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Mathematica Slovaca, Vol. 57 (2007), No. 6, [523]--546

Persistent URL: http://dml.cz/dmlcz/136976

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0044-1
Math. Slovaca **57** (2007), No. 6, 523–546

UPPER INTEGRAL AND ITS GEOMETRIC MEANING

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(*Communicated by Miloslav Duchoň*)

ABSTRACT. The Hahn definition of the integral is recalled, the requirement of measurability of the integrand omitted. Both the upper and lower integrals comply with this definition and so does any measurable function between them.

The outer product measure of the hypograph of a nonnegative bounded nonmeasurable function is equal to the upper integral which is equal to one of the Fan integrals. The outer measure of the graph of a bounded nonmeasurable function is equal to the difference between the upper and lower integrals.

A norm for not necessarily measurable functions is defined with the upper integral. The linear space with this norm is complete. The convergence in this space implies the convergence in outer measure. The distance as an outer measure of the symmetric difference of two sets gives us a complete metric space of classes of subsets.

1. The Hahn definition of integral

Throughout the paper we use a complete nonnegative measure which means that all subsets of a set of measure zero are measurable and have measure zero. The reason is that it would be difficult to trace exactly where the level sets of nonmeasurable functions would have measure zero or would be nonmeasurable as subsets of a set of measure zero.

DEFINITION. Let S be a set. We say that (S, Σ, ϕ) is a *measure space* if Σ is a σ-algebra of some subsets of S, $S \in \Sigma$, and ϕ is a σ-additive, nonnegative, finite and complete set function defined on Σ. Such a set function is also called a *measure*.

2000 Mathematics Subject Classification: Primary 28A20, 28A35.
Keywords: Hahn integral, nonmeasurable, outer measure, product space, seminorm, Minkowski inequality, completeness, convergence in outer measure, class of sets with complete metric.

We could find a dozen definitions or so of the Lebesgue integral in the literature. They define the same thing and differ only in details. We start with the Hahn definition of the integral and find out what its relation to the upper and lower integrals is.

Even though this paper is centered around the upper integral and its applications, we recall the Hahn definition first. The way the integral is presented in [2, §12.1] requires measurability but the very definition may be given without it.

Definition. Let a measure space (S, Σ, ϕ) be given and let a function f be defined almost everywhere on S. If there is a σ-additive set function h defined on Σ such that for any $A \in \Sigma$ and arbitrary C_1 and C_2, for which $C_1 \le f(x) \le C_2$ for almost all $x \in A$ it holds that $C_1\phi(A) \le h(A) \le C_2\phi(A)$, then f is called *integrable* on S, h is called an *integral* of the function f. When the function h is uniquely determined, we write $h(A) = \int\limits_A f \,\mathrm{d}\phi$.

The measurability of the integrand may be put off until the time when it is required to prove the uniqueness of the integral.

Theorem 1. *If f is measurable and integrable on S, then the function h is determined uniquely.*

Proof. [2, §12.1.2]. □

It is important that if f is measurable, the Hahn definition of the integral is equivalent to the usual definitions of the Lebesgue integral as shown in [2, §13.1.1, §13.1.2]. It means we feel free to take for granted theorems from real analysis without any proof or reference. Only two examples are mentioned specifically because the terminology has not been unified.

As a first example we present the theorem ([2, §12.1.8]) on monotone bounded convergence and use it several times. It says that if (S, Σ, ϕ) is a measure space, $0 \le f_1, f_2, \dots$ is a sequence of measurable functions defined for almost all $x \in S$ and there is an $M > 0$ such that $-M \le f_N(x) \le f_{N+1}(x) \le M$ for all N and almost all $x \in S$, and $\int\limits_S f_N \,\mathrm{d}\phi \le M < \infty$, then not only is there an f such that $\lim f_N(x) = f(x)$ for almost all $x \in S$ but also $\lim \int\limits_S f_N \,\mathrm{d}\phi = \int\limits_S f \,\mathrm{d}\phi \le M$.

Another example is the theorem ([4, 5-5 I]) on monotone convergence with uniformly bounded integrals which says that if (S, Σ, ϕ) is a measure space, $f_1, f_2, \dots$ is a sequence of measurable functions defined for almost all $x \in S$ such that $0 \le f_N(x) \le f_{N+1}(x)$ for all N and almost all $x \in S$ and there is an $M > 0$ such that $\int\limits_S f_N \,\mathrm{d}\phi \le M$ for all N, then $f_N(x)$ converges to some $f(x)$ for almost all x and $\lim \int\limits_S f_N(x) \,\mathrm{d}\phi = \int\limits_S f(x) \,\mathrm{d}\phi \le M$.

The upper integral is defined in [2, §12.5]. We may use any equivalent definition of the integral we are used to for the definition of the upper integral.

Definition. Let f be defined almost everywhere on $A \in \Sigma$. There exists a measurable function which is greater than or equal to f almost everywhere on A. We define the *upper integral* of f on A as

$$\overline{\int_A} f \, \mathrm{d}\phi = \inf_{g \geq f} \int_A g \, \mathrm{d}\phi,$$

where the infimum is taken over all the measurable functions greater than or equal to f.

Theorem 2. *Let the function f be defined almost everywhere on $A \in \Sigma$. Among the measurable functions that are greater than or equal to f there exists almost everywhere a unique function f^*, for which*

$$\overline{\int_A} f \, \mathrm{d}\phi = \int_A f^* \, \mathrm{d}\phi.$$

Proof. [2, §12.5.2]. □

Theorem 3. *If $\overline{\int_A} f \, \mathrm{d}\phi$ is finite, then there is a measurable function $f^* \geq f$ such that*

$$\overline{\int_E} f \, \mathrm{d}\phi = \int_E f^* \, \mathrm{d}\phi$$

for all measurable $E \subset A$.

Proof. [2, §12.5.211]. □

Theorem 4. *Let a measure space (S, Σ, ϕ) and a finite nonnegative function f defined on S almost everywhere be given. The function f does not have to be measurable. Then $\overline{\int_A} f \, \mathrm{d}\phi$ satisfies the conditions in the Hahn definition of the integral of f.*

Proof. There is a measurable function $f^* \geq f$ such that $\overline{\int_E} f \, \mathrm{d}\phi = \int_E f^* \, \mathrm{d}\phi$ for any $E \in \Sigma$. For any measurable $E \subset A$ and C_1 such that $C_1 \leq f(x)$ for $x \in E$ it holds that $C_1 \leq f(x) \leq f^*(x)$ for almost all $x \in E$. Therefore

$$C_1 \phi(E) = \int_E C_1 \, \mathrm{d}\phi \leq \int_E f^* \, \mathrm{d}\phi.$$

We assume that $f(x) \leq C_2$ for $x \in E$. Let B be the set of those $x \in E$ for which $f^*(x) > C_2$. We will show that $\phi(B) = 0$. Let $\phi(B) > 0$. Then we define $f_2(x) = C_2$ for $x \in B$ and $f_2(x) = f^*(x)$ for $x \in E - B$. Therefore it holds that $f(x) \leq f_2(x)$ for $x \in E$ and $f_2(x) < f^*(x)$ for $x \in B$ and it follows that $\int_E f_2 \, \mathrm{d}\phi < \int_E f^* \, \mathrm{d}\phi$, which is a contradiction. Thus $\phi(\{x \in E : \ f^*(x) > C_2\}) = 0$ and we can write

$$\int_E f^* \, \mathrm{d}\phi = \int_{E-B} f^* \, \mathrm{d}\phi \leq \int_{E-B} C_2 \, \mathrm{d}\phi = C_2 \phi(E)$$

We can define the lower integral in a similar manner and show that it satisfies the definition of integral. We can thus conclude that there are more than one set functions satisfying the inequalities defining the integral. In the following, if f is finite and nonnegative almost everywhere, we denote by f^* a measurable function satisfying $f^*(x) \geq f(x)$ for almost all $x \in S$ and $\int_A f^* \, \mathrm{d}\phi = \inf_{g > f} \int_A g \, \mathrm{d}\phi$ f^* will also be called an upper function corresponding to f.

By f_* we denote the unique measurable function for which $f_*(x) \leq f(x)$ for almost all $x \in S$ and $\int_A f_* \, \mathrm{d}\phi = \sup_{g \leq f} \int_A g \, \mathrm{d}\phi$. f_* will be called a lower function corresponding to f. □

Lemma. *Let f_1 and f_2 be measurable and $f_1 < f_2$ almost everywhere on a measurable set A of positive measure. Then there is an $\epsilon > 0$ and a measurable set $A_1 \subset A$ with a positive measure such that $f_1 + \epsilon < f_2$ on A_1.*

Proof. Assume there is no such $\epsilon > 0$. Then for every natural $N > 0$ the set C_N of those $x \in A$ for which $f_1(x) + 1/N < f_2(x)$ has measure zero. If $f_1(x) < f_2(x)$, then $x \in C_N$ for some N. The union $\bigcup C_N$ has measure zero, therefore the set of those x for which $f_1(x) < f_2(x)$ would have measure zero, which is a contradiction. □

Theorem 5. *Let a measure space (S, Σ, ϕ) be given. Assume that a bounded function f is defined almost everywhere on S, f does not have to be measurable. Then the set function $\int_A f_m \, \mathrm{d}\phi$ satisfies the conditions of the Hahn definition of integral if and only if f_m is measurable and $f_* \leq f_m \leq f^*$ holds almost everywhere.*

Proof. First we show that if $f_* \leq f_m \leq f^*$ holds almost everywhere for a measurable f_m, then $\int_A f_m \, \mathrm{d}\phi$ satisfies the conditions of the Hahn definition of integral. Let $C_1 \leq f(x)$ for all $x \in E \in \Sigma$. Let B be the set of $x \in E$ for which $f_*(x) < C_1$. To show that $\phi(B) = 0$, we assume $\phi(B) > 0$. We define $f_2(x) = C_1$ for $x \in B$ and $f_2(x) = f_*(x)$ for $x \in E - B$. Thus $f_2(x) \leq f(x)$ for

$x \in E$ and $f_2(x) > f_*(x)$ for $x \in B$. It follows that $\int\limits_E f_2 \,\mathrm{d}\phi > \int\limits_E f_* \,\mathrm{d}\phi$, which is a contradiction. Therefore we get $\phi(\{x \in E : f_*(x) < C_1\}) = 0$ and we can write

$$\int\limits_E f_m \,\mathrm{d}\phi \geq \int\limits_E f_* \,\mathrm{d}\phi = \int\limits_{E-B} f_* \,\mathrm{d}\phi \geq \int\limits_{E-B} C_1 \,\mathrm{d}\phi,$$

the last integral being equal to $C_1\phi(E)$. If $C_2 \geq f(x)$ for $x \in E$, the proof is similar.

Now we want to show that if $f^* < f_m$ holds on a set of positive measure, then $\int\limits_A f_m \,\mathrm{d}\phi$ does not satisfy the conditions of the Hahn definition of integral.

According to the lemma there is an $\epsilon > 0$ such that $A_1 = \{x : f^*(x) + \epsilon < f_m(x)\}$ has positive measure. Since f^* is measurable and bounded, there is a sequence of simple functions $s_i \geq f^*$ converging uniformly to f^*. There is also a sequence of simple functions $t_i \leq f_m$ converging uniformly to f_m. We can thus pick a j such that

$$f(x) \leq f^*(x) \leq s_j(x) < f^*(x) + \epsilon/2 \leq f_m(x) - \epsilon/2 < t_j(x) \leq f_m(x)$$

on a set of positive measure A_1.

Since s_j are simple functions, it is obvious that there is a measurable set $A_2 \subset A_1$ of positive measure with s_j constant on A_2. In the same manner, because t_j is simple, we see that there is a measurable A_3 of positive measure such that $A_3 \subset A_2$ and t_j is constant on A_3. Let $C = (s_j(x) + t_j(x))/2$, where $x \in A_3$. Then for all $x \in A_3$ it holds that

$$f(x) \leq f^*(x) \leq s_j(x) < C < t_j(x) \leq f_m(x).$$

Thus $f(x) < C$ for $x \in A_3$, but, since $C < f_m(x)$, the inequality $C\phi(A_3) < \int\limits_{A_3} f_1 \,\mathrm{d}\phi$ holds and we can conclude that $\int\limits_S f_m \,\mathrm{d}\phi$ is not an integral of f.

The proof concerning the function f_* would be just a mirror image of the way we worked with f^*. □

2. The outer measure of the graph of a function

Let (S, Σ, ϕ) be a measure space, where S is a set, Σ is some σ algebra of its subsets and ϕ a measure defined on Σ. A product $C \times A$ is called a rectangle if $C \in \Sigma$ and $A \subset \mathbb{R}$ is Lebesgue measurable, $\phi(C) < \infty$, $\mu(A) < \infty$, where μ denotes the Lebesgue measure. We define a set function $\phi\mu$ on a rectangle as $\phi\mu(C \times A) = \phi(C) \times \mu(A)$. When $H \subset S \times \mathbb{R}$, we define the outer measure as $(\phi\mu)^*(H) = \inf \sum \phi\mu(C_i \times A_i)$, the infimum being taken over all the sequences of rectangles $\{C_i \times A_i\}$ such that $H \subset \cup(C_i \times A_i)$.

We assume that a nonnegative real valued function f is defined on S and bounded by some $B > f(x)$ and that the hypograph H of f, $H = \{(x,y) : x \in S,\ 0 \le y \le f(x)\}$ is covered by a countable collection of rectangles $H \subset \bigcup R_K$. A rectangle R_K is a cartesian product $R_K = C_K \times J_K$, where $C_K \in \Sigma$ and $J_K = (a_K, b_K)$ is an open interval bounded by B. To build up some intuition, we are going to discuss the way of defining a sequence of simple functions with convenient properties.

It will be useful to define a projection P_S of a subset $V \subset S \times \mathbb{R}$ as $P_S(V) = \{x : x \in S, (x,y) \in V \text{ for some } y\}$ and the projection P_R of a subset $V \subset S \times \mathbb{R}$ as $P_R(V) = \{y : y \in \mathbb{R}, (x,y) \in V \text{ for some } x\}$. Thus $P_R(C \times (a,b)) = (a,b)$ and $P_S(C \times (a,b)) = C$. It is clear that $P_R\Big(\bigcup_{K=1}^{N} R_K\Big) = \bigcup_{K=1}^{N} P_R(R_K)$ and also $P_S\Big(\bigcup_{K=1}^{N} R_K\Big) = \bigcup_{K=1}^{N} P_S(R_K)$.

If a subset $V \subset S \times \mathbb{R}$ and $C \subset S$ is given, we can define an inverse projection of C as $\{(x,y) : x \in C, (x,y) \in V\}$. But we are interested only in the inverse projection of a one element set $\{x_0\} \subset S$ giving us $\{(x,y) : x = x_0 \text{ and } (x,y) \in V\}$. We define a cross-section $V_{x_0} \subset \mathbb{R}$ at x_0 as

$$V_{x_0} = P_R\big(\{(x,y) : x = x_0 \text{ and } (x,y) \in V\}\big) = \{y : x = x_0 \text{ and } (x,y) \in V\}.$$

We can now see that, if $V = \bigcup_{K=1}^{N} R_K$, then $V_x = \bigcup_{K=1}^{N} (R_K)_x$.

We recall that the notion of a cross-section $V_x = \{y : (x,y) \in V\}$ is also used for a different purpose in the proof of the Fubini theorem. A picture is helpful to show that $\{x\} \subset S$ is mapped to $V_x = \bigcup_{K=1}^{N} (R_K)_x$ as a union of a finite number of open intervals in $\mathbb{R}$ or as an empty set.

If a countable set of rectangles is given, we assume that the rectangles may be enumerated in some fixed way. The aim is to define a function $f_N(x)$ at a fixed $x \in S$ if a finite collection $R_1, R_2, \dots, R_N$ is given, where $R_K = C_K \times J_K$. We use the projection V_x of $V = \bigcup_{K=1}^{N} R_K$

$$V_x = \bigcup_{K=1}^{N} (R_K)_x = \bigcup_{i=1}^{L} J_{K_i},$$

where the subscripts K_i are precisely those for which $x \in C_{K_i} = P_S(R_{K_i})$. The subscripts K_i correspond to a subcollection of $L \le N$ rectangles $R_{K_1}, R_{K_2}, \dots$ $\dots, R_{K_L}$ for which $x \in \bigcap_{i=1}^{L} P_S(R_{K_i})$. If for any subcollection with $x \in \bigcap_{i=1}^{L} P_S(R_{K_i})$

we have $0 \notin \bigcup_{i=1}^{L} J_{K_i}$, we define $f_N(x) = 0$. If $0 \in J_{K_i}$ for some K_i, we want to define $f_N(x)$ in a convenient way.

It might be wrong to be greedy. We could take $f_N(x) = \sup\Big(\bigcup_{K_i}^{L} J_{K_i}\Big)$, but it would not work, because we could pick a $\delta > 0$ and consider $R_1 = S \times (B-\delta, B)$, where $B > f(x)$ for all $x \in S$. If the measure of S were finite, the product measure of such a rectangle could be made arbitrarily small by picking small δ. We could get $f_N(x) = B$ for all $x \in S$ and such a definition would be useless.

If we want to stay on the safe side, we write $\bigcup_{K_i}^{L} J_{K_i}$ as a disjoint union of intervals, we pick the interval containing zero, and set $f_N(x)$ equal to the right endpoint of this interval.

To discuss the definition of the function $f_N(x)$ further, we check all the subcollections of $R_1, R_2, \dots, R_N$ and consider only those consisting of $L \le N$ rectangles for which

1) $x \in \bigcap_{i=1}^{L} P_S(R_{K_i})$,

2) $0 \in P_R\Big(\bigcup_{i=1}^{L} R_{K_i}\Big)$,

3) $P_R\Big(\bigcup_{i=1}^{L} R_{K_i}\Big)$ is an interval.

If these conditions are satisfied for the two projections, we call such a subcollection p-connected with $(x, 0)$, where the letter p stands for projection.

Out of all such p-connected subcollections we pick the one with the largest projection $(a, b) = P_R\Big(\bigcup_{i=1}^{L} R_{K_i}\Big)$. We then define $f_N(x)$ to be equal to b which is the right endpoint of the interval (a, b).

Theorem 6. *Let (S, Σ, ϕ) be a measure space. Assume that $f(x)$ is a bounded nonnegative real valued function defined on S. Let $H = \big\{(x, y) : x \in S, 0 \le y \le f(x)\big\}$. Then $(\phi\mu)^*(H) = \overline{\int_S} f(x)\,\mathrm{d}\phi$.*

Proof. It is obvious that $(\phi\mu)^*(H) \le \overline{\int_S} f(x)\,\mathrm{d}\phi$.

We want to show that the assumption

$$0 < \epsilon = \overline{\int_S} f(x)\,\mathrm{d}\phi - (\phi\mu)^*(H)$$

leads to a contradiction.

Assume that $f(x)$ is bounded by some real $B > \sup f(x)$. There is a sequence of measurable rectangles $\{A_i \times B_i\}$ such that $A_i \in \Sigma$, B_i is Lebesgue measurable and $\sup B_i < B$, $H \subset \bigcup (A_i \times B_i)$, and $(\phi\mu)^*(H) \leq \sum \phi(A_i)\mu(B_i) < (\phi\mu)^*(H) + \epsilon/4$.

Define d_i as

$$d_i = \begin{cases} 1 & \text{if } \phi(A_i) = 0, \\ \frac{1}{4\phi(A_i)2^i} & \text{if } \phi(A_i) > 0. \end{cases}$$

Since each B_i is Lebesgue measurable, there is a sequence of open intervals $\{I_{ij}\}$ such that $B_i \subset \bigcup I_{ij}$, $\sup I_{ij} \leq B$, and

$$\mu(B_i) \leq \sum_j \mu(I_{ij}) \leq \mu(B_i) + \epsilon d_i.$$

Then, since we can discard the terms with $\phi(A_i) = 0$, we have

$$\begin{aligned} (\phi\mu)^*(H) &\leq \sum_i \phi(A_i)\mu(B_i) \leq \sum_i \phi(A_i)\Big(\sum_j \mu(I_{ij})\Big) \leq \sum_i \phi(A_i)(\mu(B_i) + \epsilon d_i) \\ &\leq \sum_i \phi(A_i)\mu(B_i) + \epsilon \sum_i \phi(A_i) d_i \leq \sum_i \phi(A_i)\mu(B_i) + \epsilon \sum_{i:\, \phi(A_i)>0} \frac{\phi(A_i)}{4\phi(A_i)2^i} \\ &\leq \sum_i \phi(A_i)\mu(B_i) + \frac{\epsilon}{4} \leq (\phi\mu)^*(H) + \epsilon/4 + \epsilon/4 = (\phi\mu)^*(H) + \epsilon/2. \end{aligned}$$

Since the collection of rectangles $A_i \times I_{ij}$ is countable, the rectangles may be numbered to form a sequence $R_1, R_2, \dots$ with rectangles denoted as $R_K = C_K \times J_K$, where $C_K \subset S$ is measurable, $J_K = (a_K, b_K)$, $a_K < b_K$.

We study a finite collection $R_1, R_2, \dots, R_N$. As indicated above, its subcollection $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ is p-connected with $(x, 0)$, where $x \in S$, that is, if $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_L}$ is an interval (a, b) containing zero and $x \in \bigcap_{i=1}^{L} C_{K_i}$, then a is called the left endpoint of the subcollection $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ and b is called the right endpoint.

For $x \in S$ we define a function $f_N(x)$ as zero if there is no subcollection p-connected with $(x, 0)$ among the rectangles $R_1, R_2, \dots, R_N$, that is, with the left endpoint negative and right endpoint positive. If there is a subcollection of $R_1, R_2, \dots, R_N$, p-connected with $(x, 0)$, we define the value of $f_N(x)$ as the maximum of the right endpoints over all the subcollections p-connected with $(x, 0)$. This maximum is well defined since there is a finite number, $2^N - 1$, of nonempty subcollections of N rectangles.

For each N, the function $f_N(x)$ is simple, because the number of rectangles $R_1, R_2, \dots, R_N$ is finite and so is the number of values $f_N(x)$ may attain.

From the definition of $f_N(x)$ it follows that $f_{N+1}(x) \geq f_N(x)$ for every $x \in S$, for we have $N+1$ rectangles instead of just N to consider. We define $F(x) = \lim f_N(x)$, where $F(x)$ is measurable and bounded. We may now use the theorem on monotone bounded convergence to get $\lim \int\limits_S f_N(x)\,\mathrm{d}\phi = \int\limits_S F(x)\,\mathrm{d}\phi$.

Since $f_N(x)$ is simple and nonnegative, there is a finite number M of real numbers $s_1, s_2, \dots, s_M$ as the values the function f_N takes on. We define $D_i = \{x \in S:\ f_N(x) = s_i\}$. The integral

$$\int\limits_S f_N(x)\,\mathrm{d}\phi = \sum_{i=1}^{M} s_i \phi(D_i) = \sum_{i=1}^{M} \phi(D_i)(s_i - 0)$$

is the sum of product measures of rectangles $D_i \times (0, s_i)$.

To show that $\bigcup(D_i \times (0, s_i)) \subset \bigcup\limits_{K=1}^{N} R_K$, where $R_K = C_K \times J_K$, let $x \in S$, $y > 0$, $(x, y) \in \cup(D_i \times (0, s_i))$. Then $(x, y) \in D_i \times (0, s_i)$ for some i, that is, $x \in D_i$ and $0 < y < s_i$, where $0 < s_i = f_N(x)$. By definition of $f_N(x)$ at $x \in S$, there are some rectangles $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ such that $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_L} = (a, b)$, $a < 0$, and $b = s_i$. It implies that $y \in J_j$ and $x \in C_j$ for some j, $1 \leq j \leq N$.

Now we can see that

$$\begin{aligned}\int\limits_S f_N(x)\,\mathrm{d}\phi &= (\phi\mu)\Big(\bigcup(D_i \times (0, s_i))\Big) \leq (\phi\mu)\left(\bigcup_{K=1}^{N} R_K\right) \\ &\leq (\phi\mu)\left(\bigcup_{K=1}^{\infty} R_K\right) \leq (\phi\mu)^*(H) + \frac{\epsilon}{2}\end{aligned}$$

and when we pass to the limit

$$\int\limits_S F(x)\,\mathrm{d}\phi = \lim_{N\to\infty} \int\limits_S f_N(x)\,\mathrm{d}\phi \leq (\phi\mu)^*(H) + \frac{\epsilon}{2}.$$

The last step is to show that $F(x) \geq f(x)$ for all $x \in S$. If we assumed $0 = F(x) = f_N(x)$ for all N, we would get a contradiction because there is an $R_i = C_i \times (a_i, b_i)$ with $x \in C_i$ and $0 \in (a_i, b_i)$. We may assume that $0 < \lim f_N(x) = F(x) < f(x)$. Then there is a rectangle $R_K = C_K \times J_K$ such that $J_K = (a_K, b_K)$, $F(x) \in (a_K, b_K)$, and $x \in C_K$. Since $f_N(x) \to F(x)$, there is an i such that $f_i(x) > a_K$. By definition of $f_N(x)$, there are rectangles p-connected with $(x, 0)$ $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ as a subcollection of $R_1, R_2, \dots, R_i$ such that $f_i(x)$ is the right endpoint of the interval $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_i}$. If we now add R_i to this collection of rectangles, we obtain $R_{K_1}, R_{K_2}, \dots, R_{K_i}, R_i$, for which $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_i} \cup J_i$ is an interval (a, b). Clearly $b > F(x)$, which is a contradiction. This finishes the proof. □

In the following theorem we investigate the graph and its countable covering $G = \{(x, f(x)) : x \in S\} \subset \bigcup R_K$, $R_K = C_K \times J_K$, C_K is measurable, $J_K -(a_K, b_K)$ is an open interval bounded by B. When a finite number of such rectangles $R_1, R_2, \dots, R_N$ is given, we will again consider projections for $V \bigcup_{K-1}^{N} R_K$

$$V_x = \bigcup_{K-1}^{N} (R_K)_x = \bigcup_{i=1}^{L} J_{K_i}$$

where K_i are the subscripts for which $x \in C_{K_i}$. We will say that a subcollection $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ is p-connected with (x, y) if $x \in \bigcap_{i=1}^{L} P_S(R_{K_i}) \subset S$ and $P_R\Big(\bigcup_{i=1}^{L} R_{K_i}\Big)$ is an interval containing y. We will obviously be interested in the subcollection that gives the largest interval containing y to be able to define sequences of simple functions.

Theorem 7. *Let (S, Σ, ϕ) be a measure space. Assume that $f(x)$ is a bounded nonnegative real valued function defined on S. Let $G = \{(x, f(x)) : x \in S\}$. Then $(\phi\mu)^*(G) = \int_S f^*(x)\,\mathrm{d}\phi - \int_S f_*(x)\,\mathrm{d}\phi$, where f^* and f_* are the upper and lower functions corresponding to f.*

Proof. It is obvious that $(\phi\mu)^*(G) \le \int_S f^*(x)\,\mathrm{d}\phi - \int_S f_*(x)\,\mathrm{d}\phi$. We want to show that the assumption

$$0 < \epsilon = \int_S f^*(x)\,\mathrm{d}\phi - \int_S f_*(x)\,\mathrm{d}\phi - (\phi\mu)^*(G)$$

leads to a contradiction.

Assume that $f(x)$ is bounded by some real $B > 0$. There is a sequence of rectangles $\{A_i \times B_i\}$ such that $A_i \in \Sigma$, B_i is Lebesgue measurable and $\sup B_i < B$, $\inf B_i > -1/2$, $G \subset \cup(A_i \times B_i)$, and $(\phi\mu)^*(G) \le \sum \phi(A_i)\mu(B_i) < (\phi\mu)^*(G) + \epsilon/4$.

Define d_i as

$$d_i - \begin{cases} 1 & \text{if } \phi(A_i) = 0, \\ \frac{1}{4\phi(A_i)2^i} & \text{if } \phi(A_i) > 0. \end{cases}$$

Since each B_i is Lebesgue measurable, there is a sequence of open intervals $\{I_{ij}\}$ such that $B_i \subset \bigcup I_{ij}$, $\sup I_{ij} \le B$, $\inf I_{ij} \ge -1/2$, and

$$\mu(B_i) \le \sum_j \mu(I_{ij}) \le \mu(B_i) + \epsilon d_i.$$

Then

$$\begin{aligned}(\phi\mu)^*(G) &\le \sum_i \phi(A_i)\Big(\sum_j \mu(I_{ij})\Big) \le \sum_i \phi(A_i)(\mu(B_i)+\epsilon d_i)\\ &\le \sum_i \phi(A_i)\mu(B_i) + \epsilon \sum_{i:\,\phi(A_i)>0} \frac{\phi(A_i)}{4\phi(A_i)2^i} \le \sum_i \phi(A_i)\mu(B_i) + \frac{\epsilon}{4}\\ &\le (\phi\mu)^*(G) + \epsilon/4 + \epsilon/4 = (\phi\mu)^*(G) + \epsilon/2.\end{aligned}$$

We may number the rectangles and form a sequence. We have to use the notation $R_1, R_2, \ldots$ where $R_K = C_K \times J_K$, $C_K \subset S$ is measurable, $J_K = (a_K, b_K)$, $a_K < b_K$. Let a finite collection $R_1, R_2, \ldots, R_N$ be given. If a subcollection $R_{K_1}, R_{K_2}, \ldots, R_{K_L}$ is p-connected with $(x, f(x))$, that is, if $x \in \bigcap_{i=1}^{L} C_{K_i} \subset S$ and $J_{K_1} \cup J_{K_2} \cup \cdots \cup J_{K_L}$ is an interval (a, b) containing $f(x)$, then a is called the left endpoint of the subcollection $R_{K_1}, R_{K_2}, \ldots, R_{K_L}$ and b is called the right endpoint.

For $x \in S$ we define a function $f_N(x)$ as -1 if there is no subcollection p-connected with $(x, f(x))$ among the rectangles $R_1, R_2, \ldots, R_N$, that is, with the left endpoint less than $f(x)$ and right endpoint greater than $f(x)$. If there is a subcollection p-connected with $(x, f(x))$ among $R_1, R_2, \ldots, R_N$, we define the value of $f_N(x)$ as the maximum of the right endpoints over all the subcollections p-connected with $(x, f(x))$ of the collection $R_1, R_2, \ldots, R_N$.

We also define for $x \in S$ a function $g_N(x)$ as $B+1$ if there is no subcollection p-connected with $(x, f(x))$ among the rectangles $R_1, R_2, \ldots, R_N$. If there is a subcollection p-connected with $(x, f(x))$ among $R_1, R_2, \ldots, R_N$, we define the value of $g_N(x)$ as the minimum of the left endpoints over all the subcollections p-connected with $(x, f(x))$ of the collection $R_1, R_2, \ldots, R_N$.

It is easy to see that $f_N(x) = -1$ if and only if $g_N(x) = B+1$. The functions $f_N(x)$ and $g_N(x)$ are simple, because the number of rectangles $R_1, R_2, \ldots, R_N$ is finite and so is the number of values $f_N(x)$ or $g_N(x)$ may attain.

From the definition of $f_N(x)$ and $g_N(x)$ it follows that $f_{N+1}(x) \ge f_N(x)$ and $g_{N+1}(x) \le g_N(x)$ for every $x \in S$. Since f_N and g_N are bounded, we define $F(x) = \lim f_N(x)$ and $G(x) = \lim g_N(x)$. We use the theorem on monotone bounded convergence on $f_N(x)+1$ to get $\int_S F(x)\,\mathrm{d}\phi = \lim \int_S f_N(x)\,\mathrm{d}\phi$. The same theorem may be used on $B + 1 - g_N(x)$ to obtain $\int_S G(x)\,\mathrm{d}\phi = \lim \int_S g_N(x)\,\mathrm{d}\phi$.

We define $E_N = \{x \in S : f_N(x) = -1 \text{ or } g_N(x) = B+1\}$. Since $f_N(x)$ is simple and nonnegative, there is a finite number M_s of real numbers $s_1, s_2, \ldots, s_{M_s}$ as the values the function f_N attains on $S - E_N$. There is also a finite number M_t of values $t_1, t_2, \ldots, t_{M_t}$ the function $g_N(x)$ attains on $S - E_N$.

We define $D_{i,j} - \{x \in S: \; f_N(x) = s_i, \;\; g_N(x) = t_j\}$. We write

$$\int_S (f_N - g_N)\,\mathrm{d}\phi = \int_{E_N} (f_N - g_N)\,\mathrm{d}\phi + \int_{S-E_N} (f_N - g_N)\,\mathrm{d}\phi.$$

The value of the first of the two integrals on the right hand side is

$$\int_{E_N} (f_N - g_N)\,\mathrm{d}\phi = -\int_{E_N} (B+2)\,\mathrm{d}\phi = -(B+2)\phi(E_N).$$

We know that $E_{N+1} \subset E_N$, and $\lim E_N = \bigcap E_N$. To show that $\bigcap E_N$ is empty we assume that $x \in \bigcap E_N$. Then there is a subscript, say K, such that $(x, f(x)) \in R_K = C_K \times J_K$, where $C_K \subset S$, $J_K = (a_K, b_K)$, $a_K < f(x) < b_K$. Therefore $x \notin E_K$. It follows that $\lim (B+2)\phi(E_N) = 0$.

The second integral on the right hand side is

$$\int_{S-E_N} (f_N - g_N)\,\mathrm{d}\phi = \sum_{i=1}^{M_s} \sum_{j\;1}^{M_t} \phi(D_{i,j})(s_i - t_j),$$

the sum of product measures of rectangles $D_{i,j} \times (t_j, s_i)$.

To show that $\bigcup (D_{i,j} \times (t_j, s_i)) \subset \bigcup\limits_{K=1}^{N} R_K$, let $x \in S$, $(x,y) \in \bigcup (D_{i,j} \times (t_j, s_i))$. Then $(x,y) \in D_{i,j} \times (t_j, s_i)$ for some i, j, that is, $x \in D_{i,j}$ and $t_j < y < s_i$, where $s_i = f_N(x)$ and $B \geq t_j = g_N(x)$. By definition of $f_N(x)$ and $g_N(x)$ at $x \in S$, there are some rectangles $R_{K_1}, R_{K_2}, \ldots, R_{K_L}$, $R_K = C_K \times J_K$ such that $J_{K_1} \cup J_{K_2} \cup \cdots \cup J_{K_L} = (a, b)$, $a = t_j$, and $b = s_i$. It implies that $y \in J_j$ and $x \in C_j$ for some j, $1 \leq j \leq N$.

We may now conclude that

$$\begin{aligned} \int_{S\;E_N} (f_N - g_N)\,\mathrm{d}\phi &= \sum_{i=1}^{M_s} \sum_{j-1}^{M_t} \phi(D_{i,j})(s_i - t_j) \\ &\leq (\phi\mu)\Big(\bigcup_{K=1}^{N} R_K\Big) \leq (\phi\mu)\Big(\bigcup_{K-1}^{\infty} R_K\Big) \leq (\phi\mu)^*(G) + \frac{\epsilon}{2} \end{aligned}$$

and it follows that

$$\int_S (F(x) - G(x))\,\mathrm{d}\phi - \lim_{N\to\infty} \int_S (f_N - g_N)\,\mathrm{d}\phi \leq (\phi\mu)^*(G) + \frac{\epsilon}{2}.$$

The last step is to prove that $F(x) \geq f(x)$ and $G(x) \leq f(x)$ for all $x \in S$. First we assume that $\lim f_N(x) = F(x) = -1$. But there exists a rectangle $R_K - C_K \times (a_K, b_K)$ such that $x \in C_K$ and $f(x) \in (a_K, b_K)$ and it follows that $F(x) \geq b_K > 0$, which is a contradiction. We may now assume that

$\lim f_N(x) = F(x) < f(x)$. There is a rectangle $R_K = C_K \times J_K$ such that $x \in C_K$, $J_K = (a_K, b_K)$, and $F(x) \in (a_K, b_K)$. Since $f_N(x)$ converges to $F(x)$, there is an i such that $f_i(x) > a_K$. By definition of $f_N(x)$, there are rectangles $R_{K_1}, R_{K_2}, \dots, R_{K_L}$ as a subset of $R_1, R_2, \dots, R_i$ such that $f_i(x)$ is the right endpoint of $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_L}$. If we now add R_K to this collection of rectangles, we obtain $R_{K_1}, R_{K_2}, \dots, R_{K_L}, R_K$, for which $J_{K_1} \cup J_{K_2} \cup \dots \cup J_{K_L} \cup J_K$ is an interval (a,b). Clearly $b > F(x)$, which is a contradiction.

The proof that $G(x) \le f(x)$ would be based on the same idea. □

Theorem 8. *Let (S, Σ, ϕ) be a measure space. Assume that $f(x)$ is a bounded nonnegative real valued function defined on S. Let $H = \{(x,y) :\ x \in S,\ 0 \le y \le f(x)\}$ and $P = \{(x,y) :\ x \in S,\ 0 \le y < f(x)\}$. Then $(\phi\mu)^*(P) = (\phi\mu)^*(H)$.*

Proof. Since $P \subset H$, we have $(\phi\mu)^*(P) \le (\phi\mu)^*(H)$.

Assume that $\epsilon > 0$ is given. Since $f(x)$ is bounded by some real $B > \sup f(x)$, there is a sequence of measurable rectangles $\{A_i \times B_i\}$ such that $A_i \in \Sigma$, B_i is Lebesgue measurable and $\sup B_i < B$, $P \subset \bigcup (A_i \times B_i)$, and $(\phi\mu)^*(P) \le \sum \phi(A_i)\mu(B_i) < (\phi\mu)^*(P) + \epsilon/4$.

Define d_i as

$$d_i = \begin{cases} 1 & \text{if } \phi(A_i) = 0, \\ \frac{1}{4\phi(A_i)2^i} & \text{if } \mathrm{d}\phi(A_i) > 0. \end{cases}$$

Since each B_i is Lebesgue measurable, there is a sequence of open intervals $\{I_{ij}\}$ such that $B_i \subset \cup I_{ij}$, $\sup I_{ij} \le B$, and

$$\mu(B_i) \le \sum_j \mu(I_{ij}) \le \mu(B_i) + \epsilon d_i.$$

Then

$$\begin{aligned} (\phi\mu)^*(P) &\le \sum_i \phi(A_i)\Big(\sum_j \mu(I_{ij})\Big) \le \sum_i \phi(A_i)(\mu(B_i) + \epsilon d_i) \\ &\le \sum_i \phi(A_i)\mu(B_i) + \epsilon \sum_{i:\,\phi(A_i)>0} \frac{\phi(A_i)}{4\phi(A_i)2^i} \le \sum_i \phi(A_i)\mu(B_i) + \frac{\epsilon}{4} \\ &\le (\phi\mu)^*(P) + \epsilon/4 + \epsilon/4 = (\phi\mu)^*(P) + \epsilon/2. \end{aligned}$$

We investigate an enumeration of all the rectangles $A_i \times J_{ij}$ denoted as $R_K = C_K \times (a_K, b_K)$, where $C_K \in \Sigma$ and $a_K < b_K$. We construct a new collection of rectangles covering P consisting of rectangles $C_K \times (a_K + \delta, b_K + \delta)$ and $S \times (-\delta, \delta)$, where $0 < \delta = \epsilon/(4\phi(S))$.

Assume $(x,y) \in H$ and $0 \le y < \delta$. Then clearly $(x,y) \in S \times (-\delta, \delta)$. Assume $(x,y) \in H$ and $\delta \le y$. Then $(x, y-\delta) \in P$ and there is a K such that $(x, y-\delta) \in C_K \times (a_K, b_K)$ and it follows that $(x,y) \in C_K \times (a_K + \delta, b_K + \delta)$.

We now calculate

$$\begin{aligned}(\phi\mu)^*(H) &\le \phi(S)\mu((-\delta,\delta)) + \sum \phi(C_K)\mu((a_K+\delta, b_K+\delta))\\ &= 2\delta\phi(S) + \sum \phi(C_K)(b_K - a_K)\\ &\le 2\delta\phi(S) + (\phi\mu)^*(P) + \epsilon/2 \le (\phi\mu)^*(P) + \epsilon.\end{aligned}$$

Since this inequality holds for any $\epsilon > 0$, we have $(\phi\mu)^*(H) \le (\phi\mu)^*(P)$. □

3. Ward theorem, the equivalence to the Fan integral

Fan [1] defined several types of integrals for nonmeasurable functions and presented some of their properties.

Definition. Let (S,Σ,ϕ) be a measure space. Let f be a real valued function defined on S, not necessarily measurable, nonnegative, and bounded by some $B > 0$. The Fan integral is defined as

$$\int_0^B \phi^*\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y,$$

where $\phi^* = \inf_{E\subset A\in\Sigma} \phi(A)$ is the outer measure. Since $\big\{x \in S:\ f(x) > y_2\big\} \subset \big\{x \in S:\ f(x) > y_1\big\}$ for $y_1 < y_2$ means the integrand is nondecreasing thus the Riemann integral exists.

Further properties of Fan integrals were derived in Ward [5]. The most important property, from our point of view, is the one that connects the Fan integrals to the outer product measure of the hypograph of the integrand, not necessarily measurable. Once we know how the definitions are connected, the properties derived in [1] and [5] immediately apply.

We have to rephrase the theorem to comply with our notation. The idea of the proof is due to [5].

Theorem 9 (Ward). *Let (S,Σ,ϕ) be a measure space, $0 < \phi(S) < \infty$. Let f be a nonnegative and bounded function, $0 \le f(x) \le B$ for all $x \in S$, defined on S, not necessarily measurable. Let $H = \big\{(x,y):\ x \in S,\ 0 \le y \le f(x)\big\}$. Then*

$$(\phi\mu)^*(H) = \int_0^B \phi^*\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y,$$

where we define the outer measure as $\phi^*(C) = \inf\{\phi(F) : \; C \subset F, \; F \in \Sigma\}$. *The outer product measure* $(\phi\mu)^*(H)$ *is defined as*

$$(\phi\mu)^*(H) = \inf\Big\{ \sum \phi(C_N)\mu(B_N) \Big\},$$

where $H \subset \bigcup (C_N \times B_N)$, $C_N \in \Sigma$, *and* B_N *is Lebesgue measurable.*

Proof. Let $\bigcup(C_N \times B_N)$ be any covering of H, where C_N are measurable and B_N Lebesgue measurable. For any $\epsilon > 0$ each B_N can be covered by a countable number of open intervals $I_{N,i}$ in such a way that $\sum \mu(I_{N,i}) \le \mu(B_N) + \epsilon/2^N$. It follows that we may consider only the coverings of H by $\bigcup(C_K \times J_K)$ where J_K are open intervals.

Let $\chi_{J_K}(y)$ be the characteristic function of J_K, that is, $\chi_{J_K}(y) = 1$ if $y \in J_K$ and $\chi_{J_K}(y) = 0$ if $y \notin J_K$.

Let y_0 be fixed. The set $\{x \in S : \; f(x) \ge y_0\}$ is covered by the sets C_K for which $y_0 \in J_K$, that is, if $\chi_{J_K}(y_0) = 1$.

We may write

$$\sum \phi(C_K)\chi_{J_K}(y_0) \ge \phi^*\big(\{x \in S : \; f(x) \ge y_0\}\big) \ge \phi^*\big(\{x \in S : f(x) > y_0\}\big).$$

If for some $J_K = (a_K, b_K)$ we have $B < b_K$, we have $\mu(J_K) > \int_0^B \chi_{J_K}(y)\,\mathrm{d}y$. If $a_K < 0$, we also have $\mu(J_K) > \int_0^B \chi_{J_K}(y)\,\mathrm{d}y$. Thus for each K, we write

$$\phi(C_K)\mu(J_K) \ge \phi(C_K)\int_0^B \chi_{J_K}(y)\,\mathrm{d}y = \int_0^B \phi(C_K)\chi_{J_K}(y)\,\mathrm{d}y.$$

Since $\sum \phi(C_K)\mu(J_K)$ is finite and the sequence of simple functions

$$f_L(y) = \sum_{K=1}^{L} \phi(C_K)\chi_{J_K}(y)$$

is nondecreasing for each y, we may use the Lebesgue theorem on monotone convergence.

$$\begin{aligned}\sum \phi(C_K)\mu(J_K) &\geq \sum \phi(C_K) \int_0^B \chi_{J_K}(y)\,\mathrm{d}y \\ &= \sum \int_0^B \phi(C_K)\chi_{J_K}(y)\,\mathrm{d}y = \int_0^B \sum \phi(C_K)\chi_{J_K}(y)\,\mathrm{d}y \\ &\geq \int_0^B \phi^*\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y.\end{aligned}$$

Since $\bigcup(C_N \times B_N)$ may be any covering of H we have the inequality $\int_0^B \phi\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y \leq (\phi\mu)^*(H) + \epsilon$ for $\epsilon > 0$ arbitrarily small and we have $\int_0^B \phi\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y \leq (\phi\mu)^*(H)$.

If any $\epsilon > 0$ is given, we form a partition of $[0, B]$ by $0 = y_0 < y_1 < y_2 < \cdots < y_N = B$, where $N > 2B$, such that

$$\sum_{K=0}^{N-1} \phi^*\big(\{x \in S:\ f(x) > y_K\}\big)(y_{K+1} - y_K) < \int_0^B \phi^*\big(\{x \in S:\ f(x) > y\}\big)\,\mathrm{d}y + \frac{\epsilon}{3}.$$

Let us write $C_K = \{x \in S : f(x) > y_K\}$. For $\epsilon/2$ there are measurable sets $F_K \in \Sigma$ such that $C_K \subset F_K$ and $\phi(F_K) \leq \phi^*(C_K) + \epsilon/(2N)$. Let

$$\delta = \frac{\epsilon}{3\epsilon + 6 \sum\limits_{K=0}^{N-1} \phi^*(C_K)}.$$

We assign an open interval (a_K, b_K) to every F_K such that $a_K = y_K - \delta$ and $b_K = y_{K+1} + \delta$. Obviously $H \subset \bigcup(F_K \times (a_K, b_K))$. We calculate

$$
\begin{aligned}
&\sum_{K=0}^{N-1} \phi(F_K)(b_K - a_K) \\
&\le \sum_{K=0}^{N-1} \Big(\phi^*(C_K) + \frac{\epsilon}{2N}\Big)(b_K - a_K) \\
&\le \sum_{K=0}^{N-1} \phi^*(C_K)(b_K - a_K) + \frac{\epsilon}{2N} \sum_{K=0}^{N-1} (b_K - a_K) \\
&= \sum_{K=0}^{N-1} \phi^*(C_K)(y_{K+1} + \delta - (y_K - \delta)) + \frac{\epsilon}{2N} \sum_{K=0}^{N-1} (y_{K+1} + \delta - (y_K - \delta)) \\
&= \sum_{K-0}^{N-1} \phi^*(C_K)(y_{K+1} - y_K) + \delta\Big(\epsilon + 2\sum_{K=0}^{N-1} \phi^*(C_K)\Big) + \frac{B\epsilon}{2N} \\
&< \sum_{K=0}^{N-1} \phi^*(C_K)(y_{K+1} - y_K) + \frac{\epsilon}{3\epsilon + 6\sum\limits_{K=0}^{N-1} \phi^*(C_K)} \Big(\epsilon + 2\sum_{K=0}^{N-1} \phi^*(C_K)\Big) + \frac{B\epsilon}{4B} \\
&< \sum_{K=0}^{N-1} \phi^*(C_K)(y_{K+1} - y_K) + \frac{\epsilon}{3} + \frac{\epsilon}{4} < \int_0^B \phi^*\big(\{x \in S :\ f(x) > y\}\big)\,\mathrm{d}y + \epsilon.
\end{aligned}
$$

Since this inequality holds for any ϵ, we have

$$(\phi\mu)^*(H) = \int_0^B \phi^*\big(\{x \in S :\ f(x) > y\}\big)\,\mathrm{d}y.$$

□

THEOREM 10. *Let (S, Σ, ϕ) be a measure space, $0 < \phi(S) < \infty$. Let $A \subset S$ be nonmeasurable. Let χ_A denote the characteristic function of A and χ_A^* the upper function corresponding to χ_A. Then $\int_S \chi_A^* \,\mathrm{d}\phi = \phi^*(A)$ where ϕ^* is the outer measure corresponding to ϕ.*

Proof. Let $H = \big\{(x,y) :\ x \in S,\ 0 \le y \le \chi_A(x)\big\}$ Then the previous theorems give us

$$\overline{\int_S} \chi_A(x)\,\mathrm{d}\phi = (\phi\mu)^*(H) = \int_0^1 \phi^*\big(\{x \in S : \chi_A(x) > y\}\big)\,\mathrm{d}y.$$

Since $\{x \in S : \ \chi_A(x) > y\} = A$ for $y < 1$, we have

$$\int_0^1 \phi^*(\{x \in S : \ \chi_A(x) > y\})\,\mathrm{d}y = \int_0^1 \phi^*(A)\,\mathrm{d}y = \phi^*(A).$$

□

4. Minkowski type inequality and completeness

Let (S, Σ, ϕ) be a measure space. Assume f is defined on S and measurable. A seminorm is defined as

$$\|f\|_p = \left(\int_S |f|^p \,\mathrm{d}\phi \right)^{1/p},$$

where $1 \le p < \infty$. We will extend this definition to nonmeasurable functions.

Definition. Let (S, Σ, ϕ) be a measure space. Let f be a function defined on S, measurability of f is not required. We define the upper seminorm as

$$\|f\|_p^* = \left(\int_S |f|^{*p} \,\mathrm{d}\phi \right)^{1/p},$$

where $|f|^*$ stands for the upper function corresponding to $|f|$.

When the upper norm is defined in this way, it is nothing but $\|f\|_p^* = \| |f|^* \|_p$. It means that $\|f\|_p^*$ is the same as $\|f\|_p$ for measurable f. To show that $\|f\|_p^*$ is also a seminorm, we prove the following theorem.

Theorem 11. *Let (S, Σ, ϕ) be a measure space. Assume f and g, not necessarily measurable, are defined on S. Then the Minkowski inequality holds for the upper seminorm*

$$\|f + g\|_p^* \le \|f\|_p^* + \|g\|_p^*.$$

Proof. First we apply the elementary inequality to real numbers

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le |f(x)|^* + |g(x)|^*,$$

where the second inequality holds for almost all $x \in S$, to see, by the definition of the upper function, that

$$|f(x) + g(x)|^* \le |f(x)|^* + |g(x)|^*$$

for almost all x. By integrating both sides, we get

$$\||f+g|^*\|_p \le \||f|^* + |g|^*\|_p$$

and applying the Minkowski inequality to the right hand side yields

$$\||f+g|^*\|_p \le \||f|^*\|_p + \||g|^*\|_p$$

which proves the theorem. □

The inequality we have just proved enables us to define a normed linear space of classes of functions. We say that f and g are in the same equivalence class if $\|f-g\|_p^* = 0$. We know that $\|f\|_p^* = 0$ if and only if $f = 0$ almost everywhere. A class of functions is in the L_p^* space if for its representative f the norm is finite, $\|f\|_p^* < \infty$. Further details may be obtained in the similar manner as in the case of classes of measurable functions, [2, §15.1], [3, 6.1], or [4, 5-8].

Theorem 12. *Let (S, Σ, ϕ) be a measure space. Let $f_1, f_2, \dots$ be a sequence of real valued functions on S, not necessarily measurable, such that $f_N(x) \le f_{N+1}(x)$ for almost all $x \in S$. Let f denote the limit of $f_1, f_2, \dots$, that is, $f(x) = \lim f_N(x)$ for almost all $x \in S$. Let f_N^* be the upper function of f_N and f^* the upper function of f. Then, for almost all $x \in S$, $f_N^*(x) \le f_{N+1}^*(x)$ and*

$$\lim f_N^*(x) = f^*(x).$$

Proof. We denote the limit of $f_N(x)$ by $f(x)$, which is defined for almost all $x \in S$. Let f^* denote the upper function of f. Then $f^*(x) \ge f_N(x)$ for almost all x. If f_N^* denotes the upper function of f_N, then it is obvious that $f_N^*(x) \le f^*(x)$ for almost all x.

We also have $f_N^*(x) \le f_{N+1}^*(x)$, for otherwise there would be a measurable subset $E \subset S$ with $\mu(E) > 0$ for which $f_N^*(x) > f_{N+1}^*(x)$. Then for the function $g(x) = \min\big(f_N^*(x), f_{N+1}^*(x)\big)$ we have $\int_S g \,\mathrm{d}\phi < \int_S f_N^* \,\mathrm{d}\phi$ while $g(x) \ge f_N(x)$ for almost all x.

The limit of $f_N^*(x)$ is defined almost everywhere and we denote it by F. Since $f_N^*(x) \le f^*(x)$ almost everywhere, we have $F(x) \le f^*(x)$ for almost all x. But since $F(x) \ge f_N^*(x) \ge f_N(x)$ for all N and almost all x, we have $F(x) \ge f(x)$ for such x and it follows that $F(x) \ge f^*(x)$ for almost all x. □

Fact. *Let (S, Σ, ϕ) be a measure space, g a function defined for almost all $x \in S$, and $1 \le p \le \infty$. Then $(g^p)^* = (g^*)^p$.*

Proof. Since $(g^*)^p$ is measurable, we have $(g^*)^p = ((g^*)^p)^*$. Thus $g \le g^*$ implies $g^p \le (g^*)^p$ which means $(g^p)^* \le ((g^*)^p)^* = (g^*)^p$.

If we assume there is an $A \in \Sigma$ with $\phi(A) > 0$ such that $(g^p)^* < (g^*)^p$ for $x \in A$, then $((g^p)^*)^{1/p} < g^*$ on A, which is a contradiction.

A normed linear space is complete if and only if every absolutely summable series is summable, see [3, 6.3].

Theorem 13. *Let (S, Σ, ϕ) be a measure space. Let $f_1, f_2, \dots$ be a sequence of real valued functions on S, not necessarily measurable, such that $\sum_{N=1}^{\infty} \| f_N \|_p^* = M < \infty$, where $1 \le p < \infty$. Then the sequence of partial sums is convergent, that is, there is a real valued function s defined on S such that* $\lim \|s - \sum_{i}^{N} f_i\|_p^* = 0$.

Proof. Let $f_1, f_2, \dots$ be a sequence of real valued functions on S with $\sum_{i=1}^{\infty} \|f_i\|_p^* = M < \infty$. We define functions g_N by setting $g_N(x) = \sum_{i=1}^{N} |f_i(x)|$. From the Minkowski inequality we have

$$\|g_N\|_p^* \le \sum_{i=1}^{N} \|f_i\|_p^* \le M.$$

Hence

$$\int_S (g_N^p)^* \, \mathrm{d}\phi = \int_S (g_N^*)^p \, \mathrm{d}\phi \le M^p.$$

For each $x \in S$, the sequence $g_1(x), g_2(x), \dots$ is nondecreasing and must converge to some finite $g(x)$ or infinity. Since the sequence $g_1(x), g_2(x), \dots$ is nondecreasing we can use the theorem on monotone convergence with uniformly bounded integrals to see that

$$\int_S (g^p)^* \, \mathrm{d}\phi = \int_S (g^*)^p \, \mathrm{d}\phi \le M^p.$$

Thus $g(x)$ is finite for almost all x.

For each x for which $g(x)$ is finite the series of real numbers $\sum_{i=1}^{\infty} f_i(x)$ converges absolutely to a real number $s(x)$. We set $s(x) = 0$ for those x for which $g(x)$ is infinite. The partial sums are denoted as $s_N(x) = \sum_{i=1}^{N} f_i(x)$. Then

$$|s(x) - s_N(x)| = \left| \sum_{i=N+1}^{\infty} f_i(x) \right| \leq \sum_{i=N+1}^{\infty} |f_i(x)|$$

for almost all $x \in S$.

To avoid confusion with the subscripts, we write $Q_K(x) = \sum_{N+1}^{N+K} |f_i(x)|$. Thus $Q_K(x)$ is nondecreasing for almost all fixed x and converges to some $Q(x) = \sum_{i-N+1}^{\infty} |f_i(x)| \leq g(x)$ as K goes to infinity. By the previous theorem we must have $\lim_{K\to\infty} Q_K^*(x) = Q^*(x) \leq g^*(x)$ for almost all x and it follows that $\lim_{K\to\infty} \|Q_K\|_p^* = \|Q\|_p^*$ for $1 \leq p < \infty$.

$$\begin{aligned}\|s - s_N\|_p^* &\leq \left\| \sum_{i=N+1}^{\infty} |f_i| \right\|_p^* = \|Q(x)\|_p^* = \lim_{K\to\infty} \|Q_K(x)\|_p^* \\ &= \lim_{K\to\infty} \left\| \sum_{i=1}^{K} |f_{N+i}| \right\|_p^* \leq \lim_{K\to\infty} \sum_{i=1}^{K} \|f_{N+i}\|_p^* = \sum_{i=N+1}^{\infty} \|f_i\|_p^*.\end{aligned}$$

The sum $\sum_{i=N+1}^{\infty} \|f_i\|_p^*$ converges to zero as N goes to infinity since it is a remainder of a convergent series. □

5. Chebyshev inequality, convergence in outer measure

We want to show that the Chebyshev inequality may be generalized for nonmeasurable functions.

Assume that $|f|$ does not have to be measurable. Then, if $\epsilon > 0$ and $E = \{x \in S : |f(x)|^{*p} > \epsilon\}$,

$$\int_S |f|^{*p}\, d\phi \geq \int_E |f|^{*p}\, d\phi \geq \int_E \epsilon^p\, d\phi = \epsilon^p \phi(E) \geq \epsilon^p \phi^*\big(\{x \in S : |f(x)|^p > \epsilon\}\big).$$

Theorem 14. *Let (S, Σ, ϕ) be a measure space. Assume that the functions $f_1, f_2, \dots,$ not necessarily measurable, converge in L_p^* to f, where $1 < p < \infty$, then, for each fixed $\epsilon > 0$, $\phi^*\big(\{x \in S : |f_N(x)|^p > \epsilon\}\big)$ converges to zero.*

Proof. We use the Chebyshev inequality for $|f_N - f|$ for a fixed ϵ to see that

$$\int\limits_S |f_N - f|^{*p} \,\mathrm{d}\phi \geq \epsilon^p \phi^*\big(\{x \in S:\ |f_N(x) - f(x)|^p > \epsilon\}\big)$$

and that $\phi^*\big(\{x \in S:\ |f_N(x) - f(x)|^p > \epsilon\}\big)$ converges to zero if f_N converges to f in L^{*p} for $1 \leq p < \infty$. □

6. Application to a metric on sets

We assume that sets $A, B \in \Sigma$ are given and consider $\phi^*(A\Delta B)$. If A' is also a set and $\phi^*(A\Delta A') = 0$, we also have $\phi(A\Delta B) = \phi(A'\Delta B)$ and it follows that it makes sense to define a metric on equivalence classes $[A]$ of sets defined by the equivalence relation $A \sim A'$ if and only if $\phi^*(A\Delta A') = 0$. We introduce a metric on such equivalence classes as $d([A],[B]) = \phi^*(A\Delta B)$ where A is a representative of $[A]$ and B is a representative of $[B]$. Such classes of sets are well known.

Let A, B be any subsets of S and χ_A, χ_B their characteristic functions. Then for the outer measure of the symmetric difference it holds that

$$\phi^*(A\Delta B) = \int\limits_S \chi^*_{A\Delta B} \,\mathrm{d}\phi = \int\limits_S |\chi_A - \chi_B)|^* \,\mathrm{d}\phi.$$

If we replace χ_A and χ_B by functions f_A and f_B such that $f_A(x) = \chi_A(x)$ and $f_B(x) = \chi_B(x)$ for almost all $x \in S$, we get $\phi^*(A, B) = \int\limits_S |f_A - f_B|^* \,\mathrm{d}\phi$.

Theorem 15. *The space of classes of sets, for which $A \sim A'$ if and only if $\phi^*(A\Delta A') = 0$, with the metric $d([A],[B]) = \phi^*(A\Delta B)$ where A, B are representatives of $[A]$, $[B]$, is a complete metric space.*

Proof. We assume that $[A_1], [A_2], \dots$ is a Cauchy sequence. We use some representatives $A_1, A_2, \dots$ and the characteristic functions $\chi_{A_1}, \chi_{A_2}, \dots$ as the representatives of the corresponding classes of functions. Due to the completeness of the L^{*p}-spaces we know that there is a function f such that

$$\lim_{i\to\infty} \int\limits_S |f - \chi_{A_i}|^* \,\mathrm{d}\phi = 0.$$

Let f be a function and $0 < \epsilon < 1$. We define

$$C_\epsilon = \{x \in S : \ |f(x) - 1| > \epsilon, \ |f(x)| > \epsilon\}.$$

If we assume $\phi^*(C_\epsilon) > 0$, then

$$\begin{aligned}\int_S |f(x) - \chi_A(x)|^* \, \mathrm{d}\phi &\geq \int_S \min(2, |f(x) - \chi_A(x)|^*) \, \mathrm{d}\phi \\ &= \int_0^2 \phi^*\Big(\big\{x \in S : \ \min\big(2, |f(x) - \chi_A(x)| > y\big)\big\}\Big) \, \mathrm{d}y \\ &\geq \epsilon\phi^*(C_\epsilon) > 0.\end{aligned}$$

Since $\epsilon > 0$ and because $\int_S |f - \chi_{A_i}|^* \, \mathrm{d}\phi$ converges to zero, we have $\phi^*(C_\epsilon) = 0$.

We may assume that $\epsilon = 1/N$ and define $C = \bigcup C_{1/N}$. Then $\phi^*(C) \leq \sum \phi^*(C_{1/N}) = 0$.

We may now conclude that, if $\int_S |f - \chi_{A_i}|^* \, \mathrm{d}\phi$ converges to zero, the function f is equal to 0 or 1 almost everywhere because if $f(x)$ is not equal to one or zero, there is an N such that $|f(x) - 1| > 1/N$ and $|f(x)| > 1/N$ thus $x \in C_{1/N} \subset C$ and $\phi^*(C) = 0$.

If we now define a characteristic function $\chi_A(x) = 1$ for $f(x) = 1$ and $\chi_A(x) = 0$ otherwise, we have a characteristic function of a set A. This means that the metric space of classes of sets with the metric $d([A], [B]) = \phi^*(A \Delta B)$ is complete. $\square$

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Received 25. 10. 2005
Revised 22. 6. 2006

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