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# On a Generalization of Helmholtz Conditions

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**Abstract.** Helmholtz conditions in the calculus of variations are necessary and sufficient conditions for a system of differential equations to be variational 'as it stands'. It is known that this property geometrically means that the dynamical form representing the equations can be completed to a closed form. We study an analogous property for differential forms of degree 3, so-called Helmholtz-type forms in mechanics ($n = 1$), and obtain a generalization of Helmholtz conditions to this case.

## 1 Introduction

This article is a contribution to the study of properties of morphisms in the variational sequence. The variational sequence, introduced by Krupka in 1989 [8], is a quotient sequence of the De Rham sequence, such that one of the morphisms is the Euler-Lagrange mapping of the calculus of variations, assigning to a Lagrangian its Euler-Lagrange form. The idea of the variational sequence reflects and further extends a close relationship beetwen the Euler-Lagrange mapping and the exterior derivative operator, observed earlier by Lepage [17] and Dedecker [3]. The Euler-Lagrange morphism has been recently extensively investigated, and its properies are very-well known. On the other hand, much less or almost nothing is yet known about other variational morphisms. Namely, two morphisms in the sequence, the first one mapping dynamical forms (possibly Euler-Lagrange forms) to Helmholtz forms, and the next one, are very interesting, since they distinguish variational equations from the non-variational ones, and it seems that they could be used to study non-variational equations by variational techniques (see e.g. [12]). It is known that the kernel of the Helmholtz mapping is the family of differential $(n + 1)$-forms which come from a Lagrangian as its Euler-Lagrange forms. The study of the kernel then provides the solution of the so-called Covariant Inverse Problem of the Calculus of Variations: necessary and sufficient conditions for a dynamical form be variational, called Helmholtz conditions if $n = 1$. In this paper we

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study for the case of one independent variable ($n = 1$, ordinary differential equations) properies of the morphism from the third column of the first order variational sequence (Helmholtz-type forms) to 4-forms. The result is an explicit characterization of the kernel, we obtain a generalization of the Helmholtz conditions to this case. We also construct a Lepage equivalent of a Helmholtz form which is a closed 3-form. This new form represents non-variational equations, similarly as the famous Cartan 2-form (or a symplectic form) represents variational equations.

In Section 2 we recall basic structures and notations: for more details we refer to [6], [7], [14] or [18]. In Section 3 we briefly introduce the variational sequence, according to [8], [9]. In Section 4 we present the theorem which is an important result for variational sequence and the inverse problem of the calculus of variations [13], [14]. We ask a question on a possible generalization of this result to the third column of the variational sequence. We show that a Helmholtz-like form can be completed to a closed form if and only if it is a Helmholtz form, and find the corresponding necessary and sufficient conditions explicitly.

## 2 Differential Forms in Jet Bundles

Throughout this article, manifolds and mappings are smooth, summation over repeated indices is always assumed.

$Y$ is a fibred manifold with base $X$ and projection $\pi\colon Y \to X$ where $\dim X = 1$ and $\dim Y = m+1$, $m > 0$. A mapping $\gamma\colon W \to Y$, where $W$ is an open subset of $X$, is called a *section* of the manifold $\pi\colon Y \to X$ if $\pi \circ \gamma = \mathrm{id}_W$. Two sections $\gamma_1, \gamma_2$ defined on an open set $W \subset X$ are called *$r$-equivalent* at a point $t \in W$ if $\gamma_1(t) = \gamma_2(t)$, and if there is a fiber chart around $\gamma_1(t) = \gamma_2(t)$ such that the derivatives of the components of the sections $\gamma_1$ and $\gamma_2$ at the point $t$ coincide up to the order $r$. The equivalence class containing a section $\gamma$ is called the *$r$-jet* of $\gamma$ at $t$ and is denoted by $J^r_t\gamma$. Denote by $J^r_tY$ the set of all $r$-jets at $t$ and put $J^rY = \bigcup_t J^r_tY$, $t \in X$. Projection $\pi_r\colon J^rY \to X$ has the structure of a smooth manifold and it is called *$r$-jet prolongation* of $\pi$. Fibred projection of $J^rY$ onto $J^kY$, $0 \le k \le r-1$, are denoted by $\pi_{r,k}$. The mapping $t \to J^r_t\gamma$ is a section of $\pi_r$ and it is called the *$r$-jet prolongation* of the section $\gamma$ and denoted by $J^r\gamma$. A section $\delta$ of $\pi_r$ is called *holonomic* if there exists a section $\gamma$ of $\pi$ such that $\delta = J^r\gamma$. Fibred coordinates on $Y$ are denoted by $(t, q^i)$, $1 \le i \le m$, associated coordinates on $J^rY$ are denoted by $(t, q^i_k)$, $1 \le i \le m$, $0 \le k \le r$. We usually use the notation $q^i_0 = q^i$, $q^i_1 = \dot{q}^i$, $q^i_2 = \ddot{q}^i$, $q^i_3 = \dddot{q}^i$.

A vector field $\xi$ on $J^rY$ is called *$\pi_r$-vertical* if $T\pi_r \cdot \xi = 0$, and *$\pi_r$-projectable* if there exist a vector field $\xi_0$ on the base $X$ such that $T\pi_r \cdot \xi = \xi_0 \circ \pi_r$. In local coordinates, projectable vector fields have their $\partial/\partial t$ component dependent on $t$ only, and vertical vector fields have this component equal to zero.

Let $\Lambda^q(J^rY)$, $q \ge 0$, denote the module of smooth $q$-forms on $J^rY$ over the ring of functions (for $q = 0$ we have smooth functions on $J^rY$). A form $\eta \in \Lambda^q(J^rY)$, is called *$\pi_r$-horizontal* if $i_\xi\eta = 0$ for every $\pi_r$-vertical vector field $\xi$ on $J^rY$. A form $\eta \in \Lambda^q(J^rY)$, is called *$\pi_{r,k}$-horizontal*, $0 \le k < r$, if $i_\xi\eta = 0$ for every $\pi_{r,k}$-vertical vector field $\xi$ on $J^rY$. The module of $\pi_r$-horizontal (resp. of $\pi_{r,k}$-horizontal) $q$-forms on $J^rY$ is a submodule of $\Lambda^q(J^rY)$ and is denoted by $\Lambda^q_X(J^rY)$ (resp. $\Lambda^q_{J^kY}(J^rY)$). We get that $\eta$ is $\pi_r$-horizontal if and only if in coordinates it is represented by

$\eta = f\,\mathrm{d}t$, where $f = f(t, q^i, \dots, q^i_r)$, $\pi_{r,k}$-horizontal $q$-form $\eta$ is expressed by means of $\mathrm{d}t, \mathrm{d}q^i, \dots, \mathrm{d}q^i_k$ only, with the components depend on all the $t, q^i, \dots, q^i_k$.

Let $\eta \in \Lambda^q(J^rY)$. There is a unique horizontal form $h\eta \in \Lambda^q(J^{r+1}Y)$ such that for every section $\gamma$

$$J^r\gamma^*\eta = J^{r+1}\gamma^* h\eta\,.$$

The mapping $h\colon \Lambda^q(J^rY) \to \Lambda^q(J^{r+1}Y)$ is homomorphism of the exterior algebras and is called *horizontalization operator*. In particular,

$$h\,\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t, \quad \text{where} \quad \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial \dot{q}^i}\ddot{q}^i + \dots + \frac{\partial f}{\partial q^i_r}q^i_{r+1}\,.$$

Let now $r \geq 1$. A form $\eta \in \Lambda^q(J^rY)$ is called *contact* if

$$J^r\gamma^*\eta = 0$$

for every section $\gamma$ of $\pi$. The form $\eta$ is contact if and only if $h\eta = 0$. On our fibred manifolds every $q$-form for $q \geq 2$ is contact. Contact forms form a closed ideal in the exterior algebra on $J^rY$ locally generated by the 1-forms

$$\omega_0 = \mathrm{d}t, \quad \omega^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t, \quad \dot{\omega}^i = \mathrm{d}\dot{q}^i - \ddot{q}^i\,\mathrm{d}t, \quad \dots, \quad \omega^i_{r-1} = \mathrm{d}q^i_{r-1} - q^i_r\,\mathrm{d}t,$$

and their exterior derivatives.

Let $q \geq 1$ and let $\eta \in \Lambda^q(J^rY)$ be a contact form. We say that $\eta$ is 1-*contact* if for every $\pi_r$-vertical vector field $\xi$ on $J^rY$ the $(q-1)$-form $i_\xi\eta$ is $\pi_r$-horizontal. We say that $\eta$ is *k-contact*, $2 \leq k \leq q$, if $i_\xi\eta$ is $(k-1)$-contact. The following theorem describes the structure of differential forms on fibred manifolds.

**Theorem 1.** [6] *Every $q$-form $\eta$ on $J^rY$, $r \geq 0$, admits the unique decomposition*

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta + \dots + p_q\eta$$

*where $p_i\eta$ is a $i$-contact $q$-form on $J^{r+1}Y$, $1 \leq i \leq q$.*

The form $p_i\eta$ is called *$i$-contact part* of $\eta$. We shall consider operators

$$p_i\colon \Lambda^q(J^rY) \to \Lambda^q(J^{r+1}Y)\,,$$

$1 \leq i \leq q$, assigning to every form its $i$-contact part. Since we consider the base manifold one dimensional, we have $p_i\eta = 0$ for $i < q-1$. A contact $q$-form is called *strongly contact* if $\pi^*_{r+1,r}\eta = p_q\eta$.

Contact 1-forms can be completed to a basis of linear forms that is well adapted to the fibred structure. In what follows, we shall often use for a local expression of forms on $J^1Y$ the adapted basis $(\mathrm{d}t, \omega^i, \mathrm{d}\dot{q}^i)$ instead of the canonical basis $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}\dot{q}^i)$, and similarly, for forms on $J^2Y$, the adapted basis $(\mathrm{d}t, \omega^i, \dot{\omega}^i, \mathrm{d}\ddot{q}^i)$ instead of $(\mathrm{d}t, \mathrm{d}q^i, \mathrm{d}\dot{q}^i, \mathrm{d}\ddot{q}^i)$.

The basic objects for the calculus of variations are horizontal 1-forms on $J^rY$, called *Lagrangians* of order $r$, and 1-contact 2-forms horizontal with respect to the projection onto $Y$, called *dynamical forms*. In every fibred chart a Lagrangian $\lambda \in \Lambda^1(J^rY)$ and a dynamical form $E \in \Lambda^2(J^rY)$ take the form

$$\lambda = L\,\mathrm{d}t, \quad L = L(t, q^j, \dot{q}^j, \dots, q^j_r),$$

and

$$E = E_i\omega^i \wedge \mathrm{d}t, \quad E_i = E_i(t, q^j, \dot{q}^j, \dots, q^j_r)\,.$$

A special case is a dynamical form $E_\lambda$ associated with a Lagrangian $\lambda$, called the *Euler-Lagrange form* of $\lambda$. If $\lambda$ is of order $r$ then $E_\lambda$ is of order $\leq 2r$, and its components $E_i(L)$, called *Euler-Lagrange expression*, are defined by

$$E_i(L) = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} + \cdots + (-1)^r \frac{\mathrm{d}^r}{\mathrm{d}t^r}\frac{\partial L}{\partial q^i_r}\,.$$

The mapping $\Lambda^1(J^rY) \ni \lambda \to E_\lambda \in \Lambda^2(J^{2r}Y)$ is called the *Euler-Lagrange mapping*.

## 3 The Variational Sequence

A general framework for our exposition is the *variational sequence* [8], [9].

Let $\Omega^r_{0,c} = \{0\}$, and let $\Omega^r_{p,c}$ be the sheaf of contact $p$-forms, if $p \leq n$, or the sheaf of strongly contact $p$-forms, if $p > n$, on $J^rY$. Set

$$\Theta^r_p = \Omega^r_{p,c} + \mathrm{d}\Omega^r_{p-1,c}\,,$$

where $\mathrm{d}\Omega^r_{p-1,c}$ is the image sheaf of $\Omega^r_{p-1,c}$ by the exterior derivative d. We get an exact sequence of soft sheaves

$$0 \longrightarrow \Theta^r_1 \longrightarrow \Theta^r_2 \longrightarrow \Theta^r_3 \longrightarrow \cdots\,,$$

where the morphisms are the exterior derivative, i.e., a subsequence of the De Rham seguence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1 \longrightarrow \Omega^r_2 \longrightarrow \Omega^r_3 \longrightarrow \cdots\,.$$

The quotient sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \longrightarrow \Omega^r_1/\Theta^r_1 \longrightarrow \Omega^r_2/\Theta^r_2 \longrightarrow \Omega^r_3/\Theta^r_3 \longrightarrow \cdots$$

is also exact. It is called the *variational sequence* of order $r$ on $\pi$, see Figure 1. The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$ over $Y$. The quotient sheaves $\Omega^r_p/\Theta^r_p$ are not forms, but classes of (local) $p$-forms (of order $r$). We denote by

$$\mathcal{E}_p\colon \Omega^r_p/\Theta^r_p \to \Omega^r_{p+1}/\Theta^r_{p+1}$$

the quotient mapping. The class of a form $\rho \in \Omega^r_p$ is denoted by $[\rho]$. Hence $\mathcal{E}_p([\rho]) = [\mathrm{d}\rho]$.

The quotient mapping

$$\mathcal{E}_1\colon \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2$$

then identifies with the *Euler-Lagrange mapping*. The quotient mapping

$$\mathcal{E}_2\colon \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3$$

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & & & & \\
 & & 0 \arrow[r] & \Theta^r_1 \arrow[r, "\mathrm{d}"] \arrow[d] & \Theta^r_2 \arrow[r, "\mathrm{d}"] \arrow[d] & \Theta^r_3 \arrow[r, "\mathrm{d}"] \arrow[d] & \dots \arrow[r, "\mathrm{d}"] & \Theta^r_I \arrow[r, "\mathrm{d}"] \arrow[d] & 0 & & & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^r_0 \arrow[r, "\mathrm{d}"] \arrow[dr, "\mathcal{E}_0"'] & \Omega^r_1 \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega^r_2 \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega^r_3 \arrow[r, "\mathrm{d}"] \arrow[d] & \dots \arrow[r, "\mathrm{d}"] & \Omega^r_I \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega^r_{I+1} \arrow[r, "\mathrm{d}"] & \dots \arrow[r, "\mathrm{d}"] & \Omega^r_N \arrow[r, "\mathrm{d}"] & 0 \\
 & & & \Omega^r_1/\Theta^r_1 \arrow[r, "\mathcal{E}_1"] \arrow[d] & \Omega^r_2/\Theta^r_2 \arrow[r, "\mathcal{E}_2"] \arrow[d] & \Omega^r_3/\Theta^r_3 \arrow[r, "\mathcal{E}_3"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{I-1}"] & \Omega^r_I/\Theta^r_I \arrow[d] \arrow[ur, "\mathcal{E}_I"'] & & & & \\
 & & & 0 & 0 & 0 & & 0 & & & &
\end{tikzcd}
$$

**Figure 1** Variational sequence

is called the *Helmholtz mapping*. The image of a class $[\rho] \in \Omega_2^r/\Theta_2^r$, i.e., the class $[\mathrm{d}\rho] \in \Omega_3^r/\Theta_3^r$ is called *Helmholtz class*.

By exactness of the variational sequence, the condition $\mathcal{E}_1([\rho]) = 0$ means that there exists $f \in \Omega_0^r$ such that $[\rho] = [\mathrm{d}f]$. Hence, we get a (local) function $f$, such that $\mathcal{E}_1([\mathrm{d}f]) = 0$. In other words, the class $[\mathrm{d}f]$ has the meaning of a *null Lagrangian*. The condition $\mathcal{E}_2([\alpha]) = 0$ gives us a class $[\rho] \in \Omega_1^r/\Theta_1^r$ such that $[\alpha] = [\mathrm{d}\rho] = \mathcal{E}_1([\rho])$, i.e., $[\alpha]$ is the image by the Euler-Lagrange mapping of a class $[\rho]$. Thus, condition $\mathcal{E}_2([\alpha]) = [\mathrm{d}\alpha] = 0$ means that $[\alpha]$ is locally variational.

Classes in the variational sequence can be canonicly represented by *source forms* [1], [11], [19]. In the first column the classes are represented by horizontal forms, i.e., Lagrangians. In the second column the classes are represented by dynamical forms and in the third column the classes are represented by forms of Helmholtz type. The corresponding morphisms then take the form $\mathcal{E}_1 \colon \lambda \to E_\lambda$, $\mathcal{E}_2 \colon E \to H_E$, where $\lambda$ is a Lagrangian, $E_\lambda$ is the Euler-Lagrange form of $\lambda$, $E$ is a dynamical form, $H_E$ is a Helmholtz form of $E$. As shown in [10] we have for Helmholtz form $H_E$ the following formula

$$\begin{aligned} H_E = {} & \frac{1}{2}\left(\frac{\partial E_i}{\partial q^j} - \frac{\partial E_j}{\partial q^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_i}{\partial \dot q^j} - \frac{\partial E_j}{\partial \dot q^i}\right)\right)\omega^j \wedge \omega^i \wedge \mathrm{d}t \\ & + \frac{1}{2}\left(\frac{\partial E_i}{\partial \dot q^j} + \frac{\partial E_j}{\partial \dot q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_i}{\partial \ddot q^j} + \frac{\partial E_j}{\partial \ddot q^i}\right)\right)\dot\omega^j \wedge \omega^i \wedge \mathrm{d}t \\ & + \frac{1}{2}\left(\frac{\partial E_i}{\partial \ddot q^j} - \frac{\partial E_j}{\partial \ddot q^i}\right)\ddot\omega^j \wedge \omega^i \wedge \mathrm{d}t\,. \end{aligned}$$

A different representation of classes in the variational sequence is realized by so-called *Lepage forms* [11]. A $q$-form $\rho$, $q \geq 1$, is called *Lepage form* if $p_q\,\mathrm{d}\rho$ is a source form. If $\sigma$ is a source $q$-form, we say that $\rho$ is a *Lepage equivalent* of $\sigma$ if $\rho$ is a Lepage $q$-form and $p_{q-1}\rho = \sigma$. A *Lepage equivalent of Lagrangian* $\lambda$ is defined to be a 1-form $\rho$ such that $h\rho = \lambda$ and $p_1\,\mathrm{d}\rho$ is horizontal with respect to the projection onto $Y$ [7]. A direct computation then gives that every Lagrangian has a unique Lepage equivalent. It is denoted by $\theta_\lambda$ and called the *Cartan form*. In fibred coordinates (if $\lambda \in \Lambda^1(J^rY)$) one gets

$$\theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot q^i}\omega^i.$$

Note that by definition $p_1\,\mathrm{d}\theta_\lambda = E_\lambda$.

## 4 Closed Equivalents of Helmholtz Forms

In what follows, we consider a fibred manifold $\pi \colon Y \to X$, $\dim X = 1$, $\dim Y = m$, and its jet prolongations up to the third order, with fibred coordinates denoted $(t, q^i, \dot q^i, \ddot q^i, \dddot q^i)$, and with the contact structure generated by contact forms

$$\omega^i = \mathrm{d}q^i - \dot q^i\,\mathrm{d}t, \quad \dot\omega^i = \mathrm{d}\dot q^i - \ddot q^i\,\mathrm{d}t, \quad \ddot\omega^i = \mathrm{d}\ddot q^i - \dddot q^i\,\mathrm{d}t\,.$$

A second order dynamical form

$$E = E_i\,\mathrm{d}q^i \wedge \mathrm{d}t$$

(where summation runs over $i = 1, \ldots, m$) is *variational*, i.e., $E$ is an *Euler--Lagrange form* of a Lagrangian $L$, if and only if the components $E_i(t, q^j, \dot{q}^j, \ddot{q}^j)$ satisfy the famous *Helmholtz conditions* [4]

$$\frac{\partial E_i}{\partial \ddot{q}^j} - \frac{\partial E_j}{\partial \ddot{q}^i} = 0, \quad \frac{\partial E_i}{\partial \dot{q}^j} + \frac{\partial E_j}{\partial \dot{q}^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_i}{\partial \ddot{q}^j} + \frac{\partial E_j}{\partial \ddot{q}^i}\right) = 0,$$
$$\frac{\partial E_i}{\partial q^j} - \frac{\partial E_j}{\partial q^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_i}{\partial \dot{q}^j} - \frac{\partial E_j}{\partial \dot{q}^i}\right) = 0\,.$$

In this case the corresponding (second order ordinary) differential equations

$$E_i(t, q^j(t), \dot{q}^j(t), \ddot{q}^j(t)) = 0, \quad 1 \le i \le m,$$

are Euler-Lagrange equations, i.e.,

$$E_i = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\,.$$

It is known that variationality is *equivalent* with the possibility to complete the dynamical form $E$ to a *closed form* (see [2], [5], [6], [13], and [14], [16]). We have the following result characterizing variational dynamical forms:

**Theorem 2.** [13], [14] *Let $E$ be a dynamical form. The following conditions are equivalent:*

(i) *$E$ is locally variational, i.e., around every point, $E$ is the Euler-Lagrange form of a Lagrangian $L$.*

(ii) *The components of $E$ satisfy the Helmholtz conditions.*

(iii) *There exists a unique 2-contact form $F$ such that the form $\alpha = E + F$ is closed.*

(iv) *There exists a unique 2-contact form $F$ such that $p_2\,\mathrm{d}\alpha = p_2\,\mathrm{d}(E + F) = 0$.*

The 2-form in the above theorem is explicitly expressed by means of the dynamical form $E$. For example, if $E$ is a second order dynamical form then

$$F = \frac{1}{4}\left(\frac{\partial E_i}{\partial \dot{q}^j} - \frac{\partial E_j}{\partial \dot{q}^i}\right)\omega^i \wedge \omega^j + \frac{1}{2}\left(\frac{\partial E_i}{\partial \ddot{q}^j} + \frac{\partial E_j}{\partial \ddot{q}^i}\right)\omega^i \wedge \dot{\omega}^j.$$

The proof of the above theorem gives also necessary and sufficient conditions for a dynamical form be locally variational (the Helmholtz conditions) [4].

The 2-form $\alpha$ is defined on $J^1Y$, and explicitly expressed by means of the dynamical form $E$. It holds

$$\alpha = E_i\omega^i \wedge dt + \frac{1}{4}\left(\frac{\partial E_i}{\partial \dot{q}^j} - \frac{\partial E_j}{\partial \dot{q}^i}\right)\omega^i \wedge \omega^j + \frac{1}{2}\left(\frac{\partial E_i}{\partial \ddot{q}^j} + \frac{\partial E_j}{\partial \ddot{q}^i}\right)\omega^i \wedge \dot{\omega}^j.$$

The aim of this paper is to generalize the above theorem from dynamical forms to forms of degree 3, namely to so-called *Helmholtz-type forms*, i.e. to the third column of the variational sequence. Helmholtz-type forms are 2-contact 3-forms, belonging to the ideal generated by the contact forms $\omega^i$, $1 \le i \le m$.

Behind Lagrangians and Euler-Lagrange forms, Helmholtz-type forms are other important differential forms appearing in the calculus of variations (see [8], [12], [15]). In particular, to every dynamical form $E$ one assigns a 3-form of this kind, called *Helmholtz form*, $H_E$, with the property that *$E$ is variational if and only if $H_E$ vanishes* [8], [10].

The following theorem shows under what conditions a Helmholtz-type form $H$ can be completed to a closed form $\beta$. We find necessary and sufficient conditions for existence of a closed counterpart of $H$, generalizing the Helmholtz conditions to 3-forms), as well as the formula for $\beta$.

We use the following notation: $\mathrm{sym}(i,j) = \frac{1}{2}(ij + ji)$, $\mathrm{asym}(i,j) = \frac{1}{2}(ij - ji)$, $\mathrm{asym}(i,j,k) = \frac{1}{6}(ijk - ikj + kij - kji + jki - jik)$.

**Theorem 3.** *Let $H$ be a Helmholtz-type form of order 3. The following conditions are equivalent:*

(i) *$H$ is locally Helmholtz, i.e., around every point, $H = H_E$ for a dynamical form $E$.*

(ii) *Components of $H$, defined by*

$$H = H^0_{ij}\,\omega^i \wedge \omega^j \wedge \mathrm{d}t + H^1_{ij}\,\omega^i \wedge \dot\omega^j \wedge \mathrm{d}t + H^2_{ij}\,\omega^i \wedge \ddot\omega^j \wedge \mathrm{d}t,$$

*where $H^0_{ij} = -H^0_{ji}$, $H^1_{ij} = H^1_{ji}$, $H^2_{ij} = -H^2_{ji}$, satisfy the conditions*

$$\frac{\partial H^2_{ij}}{\partial \dddot q^k} = 0, \qquad (1)$$

$$\frac{\partial H^0_{ij}}{\partial \dddot q^k} + \frac{1}{2}\frac{\partial H^2_{ik}}{\partial \dot q^j} = 0, \qquad (2)$$

$$\left.\left(\frac{\partial H^1_{ij}}{\partial \dddot q^k} - \frac{\partial H^2_{ij}}{\partial \ddot q^k}\right)\right|_{\mathrm{asym}(j,k)} = 0, \qquad (3)$$

$$\left.\left(\frac{\partial H^1_{ij}}{\partial \ddot q^k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{ij}}{\partial \dddot q^k}\right)\right|_{\mathrm{sym}(j,k)} = 0, \qquad (4)$$

$$\left.\left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} + \frac{1}{2}\frac{\partial H^2_{ij}}{\partial q^k} + \frac{1}{2}\frac{\partial H^2_{jk}}{\partial q^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(3\frac{\partial H^0_{ij}}{\partial \dddot q^k} + \frac{\partial H^2_{ij}}{\partial \dot q^k}\right)\right)\right|_{\mathrm{asym}(j,k)} = 0, \qquad (5)$$

$$\left.\left(\frac{\partial H^1_{ij}}{\partial \dot q^k} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{ij}}{\partial \ddot q^k} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^1_{ij}}{\partial \dddot q^k}\right)\right|_{\mathrm{asym}(j,k)} = 0, \qquad (6)$$

$$\left.\left(\frac{\partial H^0_{ij}}{\partial \dot q^k} - \frac{1}{2}\left(\frac{\partial H^1_{ik}}{\partial q^j} - \frac{\partial H^1_{jk}}{\partial q^i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} - \frac{1}{2}\frac{\partial H^2_{ij}}{\partial q^k}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^0_{ij}}{\partial \dddot q^k}\right)\right|_{\mathrm{sym}(j,k)} = 0, \qquad (7)$$

$$\left(\frac{\partial H^0_{ij}}{\partial q^k} - \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^0_{ij}}{\partial \dot q^k} + \frac{1}{3}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} + \frac{\partial H^2_{ij}}{\partial q^k}\right) - \\ - \frac{1}{3}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\partial H^0_{ij}}{\partial \dddot q^k}\right)\bigg|_{\mathrm{asym}(j,k)} = 0\,. \quad (8)$$

(iii) *There exists a unique* 3*-contact form* $G$ *such that the form* $\beta = H + G$ *is closed.*

(iv) *There exists a unique* 3*-contact form* $G$ *such that* $p_3\,\mathrm{d}\beta = p_3\,\mathrm{d}(H+G) = 0$.

*The form* $\beta$ *has the following coordinate expression:*

$$\begin{aligned}\beta = {} & H^0_{ij}\,\omega^i \wedge \omega^j \wedge \mathrm{d}t + H^1_{ij}\,\omega^i \wedge \dot\omega^j \wedge \mathrm{d}t + H^2_{ij}\,\omega^i \wedge \ddot\omega^j \wedge \mathrm{d}t + \\ & + \frac{1}{3}\left(\frac{\partial H^0_{ij}}{\partial \dot q^k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} + \frac{\partial H^2_{ij}}{\partial q^k}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^0_{ij}}{\partial \dddot q^k}\right)\omega^i \wedge \omega^j \wedge \omega^k + \\ & + \left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} - \frac{\partial H^2_{ik}}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^0_{ij}}{\partial \dddot q^k}\right)\omega^i \wedge \omega^j \wedge \dot\omega^k + \frac{\partial H^0_{ij}}{\partial \dddot q^k}\,\omega^i \wedge \omega^j \wedge \ddot\omega^k + \\ & + \frac{1}{2}\left(\frac{\partial H^1_{ij}}{\partial \ddot q^k} + \frac{\partial H^2_{ij}}{\partial \dot q^k} - 2\frac{\partial H^0_{ij}}{\partial \dddot q^k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{ij}}{\partial \dddot q^k}\right)\omega^i \wedge \dot\omega^j \wedge \dot\omega^k + \\ & + \frac{\partial H^1_{ij}}{\partial \dddot q^k}\,\omega^i \wedge \dot\omega^j \wedge \ddot\omega^k\,.\end{aligned}$$

*Proof.* (i) ⇒ (ii) is proved by direct computation.

(ii) ⇒ (iii) The computations are long but standard, therefore we recall here only basic steps. Let $H$ be a source 3-form of order 3. Then in fibred coordinates $\beta$ is expressed in the form

$$\begin{aligned}\beta = H + G = {} & H^0_{ij}\,\omega^i \wedge \omega^j \wedge \mathrm{d}t + H^1_{ij}\,\omega^i \wedge \dot\omega^j \wedge \mathrm{d}t + H^2_{ij}\,\omega^i \wedge \ddot\omega^j \wedge \mathrm{d}t + \\ & + G^{000}_{ijk}\,\omega^i \wedge \omega^j \wedge \omega^k + G^{001}_{ijk}\,\omega^i \wedge \omega^j \wedge \dot\omega^k + G^{002}_{ijk}\,\omega^i \wedge \omega^j \wedge \ddot\omega^k + \\ & + G^{011}_{ijk}\,\omega^i \wedge \dot\omega^j \wedge \dot\omega^k + G^{012}_{ijk}\,\omega^i \wedge \dot\omega^j \wedge \ddot\omega^k + G^{111}_{ijk}\,\dot\omega^i \wedge \dot\omega^j \wedge \dot\omega^k + \\ & + G^{112}_{ijk}\,\dot\omega^i \wedge \dot\omega^j \wedge \ddot\omega^k + G^{022}_{ijk}\,\omega^i \wedge \ddot\omega^j \wedge \ddot\omega^k + G^{122}_{ijk}\,\dot\omega^i \wedge \ddot\omega^j \wedge \ddot\omega^k + \\ & + G^{222}_{ijk}\,\ddot\omega^i \wedge \ddot\omega^j \wedge \ddot\omega^k\,.\end{aligned}$$

We may assume $H^0_{ij} = -H^0_{ji}$, $H^1_{ij} = H^1_{ji}$, $H^2_{ij} = -H^2_{ji}$. The condition $\mathrm{d}\beta = 0$ is equivalent with $p_3\,\mathrm{d}\beta = p_4\,\mathrm{d}\beta = 0$. From $p_3\mathrm{d}\beta = 0$ we get for the components of $G$:

$$\begin{aligned}G^{000}_{ijk} &= \frac{1}{3}\left(\frac{\partial H^0_{ij}}{\partial \dot q^k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H^0_{ij}}{\partial \ddot q^k} - \frac{\partial H^2_{ik}}{\partial q^j}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial H^0_{ij}}{\partial \dddot q^k}\right)\bigg|_{\mathrm{asym}(i,j,k)} \\ G^{001}_{ijk} &= \frac{\partial H^0_{ij}}{\partial \ddot q^k} - \frac{1}{2}\left(\frac{\partial H^2_{ik}}{\partial q^j} - \frac{\partial H^2_{jk}}{\partial q^i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^0_{ij}}{\partial \dddot q^k} \\ G^{002}_{ijk} &= \frac{\partial H^0_{ij}}{\partial \dddot q^k}\end{aligned}$$

$$G^{011}_{ijk} = \frac{1}{2}\left(\frac{\partial H^1_{ij}}{\partial \ddot{q}^k} + \frac{\partial H^2_{ij}}{\partial \dot{q}^k} - 2\frac{\partial H^0_{ij}}{\partial \dddot{q}^k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H^1_{ij}}{\partial \dddot{q}^k}\right)\Bigg|_{\mathrm{asym}(j,k)}$$
$$G^{012}_{ijk} = \frac{\partial H^1_{ij}}{\partial \ddddot{q}^k}$$
$$G^{022}_{ijk} = G^{111}_{ijk} = G^{112}_{ijk} = G^{122}_{ijk} = G^{222}_{ijk} = 0$$

and for $H$'s we get conditions (1)–(8). Finally, one has to check that the relation $p_4 \mathrm{d}\beta = 0$ is fulfilled identically.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i) We have $p_3\, \mathrm{d}\beta = p_3\, \mathrm{d}(H + G) = 0$. It means that $[\mathrm{d}\beta] = 0$ and there exists $\alpha$ such that $[\mathrm{d}\alpha] = [\beta]$. From $\alpha = \alpha_E$ we obtain $H = H_E$. $\square$

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