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MINIMAL CLAW-FREE GRAPHS

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Abstract. A graph G is a minimal claw-free graph (m.c.f. graph) if it contains no $K_{1,3}$ (claw) as an induced subgraph and if, for each edge e of G, $G-e$ contains an induced claw. We investigate properties of m.c.f. graphs, establish sharp bounds on their orders and the degrees of their vertices, and characterize graphs which have m.c.f. line graphs.

Keywords: minimal claw-free, degree, bow-tie, line graph

MSC 2010: 05C75, 05C07

1. Introduction

Graphs which do not contain a star on four vertices (claw) as an induced subgraph have received much attention, especially since the publication of the excellent survey paper [3] in 1997. This class of graphs includes, among others, line graphs, interval graphs, middle graphs, inflations of graphs and graphs with independence number equal to 2. Recently, Chudnovsky and Seymour found a structural characterization of claw-free graphs; that is, they defined certain classes of "basic" claw-free graphs and then showed that all claw-free graphs can be obtained by applying certain "expansion" operations. See [2].

In this paper we consider graphs which are (edge-) minimal with respect to the property of being claw-free. This was motivated by questions about cycles in claw-free graphs, but we think this class has some interest in its own right.

If G is a graph, we will denote its vertex set by $V(G)$ and its edge set by $E(G)$. The order of G is $n(G)=|V(G)|$. For a vertex v, $N(v)$ is its neighbourhood and

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$N[v] = \{v\} \cup N(v)$. We refer to the (induced) star $K_{1,3}$ as a *claw* with the vertex of degree 3 as its *centre*.

Definition 1. Let G be a claw-free graph without isolated vertices. If the removal of any edge of G produces a graph which is not claw-free, then G is a *minimal claw-free graph*, briefly denoted as an m.c.f. graph.

That not every claw-free graph contains an m.c.f. graph as a subgraph may be seen, for example, by considering the line graph of K_4 (which we denote by $\mathscr{L}(K_4)$): it is obviously claw-free but one can repeatedly remove its edges until an empty graph is obtained without creating a claw. On the other hand, the line graph of $K(3,3)$ (equivalent to the cartesian product $K_3 \times K_3$) is an m.c.f. graph.

We mention that a closely related concept, minimal line graphs, was considered by Sumner [8]. A graph is a *minimal line graph* if it is a line graph, but removal of any edge results in a graph that is not a line graph. Sumner proved that a graph G is a minimal line graph if and only if the following four conditions hold:

(i) every edge of G lies in a triangle,

(ii) every vertex of G has degree at least three,

(iii) if an edge e lies on a triangle whose vertices have an even degree sum, then e lies on another triangle,

(iv) each 4-clique of G has at least two vertices adjacent to vertices outside the 4-clique.

Condition (i) clearly holds for m.c.f. graphs, and we will see that condition (ii) also holds for m.c.f. graphs.

In this paper we look at bounds on the minimum, average and maximum degrees of an m.c.f. graph. In particular, we show that an m.c.f. graph has minimum degree at least 3, average degree at least 4, and maximum degree at most $n(G) - 3$. We then look at the relationship between m.c.f. graphs and line graphs. For example, a 4-regular graph is m.c.f. if and only if it is the line graph of a $(K_4 - e)$-free cubic graph.

2. Examples

We start with some examples. The 5-regular *icosahedron* on 12 vertices is an m.c.f. graph. Indeed, if we delete one, two or three vertices from the same triangle, then the result is still an m.c.f. graph. The latter graph is depicted in Figure 1. (This is not a line graph.) An exhaustive computer search has shown that the smallest order of an m.c.f. graph is 9; apart from the above graph there are two others, namely the line graphs of the two cubic graphs of order 6.

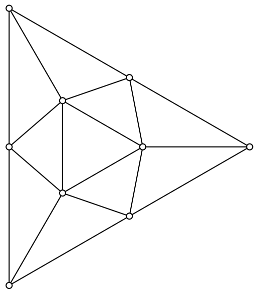

Figure 1. The m.c.f. graph I_9

A special family of m.c.f. graphs is the *bow-tie* graphs. We will denote by $\mathscr{B}$ the set of all 4-regular graphs in which the neighbourhood of each vertex has exactly two edges and these edges are independent (so a vertex and its neighbours induce the bow-tie $K_1 + 2K_2$). Such a graph is claw-free, but the removal of any edge produces two claws. The family $\mathscr{B}$ can also be defined as the line graphs of cubic triangle-free graphs. The cartesian product $K_3 \times K_3$ is the smallest member.

3. Degree bounds

We will need the following concept.

Definition 2. We will call a *near-claw* $NC(xy; c, t)$ as an induced subgraph obtained with vertex set $\{x, y, c, t\}$ and edge set $\{xy, xc, yc, tc\}$. The vertex c is the *centre* of the near-claw. See Figure 2.

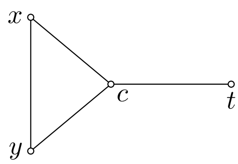

Figure 2. The near-claw $NC(xy; c, t)$

It is immediate that G is an m.c.f. graph if and only if every edge xy in G lies in a near-claw $NC(xy; c, t)$. Hence every edge of an m.c.f. graph is contained in a triangle.

3.1. Maximum degree.

Theorem 1. *Let G be an m.c.f. graph. Then the maximum degree $\Delta(G) \leqslant n(G) - 3$.*

P r o o f. Consider any vertex x. Let y be a neighbour of x. Then there exists a near-claw $NC(xy; c, t)$ with t non-adjacent to x but adjacent to c. Further, there exists a near-claw $NC(xc; c', t')$, where t' non-adjacent to x and c. Hence there are at least two vertices non-adjacent to x. It follows that $\Delta(G) \leqslant n - 3$. □

The bound is sharp, as we now show. Let A and B be the cycles a_0, a_1, a_2, a_3, a_0 and b_0, b_1, b_2, b_3, b_0 respectively. G is formed from $A \cup B$ by adding a vertex u adjacent to every vertex in $A \cup B$; adding vertices w_1 and w_2, with $N(w_1) = V(A)$, $N(w_2) = V(B)$; as well as the edges $a_i b_i$ and $a_i b_{i+1}$ for $i = 0, 1, \ldots, 3$ (addition modulo 4). It can be verified that G is an m.c.f. graph of order 11 with $\Delta(G) = \deg u = 8 = 11 - 3$. See Figure 3.

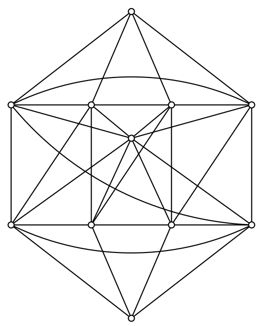

Figure 3. An m.c.f. graph with maximum Δ

One can obtain an infinite family of m.c.f. graphs with $\Delta = n - 3$ by duplicating u as follows. The *duplication of a vertex* u in G, means the addition to G of a new vertex v, adjacent to u and all vertices in $N_G(u)$ (so that $N[u] = N[v]$). Clearly, G is claw-free if and only if G' is claw-free.

Lemma 1. *Let G be a claw-free graph and suppose G' is formed by duplicating u to v. Then G' is an m.c.f. graph if and only if G is an m.c.f. graph.*

P r o o f. ($\Rightarrow$) Assume G' is an m.c.f. graph.

Let $e = ab \in E(G)$; then G' contains a near-claw $NC(ab; c, t)$ and $v \notin \{a, b\}$. If $NC(ab; c, t)$ is contained in G, then $G - e$ contains a claw.

Otherwise, suppose vertex v is on $NC(ab; c, t)$; hence $v \in \{c, t\}$. If $v = c$, then $u \notin \{a, b\}$, since otherwise, if $u = a$, then $vt \in E(G')$ and $ut \notin E(G')$, contradicting the assumption that $N[u] = N[v]$. Hence if $v = c$, then $NC(ab; u, t)$ is contained in G. On the other hand, if $v = t$, then, as $N[u] = N[v]$, $NC(ab; c, u)$ is contained in G. Hence G is an m.c.f. graph.

($\Leftarrow$) Assume G is an m.c.f. graph.

Clearly the removal of any edge of G' not incident with v produces a claw. So we need only to consider the edges incident with v. Let $v' \in V(G')$ such that $v' \in N(v)$ but $v' \neq u$, and let $e_1 = vv' \in E(G')$. So there exists the edge $uv' \in E(G)$ contained in, say, the near-claw $NC(uv'; w, x)$ in G; then vv' is contained in the near-claw $NC(vv'; w, x)$ in G', whence removal of vv' creates a claw.

Consider the edge $e_2 = uv \in V(G')$. By Theorem 1, there exists a vertex, say $w \in V(G') - N[u]$, that is adjacent to some vertex in $N(u)$, say v'. Then $wv \notin E(G')$ and $v' \in N(v)$. Moreover, since $uv', vv', v'w \in E(G')$ and $uw, vw \notin E(G')$, $\{v', u, v, w\}$ induces a claw in $G' - e_2$. Hence G' is m.c.f.. □

3.2. Minimum degree.

Theorem 2. *Let G be an m.c.f. graph. Then the minimum degree $\delta(G) \geqslant 3$.*

Proof. Since every edge of G lies in a triangle, $\delta(G) \geqslant 2$. Now suppose that G contains a vertex v_0 of degree 2, adjacent to v_1 and v_2, where $v_1v_2 \in E(G)$. Let $B = (N(v_1) \cap N(v_2)) - \{v_0\}$ and for $i = 1, 2$ $A_i = N(v_i) - (B \cup \{v_0, v_{3-i}\})$. Since v_1 and v_2 are not centres of claws, $A_1 \cup B$ and $A_2 \cup B$ induce complete subgraphs of G.

The edge v_0v_1 is contained in a near-claw with v_2 as centre, say $NC(v_0v_1; v_2, v_3)$, and v_0v_2 is contained in a near-claw $NC(v_0v_2; v_1, v_4)$; so $v_2v_3, v_1v_4 \in E(G)$ and $v_1v_3, v_2v_4 \notin E(G)$ and thus $v_3 \in A_2$ and $v_4 \in A_1$. The edge v_1v_2 is contained in a near-claw $NC(v_1v_2; v_5, v_6)$, where $v_1v_5, v_2v_5 \in E(G)$ and $v_1v_6, v_2v_6 \notin E(G)$; so $v_5 \in B$, $v_6 \notin N(v_1) \cup N(v_2)$.

Let $x \in A_2$, $y \in A_1$; then since $\langle\{v_5, v_1, x, v_6\}\rangle$ is not a claw and $v_1x, v_1v_6 \notin E(G)$, it follows that $xv_6 \in E(G)$; similarly as $\langle\{v_5, v_2, y, v_6\}\rangle$ is not a claw, it follows that $yv_6 \in E(G)$. Hence v_6 is adjacent to every vertex in $A_1 \cup A_2$. By the same argument it follows that

$$(*) \qquad \text{if } w \in N(B) - (N[v_1] \cup N[v_2]) \text{ then } w \text{ is adjacent to all of } A_1 \cup A_2.$$

The edge v_5v_6 is contained in a near-claw $NC(v_5v_6; c, t)$, say. If $c \in N(v_2)$, then as v_5 is adjacent to every vertex in $(N(v_1) \cup N(v_2)) - \{v_0\}$, it follows that $t \notin N(v_1) \cup N(v_2)$; hence as $\langle N(v_2) - \{v_0, v_1\}\rangle$ is complete, $v_2t \notin E(G)$. Also, v_2v_6, $v_6t \notin E(G)$ while c is adjacent to v_2, v_6 and t; so $\langle\{c, v_2, v_6, t\}\rangle$ is a claw, a contradiction. So $c \notin N(v_2)$. It follows similarly that $c \notin N(v_1)$.

So $c = v_7$ and $t = v_8$, where $v_7 \notin N[v_1] \cup N[v_2]$, and $v_5v_7, v_6v_7 \in E(G)$, while $v_5v_8, v_6v_8 \notin E(G)$. Note that, $v_1v_8 \notin E(G)$, since otherwise $\langle\{v_1, v_0, v_5, v_8\}\rangle$ is a claw and, similarly, $v_2v_8 \notin E(G)$. By $(*)$, v_7 is adjacent to every vertex in $A_1 \cup A_2$.

That for $x \in A_2$, $xv_8 \notin E(G)$ follows from the observation that $\langle\{x, v_2, v_6, v_8\}\rangle$ is not a claw. So v_8 is non-adjacent to each vertex in A_2 and, similarly in A_1.

Furthermore, $xy \in E(G)$ for $x \in A_2$, $y \in A_1$, since $\langle\{v_7, x, y, v_8\}\rangle$ is not a claw. See Figure 4.

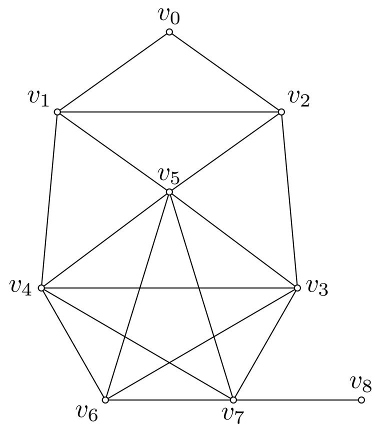

Figure 4. An induced subgraph

In conjunction with the fact that A_1 and A_2 induce complete graphs, we obtain that

$$(\dagger) \qquad \langle A_1 \cup A_2 \cup B\rangle \text{ is complete.}$$

The edge v_2v_3 is contained in a near claw, say, $NC(v_2v_3; v_9, t)$. Clearly, $v_9 \notin B$, since otherwise v_3 and t would be adjacent by $(*)$. Hence $v_9 \in A_2$.

Consider t. Since t is not adjacent to v_3, but adjacent to v_9, we have $t \notin N[v_1] \cup N[v_2] \cup \{v_6, v_7, v_8\}$, say, $t = v_{10}$ with $v_9v_{10} \in E(G)$, $v_3v_{10} \notin E(G)$.

By $(\dagger)$, $v_4v_9 \in E(G)$. Hence $v_4v_{10} \in E(G)$ since otherwise $\langle\{v_9, v_2, v_4, v_{10}\}\rangle$ is a claw. But then $\langle\{v_4, v_1, v_3, v_{10}\}\rangle$ is a claw, a contradiction. □

By Theorem 2, we have the following results.

Corollary 1. *Let G be an m.c.f. graph. Then the vertices of degree 3 form an independent set.*

P r o o f. Suppose that u, v are vertices of degree 3 in G such that $uv \in E(G)$.

If $N[u] = N[v]$, let $G' = G - \{v\}$. Then G' is m.c.f., but has a vertex of degree 2, which contradicts Theorem 2.

If the vertices u and v have different neighbourhoods, then since every edge lies in a triangle, $N(u) \cup N(v)$ induces the graph $K_1 + P_4$, where u and v are the interior vertices on P_4. Let G' be the graph obtained by adding a vertex w adjacent only to u and v. Then G' is claw-free. The removal of the edge uw produces a claw centred at v, and the removal of the edge vw produces a claw centred at u and $G' - e$ contains an induced claw for each $e \in E(G)$. So G' is a minimal claw-free graph with a vertex of degree 2, which contradicts Theorem 2.

Hence $uv \notin E(G)$, and the result follows. □

Corollary 2. *Let G be an m.c.f. graph. Then $\Delta(G) \geqslant 4$.*

P r o o f. An immediate consequence of Theorem 2 and Corollary 1. □

3.3. Average degree.

We now look at the minimum number of edges in an m.c.f. graph. We will need the following results.

Lemma 2. *If an m.c.f. graph G contains a vertex v of degree 3, then v has a neighbour of degree at least 5.*

P r o o f. Suppose to the contrary that no neighbour of v has degree exceeding 4; then it follows from Theorem 2 and Corollary 1 that the neighbours of v, say v_1, v_2, v_3, all have degree equal to 4. The edge vv_1 is contained in a near-claw, say $NC(vv_1; v_2, t_1)$, so that $t_1 \in V(G) - N[v]$, $t_1v_2 \in E(G)$ and $t_1v_1 \notin E(G)$. A near-claw $NC(v_2t_1; c_2, t_2)$ exists in G; here $c_2 = v_3$ or c_2 is a new vertex.

If $c_2 = v_3$, then $v_2v_3, t_1v_3 \in E(G)$ and t_2 is a new vertex such that $v_3t_2 \in E(G)$, but $v_2t_2, t_1t_2 \notin E(G)$. Since $\deg v_3 = 4$, it follows that $v_3v_1 \notin E(G)$. A near-claw $NC(vv_3, c_3, t_3)$ exists in G, where $c_3 = v_2$. But deg $v_2 = 4$, so $t_3 \in \{t_1, v_1\}$, a contradiction, as $vv_1, v_3t_1 \in E(G)$. Hence $c_2 \neq v_3$.

Thus c_2 is a new vertex. Then $v_2c_2, t_1c_2 \in E(G)$ and $N(v_2) = \{v, v_1, t_1, c_2\}$; hence the centre, c_3, of a near-claw $NC(vv_3; c_3, t_3)$ must be v_1 and so $v_1v_3 \in E(G)$. A near-claw $NC(v_1v_2; c_4, t_4)$ exists in G, where $c_4 \in \{v, c_2\}$. If $c_4 = v$, then $t_4 = v_3$, a contradiction, as $v_1v_3 \in E(G)$. Hence $c_4 = c_2$ and $v_1c_2 \in E(G)$. A near-claw $NC(vv_2; c_5, t_5)$ exists in G, where $c_5 = v_1$ and $t_5 \in \{v_3, c_2\}$, which yields a contradiction, as $vv_3, v_2c_2 \in E(G)$.

It follows that at least one neighbour of v is of degree exceeding 4. □

Lemma 3. *Let G be an m.c.f. graph. If v is a vertex of degree 3 in G with only one neighbour of degree at least 5, say v_3, then v_3 has no other neighbour of degree 3.*

P r o o f. Say v's neighbours are vertices v_1, v_2 and v_3, with $\deg v_1 = \deg v_2 = 4$. We show first that v_1v_3, $v_2v_3 \in E(G)$ and $v_1v_2 \notin E(G)$.

The edge vv_3 is contained in a near-claw, with say v_1 as centre, $NC(vv_3; v_1, t_1)$, where $t_1 \neq v_2$, and so $v_1v_3 \in E(G)$. Suppose $v_1v_2 \in E(G)$; then the edge v_1t_1

is contained in a near-claw which must have centre v_2 because $\deg v_1 = 4$—say $NC(v_1t_1; v_2, t_2)$ where $t_2 \neq \{v, v_3\}$, so that t_2 is a new vertex. However, $\langle\{v_2, v, t_1, t_2\}\rangle \cong K_{1,3}$, a contradiction; and so $v_1v_2 \notin E(G)$. Since vv_2 is contained in a near-claw which must have centre v_3, $v_2v_3 \in E(G)$.

Further, consider this near-claw $NC(vv_2; v_3, x_1)$. Since v_3 is not the centre of a claw, it follows that $v_1x_1 \in E(G)$. Similarly, there is a vertex x_2 such that $v_2x_2, v_3x_2 \in E(G)$ but $v_1x_2 \notin E(G)$. Again because v_3 is not the centre of a claw, it follows that $x_1x_2 \in E(G)$. That is, the set $W = \{v, v_1, x_1, x_2, v_2\}$ induces a 5-cycle.

Now suppose v_3 has another neighbour m of degree 3. If $m \notin W$, then it has only two neighbours in W, and thus is part of a claw centered at v_3. If $m \in W$, say $m = x_1$, then v_1 has no other neighbour, by the lack of claw centered at v_1. But then v_1 has degree 3, a contradiction of Corollary 1. □

Theorem 3. *Let G be an m.c.f. graph. Then G has at least $2n(G)$ edges.*

Proof. Let T denote the set of vertices of degree 3 and let U denote the set of vertices of degree at least 5. Define H as the bipartite subgraph of G with vertex set $T \cup U$ and edge set all edges with one end in T and one end in U.

By Lemma 2, in H every vertex of T has degree at least 1. Let A denote the vertices of T with degree 1 in H. By Lemma 3, the neighbours of A have degree 1 in H; let $X = N(A)$. So every vertex in $T - A$ has degree at least 2 in H. On the other hand, since T is independent in G (by Corollary 1) and G is claw-free, every vertex in $U - X$ has degree at most 2 in H. Thus $|T - A| \leqslant |U - X|$ and so $|T| \leqslant |U|$.

Now, let d_i denote the number of vertices of degree i in G. Then

$$\sum_i id_i = 4n + \sum_i (i-4)d_i \geqslant 4n + |U| - |T| \geqslant 4n,$$

as required. □

That this result is sharp follows from the bow-tie graphs $\mathscr{B}$. We will show later that 4-regular m.c.f. graphs are always line graphs (Theorem 6).

Perhaps surprisingly, there is a unique nonregular m.c.f. graph with average degree 4. We omit the laborious proof.

Theorem 4. *If G is a connected m.c.f. graph with average degree 4 but is not 4-regular, then $G = I_9$.*

4. Minimal claw-free line graphs

All line graphs are claw-free, but not all line graphs are m.c.f.. For example, the line graph of $K_{3,3}$ is m.c.f., but the line graph of K_4 is not. Theorem 5 characterizes those line graphs that are m.c.f..

Definition 3. Let G be a graph with edges $x, y \in E(G)$ so that x and y are incident. If a subgraph of G that is not necessarily induced in G, is isomorphic to the graph shown in Figure 5 (with x, y as pendant edges), then it is called an *F-graph of* (x, y).

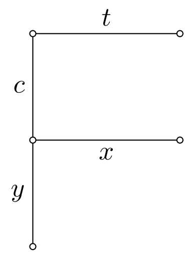

Figure 5. An F-graph for (x, y)

Lemma 4. *The line graph $\mathscr{L}(G)$ of a graph G is m.c.f. if and only if for every pair of incident edges x and y, G contains an F-graph of (x, y).*

Proof. There is only one graph up to isomorphism whose line graph is isomorphic to a near-claw $NC(xy; c, t)$, namely the F-graph of (x, y). Thus, G contains an F-graph of (x, y) if and only if its line graph contains a near-claw $NC(xy; c, t)$. □

Theorem 5. *Let $\mathscr{L}(G)$ be the line graph of graph G. The following statements are equivalent:*
(i) *$\mathscr{L}(G)$ is m.c.f.*
(ii) *No vertex v of G has neighbours u, w with*

$$(**).\qquad \text{for all } z \in N(v) - \{u, w\}\colon\ N(z) = \{v\} \text{ or } N(z) = \{u, v, w\}$$

Note that property $(**)$ holds if and only if $\deg_G v = 2$, or $N_G[v] = \{x_1, \ldots, x_k, y_1, \ldots, y_l, u, v\}$ induces the graph shown in Figure 6, where $\deg_G x_i = 1$ and $\deg_G y_j = 3$ (where, possibly $k = 0$ or $l = 0$) and where uw might or might not be an edge.

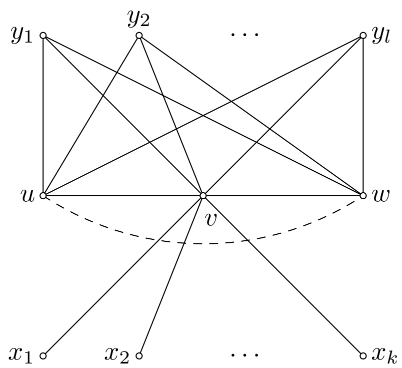

Figure 6. $\langle N_G[v] \rangle$ if v has property $(**)$

P r o o f. Assume that G contains a vertex v that has property $(**)$. Let $w, u \in N(v)$, and let $x = uv$ and $y = vw$ be edges in G. Since for any vertex $z \in N(v) - \{u, w\}$, z is only adjacent to vertices in $N_G[v]$, G cannot contain an F-subgraph of (x, y). Thus by Lemma 4, $\mathscr{L}(G)$ is not m.c.f..

Assume that G satisfies (ii). Consider any pair of incident edges, say $x = uv$ and $y = vw$. Let $z \in N(v) - \{u, w\}$; then, since G satisfies (ii), $N_G(z) \neq \{v\}$ or $\{u, w, v\}$. Hence z is adjacent to some other vertex in G, besides v and thus $\deg_G z \geqslant 2$. However, since z does not have property $(**)$, $\deg_G z \neq 2$ and therefore $\deg_G z \geqslant 3$. Since $N_G(z) \neq \{u, v, w\}$, z must be adjacent to some new vertex, say t, in G. Then the subgraph induced by the edges uv, vw, vz and zt in G is an F-subgraph of G, and hence by Lemma 4, $\mathscr{L}(G) - xy$ contains a claw. Thus $\mathscr{L}(G)$ is m.c.f.. □

By Theorem 5, we have the following corollaries.

Corollary 3. *If $\delta(G) \geqslant 4$, then $\mathscr{L}(G)$ is m.c.f.*

Corollary 4. *If $\delta(G) \geqslant 3$ and G contains no induced $K_4 - e$ or K_4, then $\mathscr{L}(G)$ is m.c.f..*

For example, the line graph of every cubic $(K_4 - e)$-free graph is a 4-regular m.c.f. graph. We now present a converse.

Theorem 6. *G is a 4-regular m.c.f. graph if and only if G is the line graph of a cubic $(K_4 - e)$-free graph.*

P r o o f. By the above corollary, one direction is true. So assume G is a 4-regular m.c.f. graph.

Suppose G contains a clique $\{u, v, w, x\}$. Let a be the fourth neighbour of u. Since ua is in a triangle, a is adjacent to some other vertex of the clique, say v. Similarly, the fourth neighbour b of w is also adjacent to x (possibly $a = b$). But then the removal of ux does not create a claw, a contradiction. Hence G does not contain K_4.

Since G is claw-free, by the characterization of line graphs by Van Rooij and Wilf [9], it suffices to show that if $H = K_4 - e$ is an induced subgraph of G, then at least one of the two triangles T of H is "even": that is, every vertex in G has an even number of neighbours on T.

So let $H = K_4 - e$ be an induced subgraph of G with vertex set $\{u, v, x, y\}$ with $xy \notin E(G)$. Let a be the fourth neighbour of u. Since there is no claw centered at u, the vertex a is adjacent to at least one of x or y, say the former. By the lack of K_4's, $av \notin E(G)$. The removal of edge xv creates a claw; so x and v have another common neighbour b. Thus the triangle $T = \{u, v, x\}$ is even: only a, y and b are adjacent to T and each has exactly two neighbours on T. This shows that G is a line graph.

Say $G = \mathscr{L}(G')$. Since G does not contain K_4, the graph G' has maximum degree 3, and is thus cubic.

Now, if G' contains an induced $K_4 - e$, then let x and y be two edges of the $K_4 - e$ that do not lie in a common triangle: these do not lie in an F-graph. Since G is m.c.f., G' is $(K_4 - e)$-free. □

5. Open questions

We list here some other thoughts and open questions.

1. What is the maximum degree in a regular m.c.f. graph? One can take $\mathscr{L}(K(3, 3))$ and duplicate each vertex: the result is approximately $5n/9$-regular.
2. What is the maximum number of edges in an m.c.f. graph? There are m.c.f. graphs with $\binom{n}{2} - 2n - O(1)$ edges; for example, take the graph of Figure 3 and repeatedly duplicate the vertex of maximum degree.
3. Which m.c.f. graphs are planar? The question of which line graphs are planar was solved by Sedláček [7]. He showed that a line graph $L(G)$ is planar iff G is planar, the maximum degree of G is at most 4 and any degree-4 vertex of G is a cut-vertex. It is known that a claw-free planar graph has maximum degree at most 6 [5].
4. On defining m.c.f. graphs, there was a brief hope that it would be easier to prove hamiltonicity results about m.c.f. graphs (perhaps generalizing results about locally-connected claw-free graphs). However, it is an open question to show the class $\mathscr{B}$ to be hamiltonian. Indeed via results of Ryjáček [6] and the fifth author [4], the question of 4-connectedness implying hamiltonicity is equivalent for (1) claw-free graphs; (2) m.c.f. graphs; (3) line graphs; and (4)

bow-tie graphs. Nevertheless, there is a series of results that 2-connected, claw-free and F-free for a particular graph F implies hamiltonicity (see [3]). Perhaps one can improve these results for m.c.f. graphs.

5. What about subgraphs and supergraphs? We have seen claw-free graphs which do not contain an m.c.f. graph. So one problem is to characterize the claw-free graphs which contain no m.c.f. graph. Alternatively, characterize the claw-free graphs where one can repeatedly remove edges and never reach an m.c.f. graph. In the other direction: Is every claw-free graph an induced subgraph of an m.c.f. graph?

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