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A SECOND ORDER η-APPROXIMATION METHOD FOR CONSTRAINED OPTIMIZATION PROBLEMS INVOLVING SECOND ORDER INVEX FUNCTIONS

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Abstract. A new approach for obtaining the second order sufficient conditions for nonlinear mathematical programming problems which makes use of second order derivative is presented. In the so-called second order η-approximation method, an optimization problem associated with the original nonlinear programming problem is constructed that involves a second order η-approximation of both the objective function and the constraint function constituting the original problem. The equivalence between the nonlinear original mathematical programming problem and its associated second order η-approximated optimization problem is established under second order invexity assumption imposed on the functions constituting the original optimization problem.

Keywords: mathematical programming, second order η-approximated optimization problem, second order invex function, second order optimality conditions

MSC 2010: 90C26, 90C30, 90C46

1. Introduction

In the theory of constrained extremum problems, optimality conditions and duality results for differentiable nonlinear constrained problems are important theoretically as well as computationally and can be formulated in several different ways. However, for many optimization problems, notably in mathematical programming the characterization of optimal solutions with the help of second order optimality conditions was always of a great interest in order to refine first order optimality conditions (for example, the need of second order information appears in numerical algorithms).

In recent years, some generalizations of convex functions have been derived which proved to be useful for extending optimality conditions and some classical duality results, previously restricted to convex programs, to larger classes of optimization problems. One of them is invexity introduced by Hanson [10]. Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the role of invexity in optimization (see, for example, [7], [9], [12]). Some of the generalizations of Hanson's definition of an invex function is a second order invexity notion introduced by Bector and Bector [3], [4], [6] (biinvexity, bonvexity in Bector and Bector terminology).

Considerable attention has been given recently to devising new methods which allow to obtain the sufficient optimality conditions for the original mathematical programming problem and its duals by solving some associated optimization problem. One of such an approach is a first order η-approximation method introduced by Antczak [1] for characterizing solvability of differentiable optimization problems involving invex functions (with respect to the same function η). In this method, an associated (first order) η-approximated optimization problem is constructed for the original nonlinear mathematical programming problem. Antczak proved the equivalence between the original mathematical programming problem and its associated (first order) η-approximated optimization problem using an invexity concept in mathematical programming. He assumed that all functions constituting the original programming problem are (first order) invex with respect to the same function η on the set of all feasible solutions in the original mathematical programming problem.

The purpose of the present paper is to extend the results proved by Antczak in [1] to the case of twice differentiable optimization problems. In other words, the (first order) η-approximation method introduced by Antczak [1] is extended to the second order η-approximation method.

In this paper, we introduce a new approach for obtaining a second order sufficient optimality conditions for a nonlinear constrained mathematical programming problem with twice differentiable functions. In this method, for the original nonlinear mathematical programming problem, an equivalent modified optimization problem is constructed by a second order η-approximation of both the objective function and the constraint function at an arbitrary but fixed feasible point $\overline{x}$. Both the second order η-approximated objective function and the second order η-approximated constraint function are characterized in terms of the Hessian. Then, we use second order invexity to prove the equivalence between the original nonlinear mathematical programming problem and its associated second order η-approximated optimization problem. Moreover, the equivalent optimization problem obtained in this approach is, in general, less complicated and its optimal solution is connected to the optimal point of the original minimization problem. In this way, we obtain the associated modified optimization problem with the same optimality solution and the optimality value equal to the optimality value in the original mathematical programming problem involving nonlinear functions. It turns out that, for a nonlinear twice differentiable mathematical programming problem, there exists more than one associated second order η-approximated optimization problem which is equivalent in the sense discussed in the paper.

2. Preliminaries

Throughout the paper we write $\nabla f(\overline{x})$ and $\nabla^2 f(\overline{x})$ for the gradient of f and for the Hessian of f evaluated at $\overline{x}$, respectively. We recall some definitions that will be used in the present paper.

Definition 1 ([10]). Let $f\colon X \to \mathbb{R}$ be a differentiable function on a nonempty open set $X \subset \mathbb{R}^n$. If there exists $\eta\colon X \times X \to \mathbb{R}^n$ such that for all $x \in X$ the following inequality

$$f(x) - f(u) \geqslant \nabla f(u)\eta(x, u) \tag{1}$$

holds, then f is said to be a first order invex function (or shortly, invex in Hanson terminology) at $u \in X$ on X with respect to η. If inequality (1) holds for each $u \in X$, then f is invex on X with respect to η.

Definition 2 ([3], [5], [6]). Let $f\colon X \to \mathbb{R}$ be a twice differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$. If there exists $\eta\colon X \times X \to \mathbb{R}^n$ such that the following inequality

$$f(x) - f(u) \geqslant [\eta(x, u)]^T[\nabla f(u) + \nabla^2 f(u)y] - \frac{1}{2}y^T\nabla^2 f(u)y \quad (>) \tag{2}$$

holds for all $y \in \mathbb{R}^n$ and for all $x \in X$, then f is said to be second order (strictly) invex (bonvex in Bector and Bector terminology) at $u \in X$ on X with respect to η. If inequality (2) holds for each $u \in X$, then f is second order invex on X with respect to η.

We consider the nonlinear constrained mathematical programming problem

$$\text{(P)} \qquad \begin{gathered} f(x) \to \min \\ \text{subject to } g_i(x) \leqslant 0,\ i \in J = \{1, \ldots, m\}, \end{gathered}$$

where $f\colon X \to \mathbb{R}$ and $g_i\colon X \to \mathbb{R}$, $i = 1, \ldots, m$, are twice continuously differentiable functions on a nonempty open set $X \subset \mathbb{R}^n$.

Let

$$D := \{x \in X\colon g_i(x) \leqslant 0,\ i \in J\}$$

denote the set of all feasible solutions in (P) and

$$J(\overline{x}) := \{i \in J\colon\ g_i(\overline{x}) = 0\}$$

denote the index set of constraints active at the feasible point $\overline{x}$.

Definition 3. We define the Lagrange function or the Lagrangian $L\colon\ D \times \mathbb{R}_+ \times \mathbb{R}^m_+ \to \mathbb{R}$ in the considered mathematical programming problem (P) as

$$L(x, \xi_0, \xi) := \xi_0 f(x) + \xi g(x).$$

Definition 4. The set

$$C(\overline{x}) := \{d \in \mathbb{R}^n\colon\ d^T \nabla f(\overline{x}) \leqslant 0 \wedge d^T \nabla g_i(\overline{x}) \leqslant 0,\ i \in J(\overline{x})\}$$

is said to be the set of critical directions at $\overline{x}$.

Definition 5. A point $\overline{x} \in D$ is said to be an optimal point in (P) if, for all $x \in D$,

$$f(x) \geqslant f(\overline{x}).$$

It is well known (see, for example, [2], [11]) that the (first order) Karush-Kuhn-Tucker conditions are necessary for optimality in such optimization problems.

Theorem 6. *Let $\overline{x}$ be an optimal solution in* (P) *and let a suitable constraint qualification* [2] *be satisfied at $\overline{x}$. Then there exist $\overline{\xi}_0 \in \mathbb{R}$ and $\overline{\xi} \in \mathbb{R}^m$ such that*

$$\begin{gathered}\overline{\xi}_0 \nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x}) = 0,\\ \overline{\xi}^T g(\overline{x}) = 0,\\ \overline{\xi}_0 > 0, \quad \overline{\xi} \geqslant 0.\end{gathered}$$

It is also known that the second-order conditions from [8] (in the so-called dual form) for a nonlinear mathematical programming problem are necessary for $\overline{x}$ to be an optimal solution in the considered mathematical programming problem.

Theorem 7. *Let $\overline{x}$ be an optimal solution in* (P) *and let a suitable constraint qualification* (CQ) *be satisfied at $\overline{x}$ (see* [8]*). Then for every $d \in C(\overline{x})$ there exist $\overline{\xi}_0 \in \mathbb{R}$ and $\overline{\xi} \in \mathbb{R}^m$ such that*

$$\nabla L(\overline{x}, \overline{\xi}_0, \overline{\xi}) = 0, \tag{3}$$

$$d^T \nabla^2 L(\overline{x}, \overline{\xi}_0, \overline{\xi}) d \geqslant 0, \tag{4}$$

$$\overline{\xi}_i g_i(\overline{x}) = 0, \quad i \in J, \tag{5}$$

$$\overline{\xi}_0 \nabla f(\overline{x}) d = 0, \tag{6}$$

$$\overline{\xi}_i \nabla g_i(\overline{x}) d = 0, \quad i \in J(\overline{x}), \tag{7}$$

$$\overline{\xi}_0 > 0, \quad \overline{\xi} \geqslant 0. \tag{8}$$

Remark 8. Whenever we assume that a suitable constraint qualification (CQ) is satisfied for the considered optimization problem (P) we shall mean that some of the constraint qualifications considered in [8] is fulfilled.

Definition 9. The nonlinear mathematical programming problem (P) is said to be second order invex at $\overline{x}$ (with respect to η) if all functions constituting the problem (P) are second order invex at $\overline{x}$ on the set of all feasible solutions D with respect to the same function η.

3. An associated second order η-approximated optimization problem and optimality

Let $\overline{x}$ be a given feasible solution in (P). We consider the following optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ given by

$$(\mathrm{P}^2_\eta(\overline{x})) \qquad f(\overline{x}) + [\eta(x,\overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 f(\overline{x}) \eta(x,\overline{x}) \to \min,$$
$$g_i(\overline{x}) + [\eta(x,\overline{x})]^T \nabla g_i(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(x,\overline{x}) \leqslant 0, \ \ i = 1,\ldots,m,$$

where f, g_i, $i = 1,\ldots,m$, X are defined as in the problem (P) and, moreover, η is a function from $X \times X$ into $\mathbb{R}^n$ satisfying $\eta(x,\overline{x}) \neq 0$ whenever $x \neq \overline{x}$. We call $(P^2_\eta(\overline{x}))$ the associated second order η-approximated optimization problem.

Let

$$D(\overline{x}) := \Big\{x \in X\colon \ g_i(\overline{x}) + [\eta(x,\overline{x})]^T \nabla g_i(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(x,\overline{x}) \leqslant 0, \ i \in J\Big\}$$

denote the set of all feasible solutions in $(\mathrm{P}^2_\eta(\overline{x}))$.

We now show that any feasible solution in the original mathematical programming problem is also feasible in its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ if each constraint function g_i, $i \in J$, is second order invex at $\overline{x}$ on D with respect to the function η satisfying the condition $\eta(\overline{x},\overline{x}) = 0$.

Proposition 10. *Let $\overline{x}$ be a feasible solution in the original mathematical programming problem* (P). *Further, assume that g is second order invex at $\overline{x}$ on D with respect to the function η. Then any feasible solution in the problem* (P) *is also feasible in its associated second order η-approximated optimization problem* $(\mathrm{P}^2_\eta(\overline{x}))$, *that is, $D \subset D(\overline{x})$.*

P r o o f. Let $\overline{x}$ be a feasible solution in the original mathematical programming problem (P). By assumption, the constraint functions g_i, $i = 1, \ldots, m$, are second order invex at $\overline{x}$ on D with respect to the same function η. Then, by Definition 2, for any $i \in J$,

$$g_i(x) \geqslant g_i(\overline{x}) + [\eta(x, \overline{x})]^T [\nabla g_i(\overline{x}) + \nabla^2 g_i(\overline{x}) y] - \frac{1}{2} y^T \nabla^2 g_i(\overline{x}) y \tag{9}$$

holds for all $y \in \mathbb{R}^n$ and for all $x \in D$. Since $x \in D$, then $g_i(x) \leqslant 0$ for $i \in J$. Thus, by (9), for any $i \in J$, the following inequality

$$g_i(\overline{x}) + [\eta(x, \overline{x})]^T [\nabla g_i(\overline{x}) + \nabla^2 g_i(\overline{x}) y] - \frac{1}{2} y^T \nabla^2 g_i(\overline{x}) y \leqslant 0$$

holds for all $y \in \mathbb{R}^n$. Hence, it is satisfied also for $y = \eta(x, \overline{x})$. Thus, for any $i \in J$,

$$g_i(\overline{x}) + [\eta(x, \overline{x})]^T \nabla g_i(\overline{x}) + \frac{1}{2} [\eta(x, \overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(x, \overline{x}) \leqslant 0. \tag{10}$$

By (10), we conclude that $x \in D(\overline{x})$ and, hence, $D \subset D(\overline{x})$. □

Now, we establish the equivalence between the original mathematical programming problem (P) and its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

First, we establish that the optimal solution $\overline{x}$ in the associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ is also optimal in the original nonlinear mathematical programming problem (P) under second order invexity assumption imposed on both the objective function f and the constraint function g and, moreover, using the condition $\eta(\overline{x}, \overline{x}) = 0$.

Theorem 11. *Let $\overline{x}$ be an optimal solution in the second order η-approximated optimization problem* $(\mathrm{P}^2_\eta(\overline{x}))$. *Moreover, assume that the objective function f and the constraint function g are second order invex at $\overline{x}$ on D with respect to the same function η satisfying the condition $\eta(\overline{x}, \overline{x}) = 0$. Then $\overline{x}$ is also optimal in the original mathematical programming problem* (P).

P r o o f. By assumption, g is second order invex at $\overline{x}$ on D with respect to the function η. Thus, by Proposition 10, we have that $D \subset D(\overline{x})$. We proceed by contradiction. Suppose that $\overline{x}$ is not optimal in (P). Then, there exists $\tilde{x} \in D$ such that

$$f(\tilde{x}) < f(\overline{x}). \tag{11}$$

By assumption, f is second order invex at $\overline{x}$ on D with respect to the function η. Then, by Definition 2, the inequality

$$f(\tilde{x}) - f(\overline{x}) \geqslant [\eta(\tilde{x}, \overline{x})]^T [\nabla f(\overline{x}) + \nabla^2 f(\overline{x}) y] - \frac{1}{2} y^T \nabla^2 f(\overline{x}) y \tag{12}$$

holds for all $y \in \mathbb{R}^n$. Thus, by (11) and (12), we get that the inequality

$$[\eta(\tilde{x}, \overline{x})]^T [\nabla f(\overline{x}) + \nabla^2 f(\overline{x}) y] - \frac{1}{2} y^T \nabla^2 f(\overline{x}) y < 0 \tag{13}$$

holds for all $y \in \mathbb{R}^n$. Hence, it is also satisfied for $y = \eta(\tilde{x}, \overline{x})$. Then, by (13), we obtain

$$[\eta(\tilde{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2} [\eta(\tilde{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\tilde{x}, \overline{x}) < 0. \tag{14}$$

By assumption, $\eta(\overline{x}, \overline{x}) = 0$. Thus, (14) implies the inequality

$$\begin{aligned} f(\overline{x}) + [\eta(\tilde{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2} [\eta(\tilde{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\tilde{x}, \overline{x}) \\ < f(\overline{x}) + [\eta(\overline{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2} [\eta(\overline{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\overline{x}, \overline{x}). \end{aligned}$$

Since $\tilde{x} \in D$ and $D \subset D(\overline{x})$, then $\tilde{x} \in D(\overline{x})$. Thus, we find the point $\tilde{x}$ feasible in $(\mathrm{P}^2_\eta(\overline{x}))$ satisfying the inequality above. But this is a contradiction to the optimality of $\overline{x}$ in the associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. $\square$

Now, we show that the optimal solution $\overline{x}$ in the original nonlinear mathematical programming problem (P) is also optimal in its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

First, we prove that the feasible solution $\overline{x}$ in the original nonlinear mathematical programming problem (P), at which the second order necessary optimality conditions (3)–(8) are fulfilled, is optimal in its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Theorem 12. *Let $\overline{x}$ be a feasible solution in the original nonlinear mathematical programming problem* (P) *and let the second order necessary optimality* (3)–(8) *be satisfied at $\overline{x}$. Moreover, assume that f and g_i, $i \in J$, are second order invex at $\overline{x}$ on D with respect to the same function η satisfying the condition $\eta(\overline{x}, \overline{x}) = 0$. Then $\overline{x}$ is also optimal in the associated second order η-approximated optimization problem* $(\mathrm{P}^2_\eta(\overline{x}))$.

P r o o f. We proceed by contradiction. Suppose that $\overline{x}$ is not an optimal solution in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. Then there exists a feasible solution $\tilde{x} \in D(\overline{x})$ such that

$$f(\overline{x}) + [\eta(\tilde{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(\tilde{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\tilde{x}, \overline{x}) < f(\overline{x}) + [\eta(\overline{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(\overline{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\overline{x}, \overline{x}).$$

Hence, using the condition $\eta(\overline{x}, \overline{x}) = 0$, the inequality above gives

$$[\eta(\tilde{x}, \overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(\tilde{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\tilde{x}, \overline{x}) < 0. \tag{15}$$

Since $\overline{x}$ is feasible in problem (P) and f is second order invex at $\overline{x}$ on D with respect to the function η satisfying $\eta(\overline{x}, \overline{x}) = 0$, therefore, the following inequality

$$[\eta(\tilde{x}, \overline{x})]^T \nabla^2 f(\overline{x}) \eta(\tilde{x}, \overline{x}) \geqslant 0 \tag{16}$$

holds. Hence, by (15) and (16),

$$[\eta(\tilde{x}, \overline{x})]^T \nabla f(\overline{x}) < 0. \tag{17}$$

By assumption, g_i, $i \in J$, are second order invex at $\overline{x}$ on D with respect to the same function η satisfying $\eta(\overline{x}, \overline{x}) = 0$. Thus,

$$[\eta(\tilde{x}, \overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(\tilde{x}, \overline{x}) \geqslant 0. \tag{18}$$

From the feasibility of $\tilde{x}$ in problem $(\mathrm{P}^2_\eta(\overline{x}))$ we have for $i \in J$,

$$g_i(\overline{x}) + [\eta(\tilde{x}, \overline{x})]^T \nabla g_i(\overline{x}) + \frac{1}{2}[\eta(\tilde{x}, \overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(\tilde{x}, \overline{x}) \leqslant 0.$$

Hence, for any $i \in J(\overline{x})$,

$$[\eta(\tilde{x}, \overline{x})]^T \nabla g_i(\overline{x}) + \frac{1}{2}[\eta(\tilde{x}, \overline{x})]^T \nabla^2 g_i(\overline{x}) \eta(\tilde{x}, \overline{x}) \leqslant 0. \tag{19}$$

Then, using (18) together with (19), we get that the inequality

$$[\eta(\tilde{x},\overline{x})]^T \nabla g_i(\overline{x}) \leqslant 0$$

holds for any $i \in J(\overline{x})$. Thus, by (17) and (19), it follows that $\eta(\tilde{x},\overline{x}) \in C(\overline{x})$, that is, $\eta(\tilde{x},\overline{x})$ is a critical direction at $\overline{x}$. By assumption, $\overline{x}$ is a feasible solution in the original mathematical programming problem (P) at which the second order necessary optimality conditions (3)–(8) are satisfied. Thus, for every $d \in C(\overline{x})$, there exist nonnegative $\overline{\xi}_0 \in \mathbb{R}$ and $\overline{\xi} \in \mathbb{R}^m$ such that the second-order necessary optimality conditions (3)–(8) (in the dual form) are fulfilled at $\overline{x}$. Since $\eta(\tilde{x},\overline{x}) \in C(\overline{x})$, then there exist $\overline{\xi}_0 \in \mathbb{R}_+$ and $\overline{\xi} \in \mathbb{R}^m_+$ such that (15) and (19) imply, respectively,

$$[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_0 \nabla f(\overline{x}) + \frac{1}{2}[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_0 \nabla^2 f(\overline{x}) \eta(\tilde{x},\overline{x}) < 0, \tag{20}$$

$$[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_i \nabla g_i(\overline{x}) + \frac{1}{2}[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_i \nabla^2 g_i(\overline{x}) \eta(\tilde{x},\overline{x}) \leqslant 0, \quad i \in J(\overline{x}). \tag{21}$$

Thus, by the second necessary optimality conditions (6) and (7), we obtain from (20) and (21), respectively,

$$[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_0 \nabla^2 f(\overline{x}) \eta(\tilde{x},\overline{x}) < 0, \tag{22}$$

$$[\eta(\tilde{x},\overline{x})]^T \overline{\xi}_i \nabla^2 g_i(\overline{x}) \eta(\tilde{x},\overline{x}) \leqslant 0, \quad i \in J(\overline{x}). \tag{23}$$

Hence, by Definition 3, we get the inequality

$$[\eta(\tilde{x},\overline{x})]^T \nabla^2 L(\overline{x},\overline{\xi}_0,\overline{\xi}) \eta(\tilde{x},\overline{x}) < 0,$$

which is a contradiction to the necessary optimality condition (4). Thus, the conclusion of theorem is proved and, therefore, $\overline{x}$ is also optimal in $(\mathrm{P}^2_\eta(\overline{x}))$. □

From the theorem above we obtain the following result.

Corollary 13. *Let $\overline{x}$ be an optimal solution in the original nonlinear mathematical programming problem* (P). *Moreover, assume that f and g_i, $i \in J$, are second order invex at $\overline{x}$ on D with respect to the same function η satisfying the condition $\eta(\overline{x},\overline{x}) = 0$. Then $\overline{x}$ is also optimal in an associated second order η-approximated optimization problem* $(\mathrm{P}^2_\eta(\overline{x}))$.

In view of Corollary 13 and Theorem 11, if we assume that both the objective function f and the constraint functions g_i, $i \in J(\overline{x})$, constituting the original twice differentiable mathematical programming problem (P), are second order invex at $\overline{x}$ on the set of all feasible solutions D with respect to the same function η satisfying

$\eta(\overline{x}, \overline{x}) = 0$ and, moreover, some suitable constraint qualification (CQ) is satisfied at $\overline{x}$, then the problems (P) and ($\mathrm{P}^2_\eta(\overline{x})$) are equivalent in the sense discussed above. This means that $\overline{x}$ being optimal in the original (second order invex (with respect to η)) mathematical programming problem (P) is also optimal in its second order η-approximated optimization problem ($\mathrm{P}^2_\eta(\overline{x})$) and conversely, if $\overline{x}$ is optimal in the problem ($\mathrm{P}^2_\eta(\overline{x})$), then it is optimal in the problem (P). Thus, the optimal value in the second order η-approximated optimization problem ($\mathrm{P}^2_\eta(\overline{x})$) is the same as the optimal value in the original mathematical programming problem (P).

Now, we give an example of a mathematical programming problem (P) which, by using the approach discussed in this paper, is transformed to an equivalent quadratic convex optimization problem ($\mathrm{P}^2_\eta(\overline{x})$).

Example 14. Consider the following nonlinear mathematical programming problem

$$\text{(P)}\qquad \begin{aligned} f(x) &= 5x_1^4 \mathrm{e}^{x_1+5} + 2x_1^3 \arctan^4(x_1+1) + \mathrm{e}^{x_1^2} + \ln(x_1^2+1) \\ &\quad + \frac{1}{2}x_2^2 + \ln^2(x_2^2+1) \to \min, \\ g(x) &= x_1^4 \mathrm{e}^{x_1+3} + x_1^2 + \frac{1}{2}\arctan(x_1^2) - x_1 + x_2^4 \leqslant 0. \end{aligned}$$

Note that $D = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 \geqslant 0 \wedge x_1^4 \mathrm{e}^{x_1+3} + x_1^2 + \frac{1}{2}\arctan(x_1^2) - x_1 + x_2^4 \leqslant 0\}$, and $\overline{x} = (0,0)$ is optimal in the considered nonlinear optimization problem (P). Moreover, f and g are second order invex at $\overline{x}$ on D with respect to the same function η, for example, defined by

$$\eta(x, \overline{x}) = \begin{bmatrix} \frac{5}{4}(x_1 - \overline{x}_1) \\ \frac{1}{2}(x_2 - \overline{x}_2) \end{bmatrix}. \tag{24}$$

Now, using the approach discussed in the paper, we construct the problem ($\mathrm{P}^2_\eta(\overline{x})$) by the second order η-approximation of both the objective function f and the constraint function g at $\overline{x}$. Thus, we obtain the following quadratic convex optimization problem

$$(\mathrm{P}^2_\eta(0))\qquad \begin{aligned} &\frac{75}{32}x_1^2 + \frac{1}{8}x_2^2 \to \min, \\ &\frac{75}{32}x_1^2 - \frac{5}{4}x_1 \leqslant 0. \end{aligned}$$

It is not difficult to see that $\overline{x} = (0,0)$ is also optimal in the above optimization problem ($\mathrm{P}^2_\eta(\overline{x})$), that is, in the associated second order η-approximated optimization problem, which is constructed by a second order η-approximation of both the objective function and the constraint function in the original optimization problem (P). Since both the objective function f and the constraint function g are second order invex at $\overline{x} = (0, 0)$ on D with respect to the same function η satisfying $\eta(\overline{x}, \overline{x}) = 0$, then the hypotheses of Theorem 11 and Corollary 13 are fulfilled. Thus, by Theorem 11 and Corollary 13, $\overline{x} = (0, 0)$ is optimal in both optimization problems. Therefore, the optimal value in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ is the same as in the original optimization problem (P) and is equal to 0.

Remark 15. Note that the function η, with respect to which all functions involved in the problem (P) are second order invex, may be nonlinear. Let us, for example, consider the following optimization problem

$$\text{(P)}\qquad \begin{aligned} f(x) &= (\arctan x)^5 + (\arctan x)^4 + (\arctan x)^3 + (\arctan x)^2 \\ &\qquad + \arctan x \to \min, \\ g(x) &= (1 + x^4)(\arctan x)^2 - \arctan x \leqslant 0. \end{aligned}$$

Note that $\overline{x} = 0$ is optimal in the considered mathematical programming problem (P) and all functions involved in (P) are second order invex at $\overline{x}$ on the set of all feasible solutions $D = \{x \in \mathbb{R}\colon\ (1 + x^4)(\arctan x)^2 - \arctan x \leqslant 0\}$ with respect to the function η defined by

$$\eta(x, \overline{x}) = \arctan(x) - \arctan(\overline{x}). \tag{25}$$

It is not difficult to see that the function η given by (25) satisfies the condition $\eta(\overline{x}, \overline{x}) = 0$. Then, we construct the following associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$,

$$(\mathrm{P}^2_\eta(\overline{x}))\qquad \begin{aligned} (\arctan x)^2 + \arctan x &\to \min, \\ (\arctan x)^2 - \arctan x &\leqslant 0. \end{aligned}$$

It is not difficult to see that $\overline{x}$ is also optimal in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. This follows from the fact that all hypotheses of Corollary 13 and Theorem 11 are fulfilled and, therefore, the original mathematical programming problem (P) and its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ are equivalent in the sense discussed above ($\overline{x} = 0$ is optimal in both problems (P) and $(\mathrm{P}^2_\eta(\overline{x}))$).

4. Conclusion

In the paper, we have presented a new method which allows to obtain the second order sufficient optimality conditions for twice differentiable nonlinear programming problems. In this way, we extended the first order η-approximation method introduced earlier by Antczak [1]. The main tool used in the introduced second order η-approximation method is second order invexity notion. To prove the main result, all functions constituting the original mathematical programming problem are assumed to be second order invex with respect to the same function η and, moreover, some constraint is imposed on the function η. However, the formulation of the second order η-approximation method requires the Lagrange multipliers of the original mathematical programming problem. As it also follows from the formulation of the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, we need a point $\overline{x}$ feasible in the original mathematical programming problem which is suspected to be optimal. More exactly, as it follows from Theorem 12, the feasible point in the original optimization problem (P), at which the second order necessary optimality conditions are satisfied, should be known. Then, in this approach, all functions constituting the original optimization problem (P) are second order η-approximated at such a selected point $\overline{x}$. In this way, we construct, at such a selected point, the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. It turns out that the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ is simpler to solve than the original mathematical programming problem. In general, we obtain, using the introduced approach, the (quadratic) convex optimization problem to solve (in the case when η is a linear function with respect to the first component). As it is known from literature, therefore, to solve such optimization problems some known computational procedures can be applied. Furthermore, there may exist more than one suitable function η with respect to which all function constituting the original mathematical programming problem (P) are second order invex at $\overline{x}$ on D. This means that there may exist more than one associated second order η-approximated optimization problem which is equivalent to the original mathematical programming problem in the sense discussed in the paper. This property is, of course, useful from the practical point of view.

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