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Some sufficient conditions for zero asymptotic density and the expression of natural numbers as sum of values of special functions

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Abstract. This paper generalizes some results from another one, namely [3]. We have studied the issues of expressing natural numbers as a sum of powers of natural numbers in paper [3]. It means we have studied sets of type $A = \{n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m} \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, 2 \ldots, m, (n_1, n_2, \ldots, n_m) \neq (0, 0, \ldots, 0)\}$, where $k_1, k_2, \ldots, k_m \in \mathbb{N}$ were given natural numbers.

Now we are going to study a more general case, i.e. sets of natural numbers that are expressed as sum of integral parts of functional values of some special functions. It means that we are interested in sets of natural numbers in the form

$$k = [f_1(n_1)] + [f_2(n_2)] + \cdots + [f_m(n_m)].$$

1. Introduction

Denotation 1.1. *Let $\mathbb{N}$ be the set of natural numbers and A be a subset of $\mathbb{N}$. We denote by $A(n)$ the number of elements of the set A which are less or equal to n, by $d(A)$ the asymptotic density of A and $[x]$ is the integral part of x.*

For subsets of real numbers $A_j \subset \mathbb{R}$, $j = 1, 2, \ldots, m$, we define

$$A_1 + \cdots + A_m = \{a_1 + \cdots + a_m \mid a_j \in A_j \cup \{0\}, \\ j = 1, 2 \ldots, m, (a_1, \ldots, a_m) \neq (0, \ldots, 0)\}.$$

For any real function $f(n)$ defined on $\mathbb{N}$ we define

$$\Phi_f = \{f(n) \mid n \in \mathbb{N}\}.$$

In [3] we have among other things proved the following theorems :

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Theorem 1.1. *Let $M \subseteq \mathbb{N}$ and $A \subseteq \mathbb{N}$ be a set which could be expressed as a union of sets $A_i \subseteq \mathbb{N}, i \in M$ i.e. $A = \cup_{i\in M} A_i$. If for every $n \in \mathbb{N}$ and $i \in M$ holds $A_i(n) = 0$ for $i > n$; exists function $f(n)$ such that for every $k \in \mathbb{N}$; for every $i \in M$ holds $A_i(k) \leq f(k)$; and if at least one of the following properties holds*

a) $f(n) = O(n^\alpha)$ and $M(n) = O(n^\beta)$, $0 \leq \alpha + \beta < 1$
b) $f(n) = o(n^\alpha)$ and $M(n) = O(n^\beta)$, $0 \leq \alpha + \beta \leq 1$
c) $f(n) = O(n^\alpha)$ and $M(n) = o(n^\beta)$, $0 \leq \alpha + \beta \leq 1$,
then $d(A) = 0$.

Theorem 1.2. *Let $k_1, k_2, \ldots, k_m \in \mathbb{N}$ be given natural numbers and $A = \{n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m} \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, 2 \ldots, m, (n_1, n_2, \ldots, n_m) \neq (0, 0, \ldots, 0)\}$. If $\sum_{i=1}^{m} \frac{1}{k_i} < 1$, then $d(A) = 0$.*

These theorems are useful for studying some properties of the set $A = \{n_1^{k_1} + n_2^{k_2} + \cdots + n_m^{k_m} \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, 2 \ldots, m, (n_1, n_2, \ldots, n_m) \neq (0, 0, \ldots, 0)\}$.

We can consider the power $n_i^{k_i}$ as a value of the function $f_i(x) = x^{k_i}$ at point $n_i \in \mathbb{N}$. It gives the opportunity to try to formulate and prove theorems like Theorem 1.1 and Theorem 1.2 using more general functions, i.e. not only in the form $f_i(x) = x^{k_i}$.

Now we are going to create a theorem similar to the Theorem 1.1 and generalise Theorem 1.2.

2. Densities of sets of natural numbers expressed as sum of values of functions

Now we define what we bear in mind when we say that a number $k \in \mathbb{N}$ is expressible in the form $k = f_1(n_1) + \cdots + f_m(n_m)$.

Definition 2.1. *We say that a natural number $k \in \mathbb{N}$ is expressible in the form $k = f_1(n_1) + \cdots + f_m(n_m)$ if and only if*

$$k \in \Phi_{f_1} + \cdots + \Phi_{f_m}.$$

Theorem 2.1. *Let A be the set of natural numbers expressible in the form $k = f_1(n_1) + f_2(n_2)$ (i.e. $A = \Phi_{f_1(n)} + \Phi_{f_2(n)}$), where $f_i : [1, \infty) \to [f_i(1), \infty), i = 1, 2$ are surjective increasing functions on $[1, \infty)$ such that $f_i(n) \in \mathbb{N}$ for $n \in \mathbb{N}$.*

If at least one of the following properties holds

a) $f_1^{-1}(n) = O(n^\alpha)$ *and* $f_2^{-1}(n) = O(n^\beta)$, $0 \leq \alpha + \beta < 1$
b) $f_1^{-1}(n) = o(n^\alpha)$ *and* $f_2^{-1}(n) = O(n^\beta)$, $0 \leq \alpha + \beta \leq 1$
c) $f_1^{-1}(n) = O(n^\alpha)$ *and* $f_2^{-1}(n) = o(n^\beta)$, $0 \leq \alpha + \beta \leq 1$,
then $d(A) = d(\Phi_{f_1(n)} + \Phi_{f_2(n)}) = 0$.

Proof. It is obvious that the inverse functions $f_i^{-1}(n)$ on $[f_i(1), \infty)$ for $i = 1, 2$ exist. Let us define

$$M = \Phi_{f_1(n)} = \{f_1(n_1) \mid n_1 \in \mathbb{N}\},$$
$$A_0 = \Phi_{f_2(n)} = \{0 + f_2(n_2) \mid n_2 \in \mathbb{N}\}$$

and for every $i \in M$ we define

$$A_i = \{i + f_2(n_2) \mid n_2 \in \mathbb{N}\} \cup \{i\}.$$

It is clear that

$$A = A_0 \cup A^*,$$

where $A^* = \cup_{i\in M} A_i$. It is obvious that $f_i(k) \le n \Leftrightarrow k \le f_i^{-1}(n)$. So it is easy to see that

a) $M(n) = [f_1^{-1}(n)] \le f_1^{-1}(n)$
b) $A_0(n) = [f_2^{-1}(n)] \le f_2^{-1}(n)$
c) For every $i \in M$ it holds that

$$i + f_2(k) \le n \Leftrightarrow f_2(k) \le n - i \Leftrightarrow k \le f_2^{-1}(n - i).$$

Therefore and because of the fact that the function $f_2^{-1}(n)$ is increasing, we can write

$$A_i(n) = [f_2^{-1}(n - i)] \le f_2^{-1}(n - i) \le f_2^{-1}(n).$$

Now we will prove that $d(A^*) = 0$. From inequalities

$$\frac{A^*(n)}{n} = \frac{(\cup_{i\in M} A_i)(n)}{n} \le \frac{\sum_{i\in M} A_i(n)}{n}$$

and because of $A_i(n) = 0$ for $i > n$ and $A_i(n) \le f_2^{-1}(n)$ for every $i \in M$, we obtain

$$\frac{\sum_{i\in M} A_i(n)}{n} = \frac{\sum_{i\in M, i\le n} A_i(n)}{n} \le \frac{\sum_{i\in M, i\le n} f_2^{-1}(n)}{n} = \frac{f_2^{-1}(n)M(n)}{n}$$
$$\le \frac{f_2^{-1}(n)f_1^{-1}(n)}{n}.$$

With respect to the assumptions of Theorem 2.1 it is clear that $d(A^*) = 0$. Since $A = A_0 \cup A^*$ and evidently $d(A_0) = 0$, we have $d(A) = 0$.

□

Example 2.1. *We can use Theorem 2.1 to prove that the set* $A = \{n_1^3 + n_2^2 \mid n_1, n_2 \in \mathbb{N} \cup \{0\}, (n_1, n_2) \ne (0, 0)\}$ *has asymptotic density* $d(A) = 0$, *which is in accordance with Theorem 1.2.*

Proof. Let us denote $f_1(x) = x^3, f_2(x) = x^2$. We can see that the functions f_1, f_2 fulfil the conditions of Theorem 2.1 and $f_1^{-1}(x) = x^{\frac{1}{3}}, f_2^{-1}(x) = x^{\frac{1}{2}}$. Then $f_1^{-1}(n) = n^{\frac{1}{3}} = O(n^{\frac{1}{3}}), f_2^{-1} = n^{\frac{1}{2}} = O(n^{\frac{1}{2}}), \frac{1}{3} + \frac{1}{2} < 1$. □

Remark 2.1. *One of the conditions of Theorem 2.1 is the statement "*$f_i(n) \in \mathbb{N}$ *for* $n \in \mathbb{N}$*". But this statement lessens the applicability of Theorem 2.1, whereas we would like to have a theorem which is able to bring similar informations about set of natural numbers in the form*

$$k = [f_1(n_1)] + [f_2(n_2)] + \cdots + [f_m(n_m)],$$

where $[f_i(n_i)]$ *is the integral part of functional value* $f_i(n_i)$ *and functions* $f_i : [1, \infty) \to [f(1), \infty), i = 1, 2 \ldots, m$ *are increasing surjective functions on the interval* $[1, \infty)$.

Now we introduce and prove the main theorem of this paper which generalizes the case a) of Theorem 2.1 and Theorem 1.2. To prove this theorem, we will need the following lemma.

Lemma 2.1. *Let A be the set of natural numbers expressible in the form*

$$k = [f_1(n_1)] + [f_2(n_2)] + \cdots + [f_m(n_m)],$$

where functions $f_i : [1, \infty) \to [f(1), \infty), i = 1, 2 \ldots, m$ are surjective increasing functions on $[1, \infty)$.

If $f_i^{-1}(n) = O(n^{\alpha_i}), i = 1, 2 \ldots, m$, then $A(n) = O(n^{\sum_{i=1}^{m} \alpha_i})$.

Proof. First we will prove that if $f_i^{-1}(n) = O(n^{\alpha_i})$, then also $f_i^{-1}(n+1) = O(n^{\alpha_i})$. Assume that $f_i^{-1}(n) = O(n^{\alpha_i})$, then there is a constant C such that

$$C \geq \limsup_{n\to\infty} \frac{f_i^{-1}(n)}{n^{\alpha_i}} = \limsup_{n\to\infty} \frac{f_i^{-1}(n+1)}{(n+1)^{\alpha_i}} =$$

$$= \limsup_{n\to\infty} \frac{f_i^{-1}(n+1)}{(n+1)^{\alpha_i}} \frac{n^{\alpha_i}}{n^{\alpha_i}} = \limsup_{n\to\infty} \frac{f_i^{-1}(n+1)}{n^{\alpha_i}} (\frac{n}{n+1})^{\alpha_i} =$$

$$= \limsup_{n\to\infty} \frac{f_i^{-1}(n+1)}{n^{\alpha_i}}$$

Hence $f_i^{-1}(n+1) = O(n^{\alpha_i})$.

We are going to prove that $A(n) = O(n^{\sum_{i=1}^{m} \alpha_i})$ by induction.

1) Let $m = 1$.

In case of $m = 1$ holds $A = \{[f_1(n_1)] \mid n_1 \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ exists $k_n \in \mathbb{N}$ such that $[f_1(k)] \leq n$ for $k = 1, \ldots, k_n$, where $k_n \geq A(n)$. (It is possible that $[f_1(n_1)] = [f_1(n_1+1)]$, so we have to write $k_n \geq A(n)$ instead of $k_n = A(n)$!) From the previous inequality we obtain

$$f_1(k_n) - 1 < [f_1(k_n)] \leq n$$

so

$$f_1(k_n) < n + 1$$

and

$$A(n) \leq k_n < f_1^{-1}(n+1) = O(n^{\alpha_1}).$$

2) Induction step. Let us suppose that Lemma 2.1 holds for $m = r - 1$. We will prove that if A is the set of natural numbers in form

$$k = [f_1(n_1)] + [f_2(n_2)] + \cdots + [f_r(n_r)],$$

it means

$$A = \Phi_{[f_1]} + \Phi_{[f_2]} + \cdots + \Phi_{[f_r]},$$

where each of the functions $f_i, i = 1, \ldots, r$, fulfils the condions of Lemma 2.1, then also $A(n) = O(n^{\sum_{i=1}^{r} \alpha_i})$.

Let us define

$$M = \Phi_{[f_1]} + \Phi_{[f_2]} + \cdots + \Phi_{[f_{r-1}]},$$

$$A_0 = \{0 + [f_r(n_r)] \mid n_r \in \mathbb{N}\}$$

and for every $i \in M$ we define

$$A_i = \{i + [f_r(n_r)] \mid n_r \in \mathbb{N}\} \cup \{i\}.$$

By assumption $M(n) = O(n^{\sum_{i=1}^{r-1} \alpha_i})$. As in case 1) we can see that

$$A_0(n) < f_r^{-1}(n+1) = O(n^{\alpha_r})$$

and

$$A_i(n) \leq A_0(n) + 1 = O(n^{\alpha_r}).$$

From this inequalities and because $M(n) = O(n^{\sum_{i=1}^{r-1} \alpha_i})$, $A = A_0 \cup (\cup_{i\in M} A_i)$ and $A_i(n) = 0$ for $i > n$, we can write

$$A(n) = (A_0 \cup (\cup_{i\in M} A_i))(n) \leq A_0(n) + \sum_{i\in M} A_i(n) = A_0(n) + \sum_{i\in M, i\leq n} A_i(n) \leq$$

$$\leq O(n^{\alpha_r}) + \sum_{i\in M, i\leq n} O(n^{\alpha_r}) = O(n^{\alpha_r}) + M(n)O(n^{\alpha_r}) =$$

$$= O(n^{\alpha_r}) + O(n^{\sum_{i=1}^{r-1} \alpha_i})O(n^{\alpha_r}) = O(n^{\sum_{i=1}^{r} \alpha_i})$$

□

Theorem 2.2. *Let A be the set of natural numbers expressible in form*

$$k = [f_1(n_1)] + [f_2(n_2)] + \cdots + [f_m(n_m)],$$

where functions $f_i : [1,\infty) \to [f_i(1), \infty), i = 1, 2 \dots, m$ are surjective increasing functions on $[1,\infty)$.

If $f_i^{-1}(n) = O(n^{\alpha_i}), i = 1, 2 \dots, m$, where $\sum_{i=1}^{m} \alpha_i < 1$, then $d(A) = 0$.

Proof. The proof of Theorem 2.2 is based on Lemma 2.2. The conditions of Lemma 2.2 are fulfiled so we can write:

$$A(n) = O(n^{\sum_{i=1}^{m} \alpha_i}) = O(n^{\epsilon})$$

where $\epsilon < 1$. Therefore

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n} = 0$$

□

Remark 2.2. *Now we have to mention that the condition $\sum_{i=1}^{m} \alpha_i < 1$ from theorem 2.2 is sufficient, but not necessary for $d(A) = 0$.*

If we denote $A = \{n_1^2 + n_2^2 \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, (n_1, n_2) \neq (0)\}$, then in [2] *it was proved (or see* [1]*) that $A(n) = \frac{c.n}{\sqrt{\log n}}(1 + o(1))$ and so*

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n} = 0$$

Example 2.2. *Using Theorem 2.2 we can prove that the set $A = \{[e^{n_1}] + [n_2^{\frac{5}{3}}] + [n_3^{\frac{20}{7}}] \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, 3\}$ has asymptotic density equal to zero.*

Proof. If we denote $f_1(n) = e^n, f_2(n) = n^{\frac{5}{3}}, f_3(n) = n^{\frac{20}{7}}$, then

$$f_1^{-1}(n) = \ln n = O(n^\epsilon), f_2^{-1}(n) = n^{\frac{3}{5}} = O(n^{\frac{3}{5}}), f_3^{-1}(n) = n^{\frac{7}{20}} = O(n^{\frac{7}{20}}),$$

where $\epsilon > 0$ is arbitrarily small. If we choose $\epsilon < 0,05$, then $\epsilon + \frac{3}{5} + \frac{7}{20} = \epsilon + 0,95 < 1$

□

Example 2.3. *Using Theorem 2.2 we can prove that the set $A = \{[q.e^n] \mid n \in \mathbb{N}\}$, where $q > 0, q \in R$ has asymptotic density equal to zero.*

Proof. If we denote $f_1(n) = q.e^n$, then

$$f_1^{-1}(n) = \ln n - \ln q = O(n^\epsilon)$$

where $\epsilon > 0$ is arbitrarily small. If we choose $\epsilon < 1$, then we can see from Theorem 2.2, that $d(A) = 0$.

□

Example 2.4. *Using Theorem 2.2 we can prove that set $A = \{[q_1.e^{n_1}] + \cdots + [q_m.e^{n_m}] \mid n_i \in \mathbb{N} \cup \{0\}, i = 1, \ldots, m, (n_1, \ldots, n_m) \neq (0, \ldots, 0)\}$, where q_i, $i = 1, \ldots, m$ are positive real constants, has asymptotic density equal to zero for arbitrarily large m.*

Proof. If we denote $f_i(n) = q_i.e^n$, for $i = 1, \ldots, m$, then we can choose $\epsilon < \frac{1}{m}$ such that

$$f_i^{-1}(n) = \ln n - \ln q_i = O(n^\epsilon)$$

Since $\sum_{i=1}^m \epsilon < 1$ we obtain $d(A) = 0$. □

Example 2.5. *We can prove similarly as in Example 2.4 that the set $A = \{[a_1 n_1^{k_1}] + \cdots + [a_m n_m^{k_m}] \mid a_i \in [0, \infty], n_i \in \mathbb{N} \cup \{0\}, i = 1, \ldots, m, (n_1, \ldots, n_m), (a_1, \ldots, a_m) \neq (0, \ldots, 0)\}$, where $\sum_{i=1}^m \frac{1}{k_i} < 1$, has asymptotic density equal to zero.*

(For example $A = \{3n_1^6 + 2n_2^5 + 81n_3^4 + n_4^3 \mid n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, 3, 4$, $(n_1, \ldots, n_4) \neq (0, \ldots, 0)\}$)

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