Mária Ždímalová; Ľubica Staneková

Asymptotic comparison of two constructions for large digraphs of given degree and diameter

Acta Mathematica Universitatis Ostraviensis, Vol. 17 (2009), No. 1, 5--9

Persistent URL: `http://dml.cz/dmlcz/137523`

Asymptotic Comparison of Two Constructions for Large Digraphs of Given Degree and Diameter

Mária Ždímalová, Ľubica Staneková

Abstract. We compare the asymptotic growth of the order of the digraphs arising from a construction of Comellas and Fiol when applied to Faber-Moore digraphs versus plainly the Faber-Moore digraphs for the corresponding degree and diameter.

1 Introduction

Determination of the largest order $n(\Delta, D)$ of a digraph of maximum in- and out- degree Δ and diameter at most D is known as the directed version of the *degree-diameter problem*. The corresponding *directed Moore bound* has the form $n(\Delta, D) \leq 1 + \Delta + \Delta^2 + \cdots + \Delta^D$. For an account on history and current development in the problem we refer to the recent survey [4].

Interest in constructions of large digraphs of a given maximum degree and diameter is motivated by potential applications to the design of large interconnection networks. From the point of view of testing properties of large digraphs it is of practical advantage to restrict to those that are *vertex-transitive*. Setting the well known Kautz digraphs of diameter 2 aside, the current largest vertex-transitive digraphs of a given degree and diameter at least 3 have been constructed in [1], [3], and [2]; cf. [4].

The construction of [1], a variant of which is equivalently described also in [3], yields 'large' digraphs of given diameter and degree from 'small' digraphs by an operation called 'digraph composition'. Various modifications of the construction give the same orders of the resulting digraphs provided that the order of the suitable starter digraphs are the same, even though their degree and diameter may slightly differ. Nevertheless, differences turn out to be negligible in asymptotic terms when analysing the magnitude of the order in terms of degree and diameter.

The authors of [1] claim to have obtained best results in the case when their construction take the digraphs of Faber and Moore [2] as input. It is then far

2000 Mathematics Subject Classification: 05C20, 05C12
Key Words and Phrases: digraph, degree, diameter, voltage assignment, lift

from clear how the order of such digraphs, compares with the order of the Faber--Moore digraphs of the corresponding degree Δ and diameter D. The aim of this contribution is to compare the asymptotic growth of the two orders. We describe the constructions of [1] and [2] in Section 2 and analyse the asymptotic growth of the order of the resulting digraphs in Section 3.

2 The Constructions

We begin with introducing the construction of Faber and Moore [2]. For any given $D \geq 2$ and $\Delta \geq D$ their construction gives a family of large vertex-symmetric digraphs $\Gamma_\Delta(D)$ which can be described as follows. The vertex set of $\Gamma_\Delta(D)$ consists of all the distinct words $x_1x_2\dots x_D$ of length D that are D-permutations of an alphabet of $\Delta+1$ letters. Adjacencies are given by:

$$x_1x_2\dots x_D \to \begin{cases} x_2x_3x_4\dots x_{D+1}, & x_{D+1} \neq x_1, x_2, \dots, x_D \\ x_2x_3x_4\dots x_Dx_1 \\ x_1x_3x_4\dots x_Dx_2 \\ x_1x_2x_4\dots x_Dx_3 \\ \dots \\ x_1x_2x_3\dots x_Dx_{D-1} \end{cases}$$

These digraphs have order

$$(\Delta+1)_D = \frac{(\Delta+1)!}{(\Delta-D+1)!},$$

diameter D and are Δ-regular. Note that for $D=2$ the digraphs $\Gamma_\Delta(2)$ reduce to the Kautz digraphs.

On the other hand, details of the basic construction of Comellas and Fiol [1] are as follows. Let $\Gamma = (V, A)$ be a digraph, serving as 'input' for the construction, and let $n \geq 2$ and $t \geq 1$ be integers. Out of this data we introduce a new digraph $\Gamma^\circ = (V^\circ, A^\circ)$ by letting

$$V^\circ = \left\{(p_0p_1\dots p_\alpha\dots p_{n-1}|\alpha); \alpha \in Z_{tn}, p_i \in V\right\}$$

and defining its arc set A° as the set consisting of all arcs of the form

$$(p_0p_1\dots p_\alpha\dots p_{n-1}|\alpha) \to (p_0p_1\dots q_\alpha\dots p_{n-1}|\alpha+1),$$

with subscripts taken modulo n, where q_α is adjacent from p_α in Γ.

The fundamental result of [1] is:

Theorem 1. [1] *Let Γ be a vertex-symmetric Δ-regular digraph of order N in which for every ordered pair of vertices u, v there exists a directed path of length exactly k from u to v. Then, for all $n \geq 2$ and $t \geq 1$ the digraph Γ° is a vertex-symmetric Δ-regular digraph of order tnN^n and diameter at most $(k+t)n-1$.*

It was shown in [1] that the Faber-Moore digraphs $\Gamma_\Delta(D)$ have the property that any ordered pair of their vertices are connected by a directed path of length D.

Therefore, any Faber-Moore digraph $\Gamma_\Delta(D)$ can serve as input for the construction of Comellas and Fiol, with $k = D$. Let $\Gamma^\circ_\Delta(D)$ be the resulting digraph. Since, in the above notation, we have

$$N = \frac{(\Delta+1)!}{(\Delta-D+1)!}\,,$$

the digraph $\Gamma^\circ_\Delta(D)$ has order

$$nt\left[\frac{(\Delta+1)!}{(\Delta-D+1)!}\right]^n,$$

degree Δ, and diameter at most $n(k+t)-1 = n(D+t)-1$. As indicated, our aim is to compare the order of this graph with the order of the corresponding Faber-Moore digraph, that is, with the order of the digraph $\Gamma_\Delta(n(D+t)-1)$.

3 Asymptotic Analysis

In the preceding notation our main result is:

Theorem 2. *Assume that the values of $D \geq 2$, $n \geq 2$, and $t \geq 1$ are fixed. Then, for all sufficiently large Δ the order of the digraph $\Gamma^\circ_\Delta(D)$ is smaller than the order of the digraph $\Gamma_\Delta(n(D+t)-1)$.*

Proof. In order to simplify the computations we let $l = \Delta+1$ and $E = n(D+t)-1$. The inequality in the statement of the theorem is then equivalent to

$$nt\left[\frac{l!}{(l-D)!}\right]^n < \frac{l!}{(l-E)!}$$

We begin with proving that for all sufficiently large l we have:

$$\left(\frac{l}{l-D}\right)^{ln} < \left(\frac{l}{l-E}\right)^{l}. \tag{1}$$

Indeed, it can be checked that (1) is equivalent with

$$-El^{n-1} < -nl^{n-1}D + g(l)\,,$$

where $g(l)$ is a polynomial in the variable l of degree $n-2$, and the last inequality is obviously valid for all sufficiently large l.

Next we show that for all sufficiently large l the following inequality holds:

$$(2\pi l)^{\frac{n-1}{2}}(l-D)^{Dn}\sqrt{2\pi(l-E)nt} < e^{Dn-E}(l-E)^{E}\sqrt{2\pi(l-D)^n}\,. \tag{2}$$

Asymptotically, the dominant terms on the left-hand side and the right-hand side of (2) are the $(Dn+\frac{n-1}{2}+\frac{1}{2})$-th and the $(E+\frac{n}{2})$-th power of l, respectively. Thus, the validity of (2) for sufficiently large l is automatic if $Dn < E$, which is obviously satisfied.

Let $A = (l - D)$, $B = (l - E)$, and $C = 2\pi l$. Combining (1) and (2) we have:

$$\frac{\left(\frac{l}{A}\right)^{ln}}{\left(\frac{l}{B}\right)} < 1 < \frac{1}{nt} \cdot C^{\frac{1-n}{2}} \cdot e^{Dn-E} \cdot \frac{B^E}{A^{Dn}} \cdot \frac{[2\pi A]^{\frac{n}{2}}}{[2\pi B]^{\frac{1}{2}}}\,.$$

Because of the fact that

$$\lim_{l\to\infty} \frac{\left(\frac{l}{A}\right)^{ln}}{\left(\frac{l}{B}\right)} \le 1 \quad \text{and} \quad \lim_{l\to\infty} \frac{1}{nt} \cdot C^{\frac{1-n}{2}} \cdot e^{Dn-E} \cdot \frac{B^E}{A^{Dn}} \cdot \frac{[2\pi A]^{\frac{n}{2}}}{[2\pi B]^{\frac{1}{2}}} = \infty$$

we obtain

$$\lim_{l\to\infty} \frac{\frac{\left(\frac{l}{A}\right)^{ln}}{\left(\frac{l}{B}\right)}}{\frac{1}{nt} \cdot C^{\frac{1-n}{2}} \cdot e^{Dn-E} \cdot \frac{B^E}{A^{Dn}} \cdot \frac{[2\pi A]^{\frac{n}{2}}}{[2\pi B]^{\frac{1}{2}}}} = 0$$

which is equivalent to

$$\lim_{l\to\infty} \frac{nt\left[\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C}}{\left(\frac{A}{e}\right)^{A} \cdot \sqrt{2\pi A}}\right]^{n}}{\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C}}{\left(\frac{B}{e}\right)^{B} \cdot \sqrt{2\pi B}}} = 0\,.$$

To be able to use Stirling's formula, we rewrite the above as follows:

$$\lim_{l\to\infty} \frac{nt\left[\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C} \cdot \frac{l!}{l!}}{\left(\frac{A}{e}\right)^{A} \cdot \sqrt{2\pi A} \cdot \frac{A!}{A!}}\right]^{n}}{\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C} \cdot \frac{l!}{l!}}{\left(\frac{B}{e}\right)^{B} \cdot \sqrt{2\pi B}\frac{B!}{B!}}} = 0\,. \tag{3}$$

From Stirling's formula we infer that for all sufficiently large l we have

$$\frac{1}{2} < \frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C}}{l!} < 2; \quad \frac{\left(\frac{A}{e}\right)^{A} \cdot \sqrt{2\pi A}}{A!} < 2 \quad \text{and} \quad \frac{\left(\frac{B}{e}\right)^{B} \cdot \sqrt{2\pi B}}{B!} > \frac{1}{2}\,.$$

Using these inequalities in combination with (3) gives

$$\begin{aligned} 0 &= \lim_{l\to\infty} \frac{nt\left[\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C} \cdot \frac{l!}{l!}}{\left(\frac{A}{e}\right)^{A} \cdot \sqrt{2\pi A} \cdot \frac{A!}{A!}}\right]^{n}}{\frac{\left(\frac{l}{e}\right)^{l} \cdot \sqrt{C} \cdot \frac{l!}{l!}}{\left(\frac{B}{e}\right)^{B} \cdot \sqrt{2\pi B}\frac{B!}{B!}}} \ge \lim_{l\to\infty} \frac{nt \cdot \left(\frac{\frac{l!}{2}}{2 \cdot A!}\right)^{n}}{\frac{2l!}{\frac{B!}{2}}} \\ &= \lim_{l\to\infty} \frac{1}{4^{n+1}} \cdot \frac{nt \cdot \left(\frac{l!}{A!}\right)^{n}}{\frac{l!}{B!}} \ge 0\,. \end{aligned}$$

Since n is constant, from this above chain of inequalities we have:

$$\lim_{l\to\infty} \frac{nt \cdot \left(\frac{l!}{A!}\right)^n}{\frac{l!}{B!}} = 0\,.$$

It follows that for all sufficiently large l we have $nt(n!/A!)^n < l!/B!$, which is equivalent to

$$nt\left[\frac{l!}{(l-D)!}\right]^n < \frac{l!}{(l-E)!}$$

which is exactly the inequality we wanted to prove. □

It follows that, asymptotically for a fixed diameter, the order of the digraphs arising from the construction of Comellas and Fiol applied to the graphs of Faber and Moore is smaller than the order of the plain Faber-Moore digraphs (for the accordingly amended diameter) for all sufficiently large degrees.

Acknowledgment

The authors acknowledge support by the APVV Research Grants No. 0040-06 and the LPP Research Grant No. 0145-06, as well as partial support from the VEGA Research Grant No. 1/0489/08.

References

[1] F. Commellas and M. A. Fiol, *Vertex-symmetric digraphs with small diameter*, Discrete Applied Mathematics 58, 1–11, 1995.

[2] V. Faber and J. W. Moore, *High-degree low-diameter interconnection networks with vertex symmetry: the directed case*, Technical Report LA-UR-88-1051, Los Alamos National Laboratory, Los Amalos, NM 1988.

[3] J. Gómez, *Large vertex-symmetric digraphs, Networks* 50 (4), 241–250, 2007.

[4] M. Miller and J. Širáň, *Moore graphs and beyond: A survey*, Electron. J. Combin., Dynamic Survey DS 14 (published on-line in December 2005), 61 pp, 2005.

Author(s) Address(es):
Department of Mathematics, Slovak University of Technology,
Faculty of Civil Engineering, Bratislava, Slovak Republic

E-mail Address: zdimalova@math.sk, ls@math.sk