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ON TESTING OF GENERAL RANDOM CLOSED SET MODEL HYPOTHESIS

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A new method of testing the random closed set model hypothesis (for example: the Boolean model hypothesis) for a stationary random closed set $\Xi \subseteq \mathbb{R}^d$ with values in the extended convex ring is introduced. The method is based on the summary statistics – normalized intrinsic volumes densities of the ε-parallel sets to Ξ. The estimated summary statistics are compared with theirs envelopes produced from simulations of the model given by the tested hypothesis. The p-level of the test is then computed via approximation of the summary statistics by multinormal distribution which mean and the correlation matrix is computed via given simulations. A new estimator of the intrinsic volumes densities from [6] is used, which is especially suitable for estimation of the intrinsic volumes densities of ε-parallel sets. The power of this test is estimated for planar Boolean model hypothesis and two different alternatives and the resulted powers are compared to the powers of known Boolean model tests. The method is applied on the real data set of a heather incidence.

Keywords: Boolean model, Boolean model hypothesis, contact distribution function, Euler–Poincaré characteristic, Intrinsic volumes, Laslett's transform

AMS Subject Classification: 60D05, 62G05

1. INTRODUCTION

Simulation based tests are widely spread in stochastic geometry for testing a model hypothesis. For example, the simulation based tests of random point process model hypothesis are described in [5]. When testing a random closed set model hypothesis (for example a Boolean model), it is important to choose a characteristic of the model which is able to distinguish between different models.

The characteristics which are commonly used for describing a closed set are intrinsic volumes. The intrinsic volumes $V_0(K), \ldots, V_d(K)$ of a convex body $K \subseteq \mathbb{R}^d$ are determined by the Steiner formula

$$V_d(K_\varepsilon) = \sum_{k=0}^{d} \varepsilon^k \omega_k V_{d-k}(K),$$

where V_d is the volume (d-dimensional Lebesgue measure), $K_\varepsilon = \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq \varepsilon\}$ the (closed) ε-parallel set to K and ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. (Under a different normalization, they are known as quermassintegrals or Minkowski functionals.) The intrinsic volumes can be extended additively to polyconvex sets (sets from the convex ring). For details see [12].

We consider a stationary random closed set Ξ in $\mathbb{R}^d$ with values in the extended convex ring (e.g. Ξ can be represented as a locally finite union of convex bodies). Under certain integrability condition, the *intrinsic volume densities* of Ξ can be defined as in [13] or [10] by

$$\overline{V}_k(\Xi) = \lim_{r\to\infty} \frac{\mathrm{E}V_k(\Xi \cap rB)}{V_d(rB)},$$

where B is any convex body with nonempty interior; for a detailed introduction see [13] or [14]. In the plane, $\overline{V}_2(\Xi)$ is the volume density, $\overline{V}_1(\Xi)$ is one half of the circumference density of border $\partial\Xi$ and $\overline{V}_0(\Xi)$ is the mean Euler number density.

The intrinsic volumes densities alone are not able to distinguish different models, therefore we use in our test the intrinsic volumes densities of ε-parallel sets to Ξ with varying radius ε. These can easily reflect the differences between regular – Boolean – cluster models.

There exist several methods for estimating intrinsic volumes densities introduced in [7, 8, 9, 11] and [6]. The last method is especially suitable for purposes of estimating the intrinsic volumes densities of ε-parallel sets due to the following two reasons:

1) It is an unbiased estimator of $\overline{V}_k(\Xi_\varepsilon),\ k = 0,\dots,d-1$ for $\varepsilon > \delta$, where δ depends on the estimation procedure.

2) When we estimate $\overline{V}_k(\Xi),\ k = 0,\dots,d-1$ using method described in [6], we use the approximation by estimating $\overline{V}_k(\Xi_\varepsilon),\ k = 0,\dots,d-1$ for as small $\varepsilon > 0$ as possible. But we are limited by the resolution of a discretized image, therefore the estimator of $\overline{V}_k(\Xi)$ is biased and it is shown that this bias is crucial for the mean square error of the estimator. Despite such a bias the estimator described in [6] is comparable with any other estimator, see [6]. Therefore, when the crucial bias is removed (due to the estimation of $\overline{V}_k(\Xi_\varepsilon),\ k = 0,\dots,d-1$), this estimator will have the smallest mean square error from all available estimators.

The chosen estimator of intrinsic volumes densities estimates only the values $\overline{V}_k(\Xi_\varepsilon),\ k = 0,\dots,d-1$ therefore the classical point-counting estimator is used for estimating volume density $\overline{V}_d(\Xi_\varepsilon)$.

Concerning the organization of the work, we give a short description of the available tests in Section 2. The proposed test is described in Section 3. Then we compare the powers of all tests with respect to the certain alternatives in Section 4. The real data study is performed in Section 5 on the heather incidence data which were studied first in [2]. The real data study is considered for illustrative purpose. Here a proposed test is applied to the data set to test the hypothesis that the data fit the Boolean model of discs with the parameters estimated in [2]. This hypothesis is rejected using the proposed test which confirms the conjecture given in [2], that the data set does not fit the Boolean model. The same hypothesis was not rejected in [2] by another simulation-based test. Furthermore another model is proposed which is not rejected by the proposed test.

2. TESTS DESCRIPTION

In this section we give a summary of the tests available for testing the Boolean model assumption and the general model assumption. For testing the Boolean model assumption one can use the following two procedures.

2.1. Laslett's test

For details, see [1] or [3].

Briefly:

1. Take the tangent points in a certain direction u.
2. Apply Laslett's transform on those points.
3. Test the Poisson property of the transformed points. If it is not Poisson then the Boolean model hypothesis is rejected.

Disadvantages: A considerably big part of the tangent points have to be omitted because of the dependencies between transformed points and the observation window. Clustering or regularity may be lost after the Laslett's transform, see [1], p. 769.

For testing the Poisson property we used, in the simulation study, the method using the reduced second-order moment function described in [14], p. 50-51.

2.2. Graphical test for contact distribution function

For details, see [14] or [3].

Usually the linear contact distribution function $\hat{H}_l(r)$ and the quadratic (spherical) contact distribution function $\hat{H}_q(r)$ are estimated. Then the normalized logarithm $\hat{H}_l^l(r) = -\frac{1}{r}\ln(1-\hat{H}_l(r))$ is graphically compared with a constant function. And the normalized logarithm $\hat{H}_q^l(r) = -\frac{1}{r}\ln(1-\hat{H}_q(r))$ is graphically compared with a linear function. If both normalized logarithms are graphically comparable with appropriate functions then the Boolean model is accepted.

Disadvantages: No significance level is available.

For the purposes of our simulation study and the comparison of the tests we need a clear decision algorithm whether the Boolean model is rejected or not. Since Prof. Stoyan maintains (in personal communication) that only $\hat{H}_q(r)$ carries any information about spatial structure of the random set, we chose $\hat{H}_q(r)$ only. We calculated the coefficient of determination (R^2) of $\hat{H}_q^l(r), r = 1, \dots, 24$ with respect to the linear regression model (R_Q^2) for the data. Then we performed 99 simulations of the Boolean model with parameters estimated using estimated intrinsic volume densities, like in [3], pp. 81–83 and we calculated R_Q^2 for each simulation. We reject the Boolean model hypothesis if R_Q^2 computed from the data is among 5 minimal values. (i. e. it is the simulation-based test with deviation measure R_Q^2 with $\hat{H}_q(r)$ as the summary statistic with the significance level 0.05, see next section for details.)

2.3. Simulation-based tests with deviation measure

For testing the general model assumption one can use the following procedures. The simulation-based tests with deviation measure have exact significance level, therefore they were used in a lot of applications. The simulation-based tests with envelopes have unknown significance level therefore they were used only as a diagnostic plots. We will compute the p-level of the simulation-based tests with envelopes in forthcoming section, which upgrade these procedures to the proper tests. Any simulation based test is applicable on any model assumption where we are able to estimate the model parameters.

Generally, when a simulation based test with deviation measure is performed, the following procedure is done:

1. A summary statistic $S(\varepsilon)$ of the random process is chosen. The summary statistic is a function $S(\varepsilon)$ of $\varepsilon \in \mathbb{R}^+$ and it is estimated from the data in n different points $\varepsilon_1, \dots, \varepsilon_n$. Denote these estimates $\hat{S}^D_{\varepsilon_1}, \dots, \hat{S}^D_{\varepsilon_n}$. Furthermore, the deviation measure Δ^D is computed.

2. The parameters θ of the assumed model in null hypothesis are estimated by an estimator $\hat{\theta}$.

3. N independent samples of the null model with estimated parameters are simulated and the deviation measures $\Delta^1, \dots, \Delta^N$ are computed.

4. We reject the hypothesis if Δ^D is among m maximal values from $\Delta^D, \Delta^1, \dots, \Delta^N$ with significance level $\frac{m}{N+1}$.

Possible deviation measures are

$$\Delta_{\max} = \max_{1 \le i \le n} |\hat{S}_{\varepsilon_i} - S_{\hat{\theta}, \varepsilon_i}|$$

or

$$\Delta_{\sum} = \sum_{i=1}^{n} (\hat{S}_{\varepsilon_i} - S_{\hat{\theta}, \varepsilon_i})^2,$$

where $S_{\hat{\theta}, \varepsilon}$ is the theoretical $S(\varepsilon)$ with the estimated parameter $\hat{\theta}$.

Remark. The disadvantage of the simulation-based tests is that there is tested the hypothesis of the given model with certain parameters, which are estimated from the data, and thus not only the hypothesis of the given model is tested. This imply that bad estimate of the parameters can cause the rejection of the true model hypothesis.

We found only one application of the test of this kind for random sets in the literature, see [2]. There the quadratic contact distribution function is used as the summary statistic.

In our simulation study we processed this test with $n = 24$, $N = 99$, $m = 5$ and with both deviation measures $\Delta_{\max}, \Delta_{\sum}$.

In our simulation study we tested the Boolean model assumption. The parameters of the Boolean model was estimated using empirical intrinsic volume densities

$\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)$ [3], pp. 81–83. The intrinsic volume densities $\overline{V}_0(\Xi), \ldots, \overline{V}_{d-1}(\Xi)$ are approximated by $\overline{V}_0(\Xi_{\varepsilon_1}), \ldots, \overline{V}_{d-1}(\Xi_{\varepsilon_1})$ as it is described in [6]. We used this estimation procedure in all simulation-based test described in this note.

2.4. Simulation-based tests with envelopes

The following procedure describes the simulation-based tests with envelopes and the computation of p-level via approximation of the summary statistics by the multinormal distribution.

1. A summary statistic $S(\varepsilon)$ of the random process is chosen. The summary statistic is a function $S(\varepsilon)$ of $\varepsilon \in \mathbb{R}^+$ and it is estimated from the data in n different points $\varepsilon_1, \ldots, \varepsilon_n$. Denote these estimates $\hat{S}^D_{\varepsilon_1}, \ldots, \hat{S}^D_{\varepsilon_n}$.

2. The parameters θ of the assumed null model are estimated by an estimator $\hat{\theta}$.

3. N independent samples of the model with estimated parameters are simulated and the summary statistics $\hat{S}^i_{\varepsilon_1}, \ldots, \hat{S}^i_{\varepsilon_n}$, $i = 1, \ldots, N$ are computed.

4. To compute the p-level of this test, the summary statistics $S_{\varepsilon_1}, \ldots, S_{\varepsilon_n}$ are approximated by the random vector $\mathbf{X}$ with multinormal distribution with mean vector $\mu = (\mu_1, \ldots, \mu_n)^T$ and the covariance matrix $\mathbf{\Sigma} = (\Sigma_{ij})_{i,j=1,\ldots,n}$ computed from N simulations. The upper envelope is constructed as $UE(s) = (\mu_1 + s\sqrt{\Sigma_{11}}, \ldots, \mu_n + s\sqrt{\Sigma_{nn}})^T$ and the lower envelope is constructed as $LE(s) = (\mu_1 - s\sqrt{\Sigma_{11}}, \ldots, \mu_n - s\sqrt{\Sigma_{nn}})^T$. The width of the envelopes depend on the parameter $s > 0$.

5. The integer number K is chosen. This is the maximum number of values $\hat{S}^D_{\varepsilon_1}, \ldots, \hat{S}^D_{\varepsilon_n}$ which fall outside of the envelopes and for which the test does not reject. The widest envelope when the test does not reject is then given by

$$s_{\text{sup}} = \sup_s(\#\{\hat{S}^D_{\varepsilon_i} \notin (LE(s)_i, UE(s)_i), i = 1, \ldots, n\} \leq K).$$

6. The p-value is then the probability that more than K component of the random vector $\mathbf{X}$ fall outside of the envelopes $(LE(s_{\text{sup}}), UE(s_{\text{sup}}))$. Since $\mathbf{X}$ is usually high dimensional, the probability is computed by Monte Carlo method. (e. g. The m random vectors are drawn from the estimated multinormal distribution of $\mathbf{X}$ and each vector is tested if more than K components fall outside of the envelopes. The ratio of the number of vectors satisfying the condition and m is then Monte Carlo estimate of the p-value. In our simulation study m was 200000. This ensures the width of 95 % confidence interval to be maximally 0.002.)

We implemented this test for the quadratic contact distribution function as the summary statistic with $n = 24$, $N = 99$, $K = 1$. The parameters were estimated in the same way as in all other cases.

3. PROPOSED TEST

We propose a simulation based test with envelopes for testing the general model assumption which is based on normalized intrinsic volumes densities of the ε-parallel sets as the summary statistics.

We consider the intrinsic volumes densities $\overline{V}_k(\Xi_{\varepsilon_i}),\ k = 0, \ldots, d,\ i = 1, \ldots, 24$ of the ε_i-parallel sets. The discretized version of the parallel set Ξ_{ε_i} is produced as dilation of the set Ξ by a discretized disc with a radius ε_i. The radii of the discs were determined as in [6], Section 6. We chose 24 different discs with radii $\varepsilon_1, \ldots, \varepsilon_{24}$ evenly spanned between 1 and 25 pixels of the image. Here we can see that the test is dependent on the resolution of the original image and the size of the particles, but the radii can be shrunk or spread to cover the interesting area of pixels (for example if the resolution is low, the parallel set for big radius can be whole white, thus the radii must be shrunk, or on the other side if the resolution is big, the neighbor parallel sets can be almost equal, thus the radii must be spread to cover whole range of interactions). When the envelopes are made from the simulations, the estimated intrinsic volumes densities $\overline{V}_k(\Xi_{\varepsilon_1})$ vary a lot for different simulations thus the envelopes are wide. Therefore we fixed first point and choose as summary statistics for the proposed test the normalized intrinsic volumes of parallel sets:

$$\frac{\overline{V}_k(\Xi_{\varepsilon_i})}{\overline{V}_k(\Xi_{\varepsilon_1})}, \quad i = 2, \ldots, 24, \quad k = 0, \ldots, d.$$

This normalized intrinsic volumes reflect the interactions among the particles and are unaffected by the number and size of the particles. The estimation of $\overline{V}_d(\Xi_{\varepsilon_i})$ is performed by the standard unbiased point counting estimator and the estimation of $\overline{V}_0(\Xi_{\varepsilon_i}), \ldots, \overline{V}_{d-1}(\Xi_{\varepsilon_i})$ is performed by the unbiased estimator described in [6].

We implemented this test with $n = 69$ (23 points for 3 normalized intrinsic volumes densities), $N = 99$, $K = 3$. The parameters were estimated in the same way as in all other cases.

The parameter K is arbitrary in simulation-based tests with envelopes. We chose $K = 3$ for the proposed test and $K = 1$ for the test with quadratic contact distribution function as the summary statistic in the following way. Denote $p(N, n, K)$ the probabilities, that more than K values of the summary statistic, which have n values, will lie outside of the envelopes (produced as maximum and minimum from N simulations), under the non-realistic assumption that the values of the summary statistic are independent. The probabilities $p(99, 69, 3) = 0.0497$ and $p(99, 24, 1) = 0.0826$ are the closest to the usual significance level 0.05. This choice cause that the tests with envelopes do not allow too many small deviations from the theoretical summary statistic (as the test with sum deviation function can allow) and that it is not too strict to allow no greater deviation (as the test with maximum deviation measure is).

For completeness we implemented the simulation-based tests with deviation measure with normalized intrinsic volumes of parallel sets as the summary statistics too. Since there are 3 different statistics in the plane $\frac{\overline{V}_k(\Xi_{\varepsilon})}{\overline{V}_k(\Xi_{\varepsilon_1})},\ k = 0, \ldots, 2$ we had to adapt used deviation measures.

$$\Delta_{\max} = \max_{0\le k\le 2}\left(\frac{1}{A_k}\max_{2\le i\le 24}\left|\frac{\widehat{\overline{V}}_k(\Xi_{\varepsilon_i})}{\widehat{\overline{V}}_k(\Xi_{\varepsilon_1})} - \frac{\overline{V}_{\hat{\theta},k}(\Xi_{\varepsilon_i})}{\overline{V}_{\hat{\theta},k}(\Xi_{\varepsilon_1})}\right|\right)$$

and

$$\Delta_{\sum} = \sum_{k=0}^{2}\left(\frac{1}{A_k}\sum_{i=2}^{24}\left(\frac{\widehat{\overline{V}}_k(\Xi_{\varepsilon_i})}{\widehat{\overline{V}}_k(\Xi_{\varepsilon_1})} - \frac{\overline{V}_{\hat{\theta},k}(\Xi_{\varepsilon_i})}{\overline{V}_{\hat{\theta},k}(\Xi_{\varepsilon_1})}\right)^2\right),$$

where A_k is the average deviation measure of kth summary statistic computed from all N simulations. The average here is computed from the simulations only.

In our simulation study we processed this test with $n = 3 \cdot 23$, $N = 99$, $m = 5$ and with both deviation measures $\Delta_{\max}, \Delta_{\sum}$.

4. SIMULATION STUDY

First we estimated the powers of all previously described tests with respect to certain alternatives. We chose two alternatives (cluster model and regular model). In both cases the model was tested on the Boolean model hypothesis.

4.1. Cluster model

As a representative of the cluster model we chose the germ-grain model in $\mathbb{R}^2$ where the germs form a Matérn cluster point process with the intensity $\lambda = 0.0012$. The cluster point process is constructed in two steps. First the Poisson point process of cluster centers is generated with the intensity $\alpha = \lambda/\nu$, where ν is the mean number of points per cluster. Then, in each cluster, there are generated points which number follows the Poisson distribution with parameter $\nu = 6$ and which are uniformly distributed in the disc with radius $R = 50$ pixels around the cluster center. The germs are discs with radii which follow the lognormal distribution with parameters $[\mu = 2.690, \sigma = 0.19]$. The point process of germs is the process of centers of discs. We made 50 simulations in the observation window $W = 500 \times 500$ pixels. The typical observation of this model is shown in Figure 1.

For each simulation we performed 8 tests.

1. Laslett's test.
2. The simulation-based test with deviation measure R_Q^2 with quadratic contact distribution function as the summary statistic.
3. The simulation-based test with deviation measure $\Delta_{\max}$ with quadratic contact distribution function as the summary statistic.
4. The simulation-based test with deviation measure $\Delta_{\sum}$ with quadratic contact distribution function as the summary statistic.
5. The simulation-based test with deviation measure $\Delta_{\max}$ with normalized intrinsic volumes densities of parallel sets as the summary statistics.

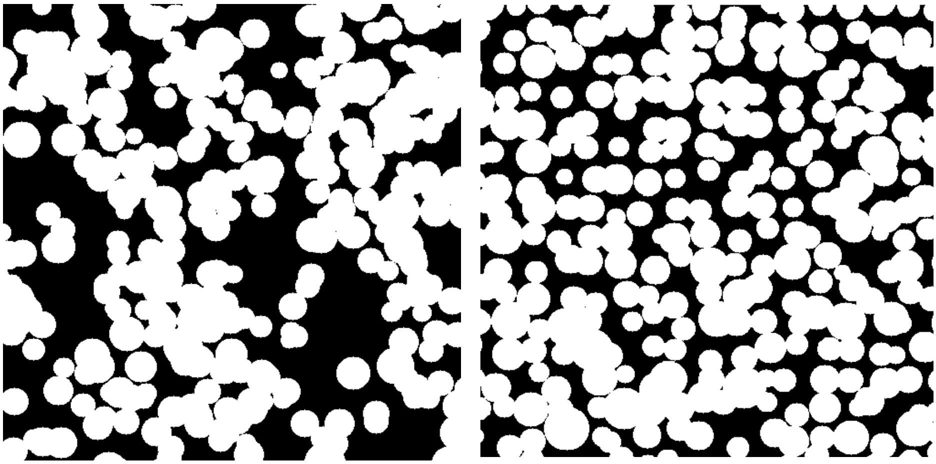

Fig. 1. Left: One observation of the chosen cluster model in the observation window $W = 500 \times 500$ pixels. Right: One observation of the chosen regular model in the observation window $W = 500 \times 500$ pixels.

6. The simulation-based test with deviation measure $\Delta_{\sum}$ with normalized intrinsic volumes densities of parallel sets as the summary statistics.
7. The simulation-based test with envelopes with quadratic contact distribution function as the summary statistic.
8. Proposed test: Simulation-based test with envelopes with normalized intrinsic volume densities of the parallel sets as the summary statistics.

For the test 1 no assumption is needed. For the simulation based tests we have to assume shape of the particles and distribution of the size of the particles. Wrong assumption can lead to a lower acceptance of the model thus we chose correct assumptions because then we can expect lower number of successive rejections of the Boolean model hypothesis for simulation-based tests. Thus only parameters (λ, μ, σ) are necessary to estimate. The approach used for the estimation of the Boolean model parameters leads to estimates of $(\lambda, \bar{A}, \bar{U})$, where $\bar{A} = \mathrm{E}A(\Xi_0)$ is the mean area of the typical grain Ξ_0 and $\bar{U} = \mathrm{E}U(\Xi_0)$ is the mean perimeter of the typical grain Ξ_0. From estimates of $(\lambda, \bar{A}, \bar{U})$ we estimated (λ, μ, σ) by comparing appropriate moments of lognormal distribution with $\bar{A}, \bar{U}$.

The numbers of successive rejections of the Boolean model hypothesis are summarized in Table 1. The proposed test 8 has the highest number of rejected simulations. To test the significance of this assertion we compare the estimated power of the proposed test with the second highest estimated power (test 2) in the following way: We assigned value 0 to the not rejected simulations and value 1 to the rejected simulations then we subtracted the values for test 2 from the values for test 8. We obtained a random sample from a distribution F and we proceed the asymptotic one sided test with the hypothesis that the expectation of F is equal or less than 0.

Table 1. The numbers of successive rejections of the Boolean model hypothesis with significance level 0.05 from 50 simulations and the mean p-values.

	Cluster	model	Regular	model
	N. reject.	Mean p	N. reject.	Mean p
1) Laslett's test	5	0.324	0	0.619
2) Qudratic CDF R^2	29	0.132	0	0.496
3) Qudratic CDF maximum	5	0.386	26	0.069
4) Qudratic CDF sum	2	0.431	22	0.074
5) Intrinsic volumes maximum	21	0.121	49	0.005
6) Intrinsic volumes sum	14	0.133	47	0.021
7) Qudratic CDF invelopes	10	0.283	31	0.051
8) Intrinsic volumes invelopes	38	0.065	49	0.008

The resulted p-level is 0.021. Thus we reject the hypothesis that both tests have the same power against the given cluster model alternative.

4.2. Regular model

As a representative of the regular model we chose the germ-grain model in $\mathbb{R}^2$ where the germs form a regular point process with the intensity $\lambda = 0.001$. The regular point process is constructed from evenly scattered points in $\mathbb{R}^2$ when each point is then shifted in random direction by a distance h. The distance h was chosen to have the uniform distribution with parameters $[0, 15]$ pixels. The germs are discs with radii which follow the lognormal distribution with parameters $[\mu = 2.6903, \sigma = 0.19]$. The point process of germs is the process of centers of discs. We made 50 simulations in the observation window $W = 500 \times 500$ pixels. The typical observation of this model is shown in Figure 1. We chose big volume fraction for the regular model because for the small volume fraction the best way to test the hypothesis would be detecting the disc centers and using the methods for point processes. Since there is little empty space around the particles of the simulated data, we used, for the dilation Ξ_ε, discretized discs with radii $\varepsilon_1, \dots \varepsilon_{24}$ evenly spanned between 1 and 12 pixels of the image. (Larger discs produce after dilation just a white rectangle.)

For each simulation we performed the same tests and the same parameter estimation procedure as for the cluster model. The numbers of successive rejections of the Boolean model hypothesis are summarized in Table 1. The proposed test 8 and all the tests based on Intrinsic volumes have the higher number of rejected simulations than the other tests. We tested the significance of this assertion in the same way as for the cluster case, we compare the estimated power of the proposed test with the estimated power of test 7. The resulted p-level is $7.6 * 10^{-8}$. Thus we reject the hypothesis that both tests have the same power against the given regular model alternative.

Table 2. The numbers of wrong rejections of the Boolean model hypothesis with significance level 0.05 from 50 simulations of Boolean model and the mean p-values.

	Boolean	model
	N. reject.	Mean p
1) Laslett's test	3	0.487
2) Qudratic CDF R^2	0	0.477
3) Qudratic CDF maximum	0	0.768
4) Qudratic CDF sum	0	0.804
5) Intrinsic volumes maximum	0	0.779
6) Intrinsic volumes sum	0	0.812
7) Qudratic CDF invelopes	0	0.819
8) Intrinsic volumes invelopes	0	0.808

4.3. Boolean model

Here, for completeness, we estimated the significance level of all studied tests.

We made 50 simulations in the observation window $W = 500 \times 500$ pixels of the Boolean model with the intensity $\lambda = 0.00018$. The germs are discs with radii which follow the lognormal distribution with parameters $[\mu = 3.383, \sigma = 0.19]$.

For each simulation we performed the same tests and the same parameter estimation procedure as for the cluster model. The numbers of wrong rejections of the Boolean model hypothesis are summarized in Table 2.

4.4. Sensitivity of the proposed test to wrong assumptions

In this subsection, we look at the sensitivity of the proposed test to

1. no wrong assumption
2. the wrong choice of the shape of the particles,
3. the wrong choice of the distribution of the particle sizes,
4. the wrong choice of the particle shapes and distribution of the particle sizes.

More specifically:

1. We estimated the significance level of the proposed test from 1000 simulation of the Boolean model with disc grains where the radii of the discs had lognormal distribution.
2. We simulated 1000 Boolean models with ellipse grains where the length a of the shorter axis of the ellipse had lognormal distribution with parameters $[\mu = 3.11, \sigma = 0.31]$ and the length of the longer axis of the ellipse was $b = aU$, where U is a random variable with uniform distribution with parameters $[1, 2]$. We tested the hypothesis that the model is Boolean with disc grains, where the radii of the discs have lognormal distribution.

Table 3. The numbers of rejections of the Boolean model hypothesis from 1000 simulations under different wrong assumptions and 95 % confidence intervals of the rejection probability produced with help of CLT.

Wrong assumption	rejections	95 % confidence intervals
1) no wrong assumption	43	(0.0295, 0.0565)
2) shape of the particles	79	(0.0655, 0.0925)
3) distribution of the particles sizes	39	(0.0255, 0.0525)
4) both	71	(0.0575, 0.0845)

3. We simulated 1000 Boolean models with disc grains where the radii of the discs had uniform distribution with parameters $[20, 40]$ pixels. We tested the hypothesis that the model is Boolean with disc grains, where the radii of the discs have lognormal distribution.

4. The same situation as in (1) but the length a of the shorter axis of the ellipse had uniform distribution with parameters $[10, 37]$.

The observation window was 500×500 pixels and the intensity was 0.00018 in all four cases. The parameters of the simulated models were set in such a way that their typical primary grains have the same mean volume and mean circumference as the model in the tested hypothesis (Boolean model with disc grains, where the radii of the discs have lognormal distribution with parameters $[\mu = 3.383, \sigma = 0.19]$). We used same envelopes for all 4000 simulations. These envelopes were constructed from 99 simulation of the model in the tested hypothesis. Thus we did not perform the parameter estimation part. This simplification was made because of the acceleration of the procedure and it could imply that the numbers of rejections was a bit higher than it would be done without applying such simplification.

The numbers of rejections of the Boolean model hypothesis with significance level 0.05 are summarized in Table 3. Furthermore, there is computed the 95 % confidence intervals for rejection probability. The number of rejections in case (1) estimates the significance level and the confidence interval contains required significance level 0.05. Furthermore the simulations showed that the wrong distribution of the radii has no influence on the result of the test and that the wrong shape of the particles has a small influence on the result of the test.

5. REAL DATA STUDY – HEATHER INCIDENCE

The heather incidence data (see Figure 2) was studied in [2]. Since individual heather plants grow into hemispherical bushes, reaching a maximum radius of about 50 cm, the Boolean model of discs was chosen in [2] as a model for the real data description. For the distribution of the discs radii the shifted Weibull distribution with parameters (0.0281, 0.8471, 144.7) was fitted. The intensity λ was estimated at 221 discs per unit area. The realization of the Boolean model fitted in [2] can be seen in Figure 3. As we can see from the realization and as was mentioned in [2]

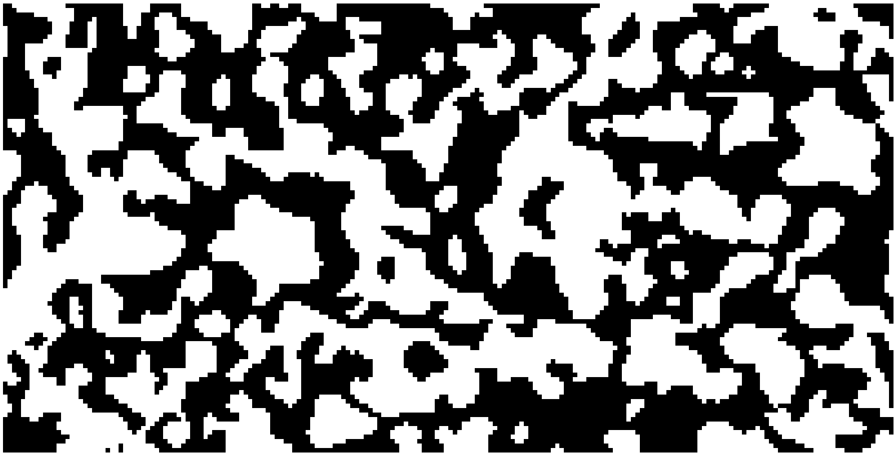

Fig. 2. The heather incidence data in a rectangular area 2×1 collected with the resolution 200×100 binary pixels. Heather is the white area.

and [1], p. 765, the Boolean model is unsatisfactory for the data description because the data contains fewer patches than the Boolean model. Nevertheless there was performed simulation based test with deviance measure which did not reject the Boolean model hypothesis in [2].

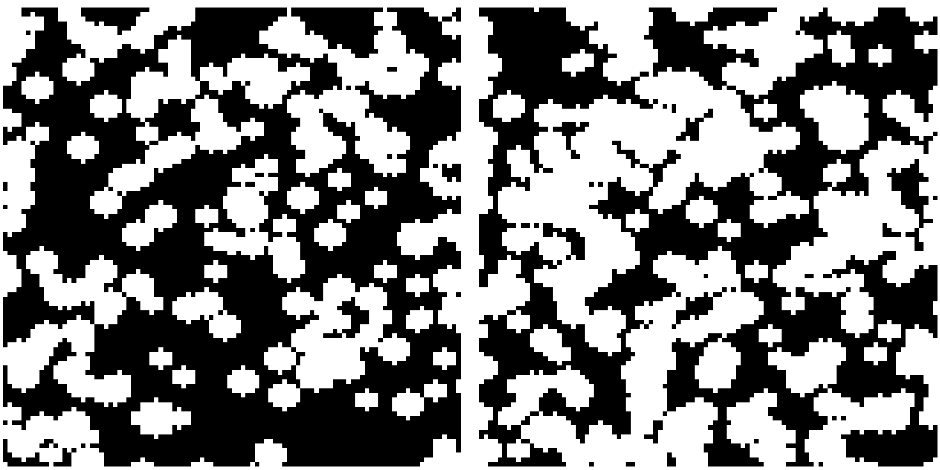

Fig. 3. Left: One realization of the Boolean model of discs with parameters fitted in [2] in the resolution 100×100 pixels. Right: One realization of the fitted Matérn Cluster model of discs in the resolution 100×100 pixels.

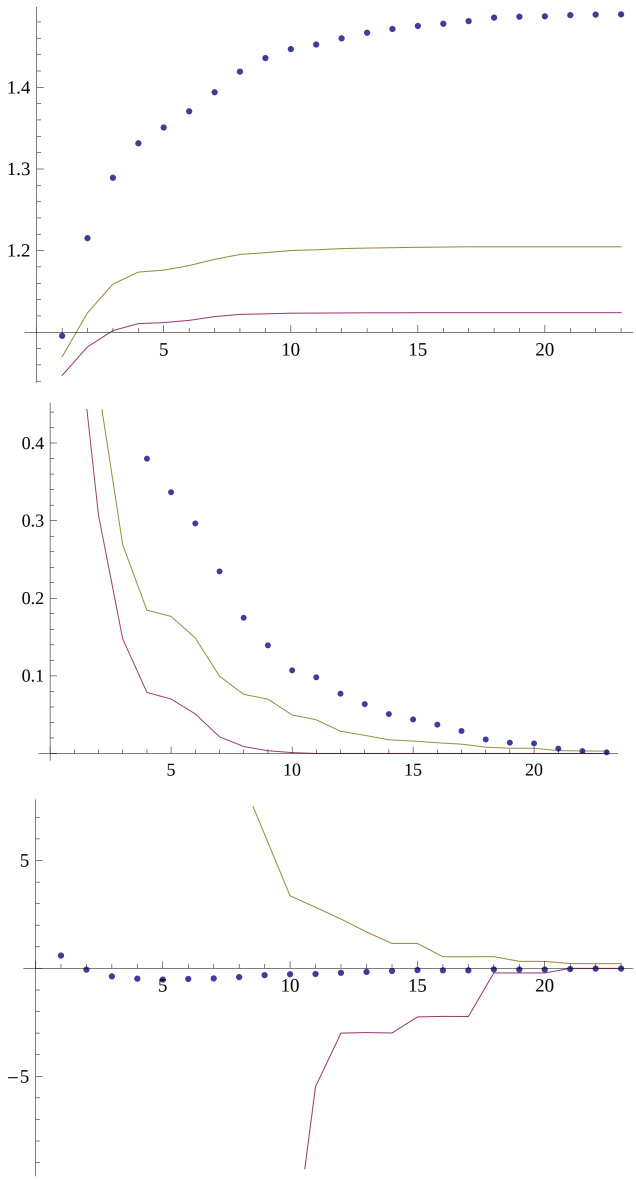

Fig. 4. The results of the proposed test on the Boolean model assumption for Heather incidence data. The points represents the estimates of normalized intrinsic volumes densities of parallel sets ($\frac{\overline{V}_2(\Xi_\varepsilon)}{\overline{V}_2(\Xi_{\varepsilon_1})}$ – upper left, $\frac{\overline{V}_1(\Xi_\varepsilon)}{\overline{V}_1(\Xi_{\varepsilon_1})}$ – upper right, $\frac{\overline{V}_0(\Xi_\varepsilon)}{\overline{V}_0(\Xi_{\varepsilon_1})}$ – bottom) for 23 different radii from the data. The envelopes are constructed from 99 simulations and they correspond to 95 % envelopes.

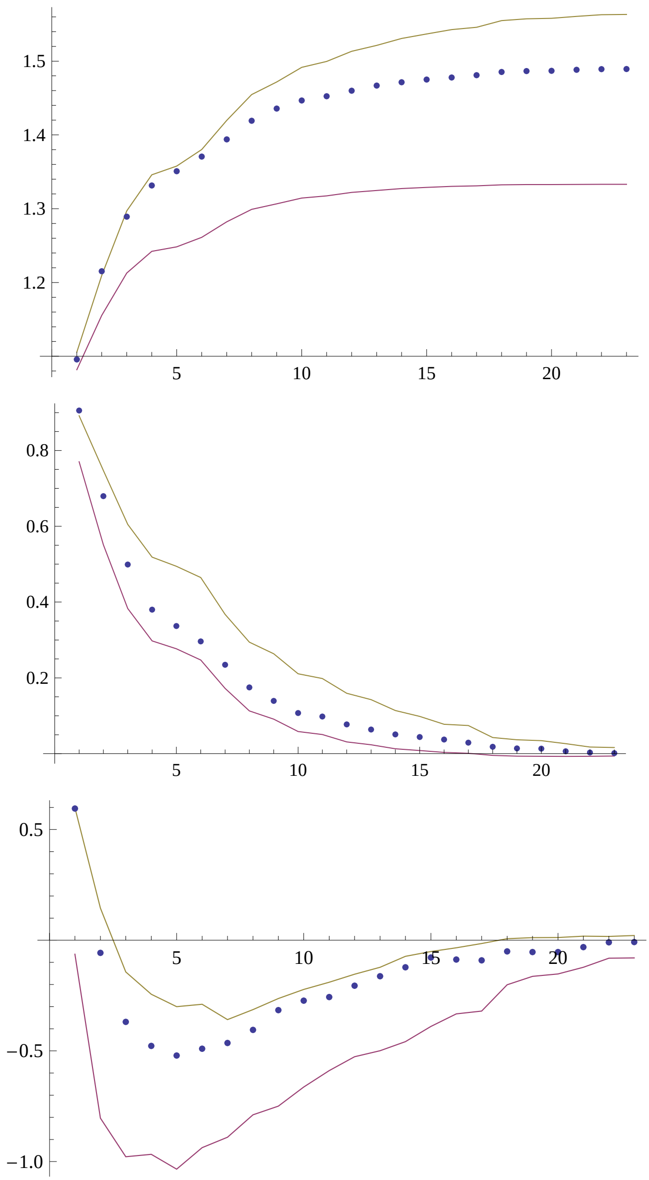

Fig. 5 The results of the proposed test on the Matérn cluster model assumption for Heather incidence data. The points represents the estimates of normalized intrinsic volumes densities of parallel sets ($\frac{\overline{V}_2(\Xi_\varepsilon)}{\overline{V}_2(\Xi_{\varepsilon_1})}$ – upper left, $\frac{\overline{V}_1(\Xi_\varepsilon)}{\overline{V}_1(\Xi_{\varepsilon_1})}$ – upper right, $\frac{\overline{V}_0(\Xi_\varepsilon)}{\overline{V}_0(\Xi_{\varepsilon_1})}$ – bottom) for 23 different radii from the data. The envelopes are constructed from 99 simulations and they correspond to 95 % envelopes.

We chose this data set to show how the proposed test works for the real data. Since the data set has the resolution 200×100, we used, for the dilation Ξ_ε, the discretized discs with radii $\varepsilon_1, \dots \varepsilon_{24}$ evenly spanned between 1 and 6 pixels of the image. (Larger discs produce after dilation just a white rectangle.) First we performed the proposed test on the Boolean model hypothesis where the Boolean model is fitted as above. As we can see from Figure 4, the Boolean model hypothesis is clearly rejected by this test with p-level 0.0034.

Thus, we tried to fit another model for the heather incidence data. We chose the Matérn cluster model (for details see [14]) for the discs centers. For the distribution of the radii we chose the same Weibull distribution as above. Thus only 3 parameters remain to be estimated, namely the intensity λ, the maximum radius of the clusters R and the mean number of discs centers in the cluster μ.

The extensive estimation procedure was recently developed in [4]. Since this work is not concerned with the estimation of the model parameters we only give an example of the parameters for which the proposed test does not reject the model assumptions. Note here that all model assumptions are: Matérn cluster disc model with parameters ($\lambda = 330$, $R = 0.035$, $\mu = 1$) with Weibull distribution of the radii with parameters (0.0281, 0.8471, 144.7). The realization of this model, shown in Figure 3, reflects a correct number of patches. And the proposed test, shown in Figure 5, does not reject the model assumptions with the p-level 0.119.

6. DISCUSSION

The comparison of the tests shows that the simulation-based tests with envelopes are more powerful than the simulation-based tests with deviation measure.

The comparison also shows that the normalized intrinsic volumes densities are more powerful summary statistics than quadratic contact distribution function.

And it also shows that the simulation-based tests with envelopes are valuable procedures even for testing the Boolean model hypothesis where other specific tests are available.

Furthermore the proposed test seems to be a sensitive tool for distinguishing the interaction among particles of the binary image. And, on the other side, the proposed test is not sensitive to the wrong assumption of the distribution of the primary grain, as was shown by simulations.

The computer programme prepared for public use is published on www.pf.jcu.cz/~mrkvicka/math.

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REFERENCES

[1] N. A. C. Cressie: Statistics for Spatial Data. Revised Edition. Wiley, New York 1997.

[2] P. J. Diggle: Binary mosaics and the spatial pattern of heather. Biometrics *37* (1981), 531–539.

[3] I. Molchanov: Statistics of the Boolean Model for Practitionars and Mathematicians. Wiley, New York 1997.

[4] J. Møller and K. Helisová: Power diagrams and interaction processes for union of discs. Adv. in Appl. Probab. (SGSA) *40* (2008), 1–27.

[5] J. Møller and R. P. Waagepetersen: Statistical Inference and Simulation for Spatial Point Processes. Chapman & Hall/CRC, London 2004

[6] T. Mrkvička and J. Rataj: On estimation of intrinsic volume densities of stationary random closed sets. Stochastic Process. Appl. *118/2* (2008), 213–231.

[7] W. Nagel, J. Ohser, and K. Pischang: An integral-geometric approach for the Euler–Poincaré characteristic of spatial images. J. Microsc. *198* (2000), 54–62.

[8] Ohser J. and F. Mücklich: Statistical Analysis of Microstructures in Materials Science. Wiley, Chichester 2000.

[9] J. Rataj: On estimation of the Euler number by projections of thin slabs. Adv. in Appl. Probab. *36* (2004), 715–724.

[10] J. Rataj: Estimation of intrinsic volumes from parallel neighbourhoods. Suppl. Rend. Circ. Mat. Palermo, Ser. II *77* (2006), 553-563.

[11] V. Schmidt and E. Spodarev: Joint estimators for the specific intrinsic volumes of stationary random sets. Stochastic Process. Appl. *115* (2005), 959–981.

[12] R. Schneider: Convex Bodies. The Brunn–Minkowski Theory. Cambridge Univ. Press, Cambridge 1993.

[13] R. Schneider and W. Weil: Stochastische Geometrie. Teubner, Stuttgart 2000.

[14] D. Stoyan, W. S. Kendall, and J. Mecke: Stochastic Geometry and Its Applications. Second edition. Wiley, Chichester 1995.

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