $p \circ F$ has compact and convex values in $\mathbb{R}$, so by Theorem 1 for each $p \in Y^*$ there exists a midconvex set-valued function $F_p\colon D \to \mathrm{ccl}(\mathbb{R})$ and a set $A_p \in \Im_1$ such that for all $x \in D \setminus A_p$ one has $p \circ F(x) = F_p(x)$. The function $G_p\colon D \to 2^Y$ defined as follows

$$G_p(x) := \{y \in Y :\ p(y) \in F_p(x)\}, \qquad x \in D,$$

has non-empty, convex and closed values.

Let $W$ be a countable family of continuous linear functionals on $Y$ as in Theorem B. Let us consider the set-valued function $G\colon D \to 2^Y$ defined by the formula

$$G(x) := \bigcap_{p\in W} G_p(x), \qquad x \in D.$$

Clearly, all the values $G(x)$ are convex and closed.

Notice that, because the set $W$ is countable, $\bigcup_{p\in W} A_p$ belongs to $\sigma$-ideal $\Im_1$.

We will show that

$$F(x) = G(x), \qquad x \in D \setminus \bigcup_{p\in W} A_p.$$

Let us take an $x \in D \setminus \bigcup_{p\in W} A_p$. For each $y \in F(x)$ we have $p(y) \in (p \circ F)(x) = F_p(x)$ so, by definition, $y \in G_p(x)$. Hence $F(x) \subset G_p(x)$ for each $p \in W$ and, as a consequence, $F(x) \subset G(x)$. Now let us take $y \in G(x)$. For every $p \in W$ we have $p(y) \in F_p(x) = (p \circ F)(x)$. Suppose that $y \notin F(x)$. Then, because $F(x)$ is compact and convex, there exists a $p_0 \in W$ and a constant $c \in \mathbb{R}$ such that $p_0(y) > c > \max_{z\in F(x)} p_0(z) \geqslant p_0(z)$ for all $z \in F(x)$. Hence $p_0(y) \notin (p_0 \circ F)(x)$, which is a contradiction. Thus $G(x) = F(x)$ for $\Im_1$-almost all $x$ from $D$.

To show that $G$ is midconvex we shall first prove the midconvexity of $G_p$ for arbitrarily chosen $p \in Y^*$. Let us fix arbitrary $x, y \in D$, $u \in G_p(x)$ and $v \in G_p(y)$. Then, by midconvexity of $F_p$,

$$p\left(\frac{u+v}{2}\right) = \frac{p(u)+p(v)}{2} \in \frac{F_p(x)+F_p(y)}{2} \subset F_p\left(\frac{x+y}{2}\right),$$

that is $\frac{u+v}{2} \in G_p\left(\frac{x+y}{2}\right)$. Hence $\frac{G_p(x)+G_p(y)}{2} \subset G_p\left(\frac{x+y}{2}\right)$. So

$$\frac{G(x)+G(y)}{2} \subset \frac{G_p(x)+G_p(y)}{2} \subset G_p\left(\frac{x+y}{2}\right),$$

for any $p \in W$. Thus

$$\frac{G(x)+G(y)}{2} \subset \bigcap_{p\in W} G_p\left(\frac{x+y}{2}\right) = G\left(\frac{x+y}{2}\right),$$

i.e. $G$ is midconvex.

The next step of our proof is to show that the values of the function $G$ are not empty. Suppose that there exists a point $x_0 \in D$ for which $G(x_0) = \emptyset$. Because for all $x \in D \setminus \bigcup_{p\in W} A_p$ we have that $G(x) = F(x)$ and the values of $F$ are not empty, there exists $p \in W$ such that $x_0 \in A_p$. Let us take a symmetric neighbourhood $V$ of zero such that $x_0 + V \subset D$. Let us choose an arbitrary point $x \in (x_0 + V) \setminus \bigcup_{p\in W} A_p$. Notice that such an element $x$ exists because otherwise $x_0 + V \subset \bigcup_{p\in W} A_p \in \Im_1$, which contradicts Lemma B. Of course $G(x) = F(x) \neq \emptyset$. Let us take $y := 2x_0 - x \in x_0 + V$. Then

$$\frac{G(x)+G(y)}{2} \subset G(x_0) = \emptyset.$$

It means that $G(y) = \emptyset$ and hence $y \in \bigcup_{p\in W} A_p$. Thus $x \in 2x_0 - \bigcup_{p\in W} A_p$. Consequently, we have

$$x_0 + V \subset \left((x_0+V) \setminus \bigcup_{p\in W} A_p\right) \cup \bigcup_{p\in W} A_p \subset \left(2x_0 - \bigcup_{p\in W} A_p\right) \cup \bigcup_{p\in W} A_p \in \Im_1,$$

which contradicts Lemma B.

Notice that all the values of $G$ are compact. To show this let us fix an arbitrary $x_0$ and take a neighbourhood $V_1 \subset D$ of $x_0$. The set $V_2 := \frac{1}{2}(V_1 - x_0) + x_0 \subset D$ is open and non-empty, so $V_2 \notin \Im_1$. Hence there exists an $x \in V_2$ such that $F(x) = G(x)$. Let us take $y = 2x - x_0 \in V_1$ and some $u \in G(y)$. Then

$$\frac{G(x_0)+u}{2} \subset \frac{G(x_0)+G(y)}{2} \subset G(x).$$

Because $G(x_0)$ is a closed subset of the compact set $2G(x)-u$, it is compact itself.

To verify the uniqueness of $G$ let us consider another midconvex set-valued function $H\colon D \to \mathrm{cc}(Y)$ equal to $F$ $\Im_1$-a.e. in $D$. Suppose that $G(x) \neq H(x)$ for some $x \in D$. Let $u \in H(x) \setminus G(x)$ (or $u \in G(x) \setminus H(x)$). By Theorem B, there is $p \in W$ for which $p(u) \notin p(G(x))$ (or $p(u) \notin p(H(x))$). Hence $p(H(x)) \setminus p(G(x)) \neq \emptyset$ (or $p(G(x)) \setminus p(H(x)) \neq \emptyset$). On the other hand, for all $p \in Y^*$ the superpositions $p \circ H$ and $p \circ G$ are midconvex and $\Im_1$-a.e. equal in $D$. Hence, by Theorem 1, they are equal, which gives a contradiction and ends the proof. □

We say that a set-valued function $F\colon D \to \mathrm{n}(Y)$ is *convex* if it satisfies the following condition

$$\lambda F(x) + (1-\lambda)F(y) \subset F\big(\lambda x + (1-\lambda)y\big) \tag{3}$$

for all $\lambda \in [0,1]$ and $x, y \in D$.

A set-valued function $F\colon D \to \mathrm{n}(Y)$ is termed $\Im_2$-*almost convex* if for all $\lambda \in [0,1]$ there exists a set $M(\lambda) \in \Im_2$ such that condition (3) is satisfied for all $(x,y) \in D^2 \setminus M(\lambda)$.

To prove the next result we will use the following theorem of R. Ger (cf. [2; Theorem 2] and its proof for $\varepsilon = 0$):

**THEOREM C.** *For each $\Im_2$-almost convex function $f\colon D \to [-\infty, +\infty)$ there exists a convex function $g\colon D \to [-\infty, +\infty)$ such that $f = g$ $\Im_1$-almost everywhere in $D$.*

In fact it was formulated for real-valued functions, but it can be proved for functions with values in $[-\infty, +\infty)$, in the same way.

**THEOREM 3.** *Let $X$ be a topological vector space and $D$ be a convex and open subset of $X$. Denote by $\Im_1$ and $\Im_2$ p.l.i. conjugate $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying conditions* (a) *and* (b).

*If a set-valued function $F\colon D \to \mathrm{ccl}(\mathbb{R})$ is $\Im_2$-almost convex, then there exists a unique convex set-valued function $G\colon D \to \mathrm{ccl}(\mathbb{R})$ such that*

$$F(x) = G(x) \qquad \Im_1\text{-a.e. in } D.$$

P r o o f . If we define functions $f$ and $g$ as in the proof of Theorem 1, it is enough to notice that functions $f$ and $g$ are $\Im_2$-almost convex and $\Im_2$-almost concave, respectively. Hence, by Theorem C, there exist a convex function $r$ equal to $f$ $\Im_1$-a.e. in $D$ and a concave function $s$ equal to $g$ $\Im_1$-a.e. in $D$. The functions $r$ and $s$ are midconvex and midconcave, respectively, so by Theorem A they are unique and like in the proof of Theorem 1 we obtain that $r(x) \leqslant s(x)$, $x \in D$. Hence we infer that $G\colon D \to \mathrm{ccl}(\mathbb{R})$ given by the formula

$$G(x) := \{y \in \mathbb{R} : \ r(x) \leqslant y \leqslant s(x)\}, \qquad x \in D,$$

is well-defined, convex and unique function equal to $F$ $\Im_1$-a.e. in $D$. □

**THEOREM 4.** *Let $X$ be a topological vector space, $D$ be a convex and open subset of $X$ and $Y$ be a separable normed space. Denote by $\Im_1$ and $\Im_2$ p.l.i. conjugate $\sigma$-ideals in $X$ and $X \times X$, respectively, satisfying conditions* (a) *and* (b).

*If a set-valued function $F\colon D \to \mathrm{cc}(Y)$ is $\Im_2$-almost convex, then there exists a unique convex set-valued function $G\colon D \to \mathrm{cc}(Y)$ such that*

$$F(x) = G(x) \qquad \Im_1\text{-a.e. in } D.$$

P r o o f . The proof of this theorem is similar to the proof of Theorem 2. Let us notice that for each $p \in Y^*$ a set-valued function $p \circ F\colon D \to 2^{\mathbb{R}}$ is $\Im_2$-almost convex. Notice that $p \circ F$ has compact and convex values in $\mathbb{R}$, so by Theorem 3 for each $p \in Y^*$ there exists a convex set-valued function $F_p\colon D \to \mathrm{ccl}(\mathbb{R})$ and a set $A_p \in \Im_1$ such that for all $x \in D \setminus A_p$ one has $p \circ F(x) = F_p(x)$. A function $G_p\colon D \to 2^Y$ defined by

$$G_p(x) := \{y \in Y : \ p(y) \in F_p(x)\}, \qquad x \in D,$$

has non-empty, convex and closed values.

Let $W$ be a countable family of continuous linear functionals on $Y$ as in Theorem B. Let us consider a set-valued function $G\colon D \to 2^Y$ defined by the formula

$$G(x) := \bigcap_{p \in W} G_p(x), \qquad x \in D.$$

Clearly, all the values $G(x)$ are convex and closed.

Notice that because the set $W$ is countable, $\bigcup_{p \in W} A_p$ belongs to the $\sigma$-ideal $\Im_1$. Similarly as in the proof of Theorem 2 we can show that

$$F(x) = G(x), \qquad x \in D \setminus \bigcup_{p \in W} A_p,$$

and that the function $G$ is convex (and of course midconvex). Hence by Theorem 2 the function $G$ has non-empty convex and compact values and it is the only convex function such that $F(x) = G(x)$ for $\Im_1$-almost all $x \in D$. □

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