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ON THE VANISHING VISCOSITY METHOD FOR FIRST ORDER DIFFERENTIAL-FUNCTIONAL IBVP

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Abstract. We consider the initial-boundary value problem for first order differential-functional equations. We present the 'vanishing viscosity' method in order to obtain viscosity solutions. Our formulation includes problems with a retarded and deviated argument and differential-integral equations.

Keywords: viscosity solutions, first order equation, parabolic equation, differential functional equations

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be any open bounded domain. For given constants $T > 0,\ a_0, r_0 \geqslant 0$ we define

$$\begin{aligned} &\Omega_{r_0} = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x,\Omega) \leqslant r_0\}, \quad \partial_0\Omega = \Omega_{r_0} \setminus \Omega, \quad \Theta = (0,T) \times \Omega, \\ &\Theta_0 = [-a_0, 0] \times \Omega_{r_0}, \quad \partial_0\Theta = (0,T) \times \partial_0\Omega, \quad \Gamma = \Theta_0 \cup \partial_0\Theta, \quad E = \Gamma \cup \Theta. \end{aligned}$$

Let $\mathbb{D} = [-a_0, 0] \times B(r_0)$, where $B(r_0) = \{x \in \mathbb{R}^n \colon |x| \leqslant r_0\}$ and $|\cdot|$ is the norm in $\mathbb{R}^n$. For every $z \colon E \to \mathbb{R}$ and $(t,x) \in \Theta$ we define a function $z_{(t,x)} \colon \mathbb{D} \to \mathbb{R}$ by $z_{(t,x)}(s,y) = z(t+s, x+y)$ for $(s,y) \in \mathbb{D}$. We call the restriction operator $z \to z_{(t,x)}$ "the Hale operator" (see [9] for ordinary differential equations).

Throughout the paper $C(A)$ stands for the space of all continuous functions $w\colon A \to \mathbb{R}$ with the supremum norm $\|\cdot\|_A$.

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Suppose that $f\colon \Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ of the variables (t, x, u, w, p) and $\Psi\colon \Gamma \to \mathbb{R}$ are given function. We will consider the initial-boundary value problem (IBVP)

$$D_t u(t,x) = f(t,x,u(t,x),u_{(t,x)},Du(t,x)) \quad \text{in } \Theta, \tag{1}$$

$$u(t,x) = \Psi(t,x) \quad \text{in } \Gamma. \tag{2}$$

We write Du for the spatial derivative $D_x u$.

Definition 1.1. A function $u \in C(E)$ is a viscosity subsolution (resp. supersolution) of (1), (2) provided for all $\varphi \in C^1(\Theta)$ if $u-\varphi$ attains a local maximum (resp. minimum) at $(t_0, x_0) \in \Theta$ then

$$D_t\varphi(t_0,x_0) \leqslant f(t_0,x_0,u(t_0,x_0),u_{(t_0,x_0)},D\varphi(t_0,x_0)) \quad (\text{resp. "} \geqslant \text{"}) \tag{3}$$

$$u(t,x) \leqslant \Psi(t,x) \quad \text{in } \Gamma, \quad (\text{resp. "} \geqslant \text{"}) \tag{4}$$

Definition 1.2. A function $u \in C(E)$ is a viscosity solution of (1), (2) if u is both a viscosity subsolution and supersolution of (1), (2).

We denote by $\mathrm{SUB}(f,\Psi), \mathrm{SUP}(f,\Psi), \mathrm{SOL}(f,\Psi)$ the sets of all viscosity subsolutions, supersolutions and solutions of problem (1), (2).

The following remark is immediate.

Remark 1.1. If $u \in C(E) \cap C^1(\Theta)$ then $u \in \mathrm{SOL}(f,\Psi)$ ($u \in \mathrm{SUB}(f,\Psi)$, $\mathrm{SUP}(f,\Psi)$) if and only if u is a classical solution (subsolution, supersolution) of (1), (2).

The notion of viscosity solution was first introduced by M. G. Crandall and P. L. Lions in [8], [15] for first order differential equations. The best general references for the second order equations (not considered here) are [2], [7].

The existence of classical solutions for first order partial differential-functional equations was considered in [3] (equation with a retarded argument) and in [4] (equation with operators of the Volterra type). The paper [12] is devoted to classical and Carathéodory solutions for a Hale type model of functional dependence in equations. This model is also studied in [20] where vanishing viscosity method is applied to the Cauchy problem. It is worth manthionig here that some expict estimates that leads to the convergence of vanishing viscosity method (for the Cauchy problem) are given in [11] for Bellman-Isaacs differential-integral equations.

The method presented in this paper (interesting in itself) is not the only one that gives existence of a viscosity solution for our problem. We believe that the existence result can be obtained also as a particular case of the theorem for a second order degenerate parabolic problem. This can be done by generalizing results obtained in [2], [7]. This problem for integro- PDE which has many applications in optimal control jump-diffusion processes was discussed in [1] for second order equation and in [16] for first order equations. Both of them deal with the Cauchy problem. Fixed point techniques combined with a results obtained for nonfunctional case can be used also to prove the existence. (see [10], [21]).

Problem (1), (2) contains as a particular case equations with a retarded and deviated argument and differential-integral (integro- PDE) equations. This can be derived from (1), (2) by specializing the function f.

Indeed, let us consider two examples,

Example 1.1. Let $g\colon \Theta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\mu\colon \Theta \to \mathbb{R}$, $\nu\colon \Theta \to \mathbb{R}^n$ be given functions such that

$$t - a_0 \leqslant \mu(t,x) \leqslant t, \quad |\nu(t,x) - x| \leqslant r_0 \quad \text{for } (t,x) \in \Theta. \tag{5}$$

Consider the equation

$$D_t u(t,x) = g(t,x,u(t,x),u(\mu(t,x),\nu(t,x)),Du(t,x)) \quad \text{in } \Theta, \tag{6}$$

with an initial-boundary condition (2). It is easy to verify that putting

$$f(t,x,u,w,p) = g(t,x,u,w(\mu(t,x)-t,\sigma(t,x)-x),p) \tag{7}$$

for $(t,x,u,w,p) \in \Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n$ we can obtain problem (1), (2).

In Section 3 we present the theorem on the existence of viscosity solution for (6), (2).

Problem in the form (1), (2) can be obtained also by transformation of the differential-integral equation.

Example 1.2. Let $\mathbb{D}_{(t,x)} = \{(t+t', x+x')\colon (t',x') \in \mathbb{D}\}$ and $h\colon \Theta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $K\colon \Theta \times \Theta \times \mathbb{R} \to \mathbb{R}$ be given functions.

Consider the equation

$$D_t u(t,x) = h\Bigl(t,x,u(t,x),\int_{\mathbb{D}(t,x)} K(t,x,s,y,u(s,y))\,\mathrm{d}s\,\mathrm{d}y, Du(t,x)\Bigr) \quad \text{in } \Theta. \tag{8}$$

Define $f\colon \Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(t,x,u,w,p) = h\Bigl(t,x,u,\int_{\mathbb{D}_{(t,x)}} K(t,x,s,y,w(s-t,y-x))\,\mathrm{d}s\,\mathrm{d}y, p\Bigr).$$

By the above formula it is evident that (8), (2) can be treated as a particular case of (1), (2).

2. Auxiliary theorems

In this section we will be concerned with the problem,

$$D_t u(t,x) - \varepsilon\Delta_x u(t,x) = f(t,x,u(t,x),u_{(t,x)},Du(t,x)) \quad \text{in } \Theta, \tag{9}$$

$$u(t,x) = \Psi(t,x) \quad \text{in } \Gamma. \tag{10}$$

Let $C^{1,2}(\bar{\Theta})$ denote the space of all functions $u \in C(\overline{\Theta})$ such that $D_t u, Du, D_x^2 u$ exist and are continuous in $\overline{\Theta}$. Write $C_*^{1,2}(E) = C^{1,2}(\overline{\Theta}) \cap C(\overline{E})$. We will write $\mathrm{CSL}(f,\Psi,\varepsilon)$ for the set of classical solutions of (9), (10) (i.e. $u_\varepsilon \in C_*^{1,2}(E)$ and u_ε satisfies (9), (10)).

The reason why we consider (9), (10) together with (1), (2) is the following. In order to obtain viscosity solutions of (1), (2) we apply the vanishing viscosity method (see [13] for entropy generalized solutions and [8], [15] for viscosity solutions, both for the nonfunctional case).

Proposition 2.1 (vanishing viscosity method)**.** *Let $u_\varepsilon \in CLS(f_\varepsilon,\Psi_\varepsilon,\varepsilon)$. Assume that $f_\varepsilon \to f$ in $C(\Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n)$ (on bounded subsets) and $\Psi_\varepsilon \to \Psi$ in $C(\Gamma)$. If $u_\varepsilon \to u$ in $C(E)$ as $\varepsilon \to 0$ then u is a viscosity solution of* (9), (10)*.*

The idea of the proof is similar to the nonfunctional case (see [8]).

To apply this method we need a theorem on global existence of classical solutions for (9), (10). This subject (for nonfunctional case) was investigated in the classical monograph [14]. The functional case was studied in [5], [6] for a special type of functional dependence. The problem for an arbitrary linear parabolic operator with a Hale's type functional dependence was considered in [19]. We wil present later a set of assumptions giving existence of classical solutions for (9), (10) (proved in [19]).

A function $\omega\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called a *modulus* if ω is nondecreasing and $\omega(0^+) = 0$. Let $K(R) = \{w \in C(\mathbb{D})\colon \|w\|_{\mathbb{D}} \leqslant R\}$. We write $G_t = \{(s,x) \in G\colon -a_0 \leqslant s \leqslant t\}$ for any $G \subseteq \mathbb{R}^{n+1}$.

Definition 2.1. Let $M \geqslant 0$ and $\sigma\colon [0,T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$. We will write $\sigma \in O_M$ if

(i) σ is continuous and nondecreasing with respect to both variables,

(ii) the right-hand maximum solution of the problem

$$z'(t) = \sigma(t,z(t)), \qquad z(0) = M \tag{11}$$

exists in $[0,T]$. (We will denote this solution by $\mu_\sigma(\cdot,M)$.)

Definition 2.2. Let $M \geqslant 0, \sigma \in O_M$. We will write $f \in X_{\sigma,M}$ if

(i) for every $(t, x, u, w) \in \Theta \times \mathbb{R} \times C(\mathbb{D})$

$$f(t, x, u, w, 0) \operatorname{sgn}(u) \leqslant \sigma(t, \max(|u|, \|w\|_{\mathbb{D}});$$

(ii) for every $R > 0$ there exists a *modulus* ω_R such that

$$|f(t, x, u, w, p) - f(t, x, u, w, 0)| \leqslant \omega_R(|p|) \quad \text{in } \Theta \times [-R, R] \times K(R) \times \mathbb{R}^n.$$

Define

$$R(\sigma, M) = \mu_\sigma(T, M). \tag{12}$$

Proposition 2.2. *Suppose that $f \in X_{\sigma, M}$, $\|\Psi\|_\Gamma \leqslant M$ and $u \in CLS(f, \Psi, \varepsilon)$ ($u \in SOL(f, \Psi)$). Then*

$$\|u\|_{E_t} \leqslant \mu_\sigma(t, M) \leqslant R(\sigma, M) \quad \text{for } t \in [0, T]. \tag{13}$$

This proposition is proved in [18] (Theorem 2).

The following corollary is easily seen.

Corollary 2.1. *If $\sigma(t, z) = C_1 + C_2 z$ for C_1, $C_2 \geqslant 0$ in Proposition 2.2 then,*

$$\|u\|_{E_t} \leqslant e^{C_2 t}(\|\Psi\|_{\Gamma_t} + C_1 t) \quad \textit{for } t \in [0, T]. \tag{14}$$

Definition 2.3. Let $(X, \|\cdot\|)$ be a real normed space and $R > 0$ any constant. We define $I_R\colon X \mapsto X$ by

$$I_R(x) = \begin{cases} x, & \text{if } \|x\| \leqslant R; \\ \dfrac{x}{\|x\|} R & \text{if } \|x\| > R. \end{cases} \tag{15}$$

It is evident that

$$\|I_R(x)\| = \min(\|x\|, R), \quad \|I_R(x) - I_R(y)\| \leqslant 2\|x - y\| \quad \text{in } X. \tag{16}$$

For any function $f\colon \Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ and $R > 0$ we define $f_R\colon \Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$f_R(t, x, u, w, p) = f(t, x, I_R(u), I_R(w), p). \tag{17}$$

By Proposition 2.2 we have

Remark 2.1. Let $\|\Psi\|_\Gamma \leqslant M, \sigma \in 0_M$ If $f \in X_{\sigma,M}$ then for every $R \geqslant R(\sigma, M)$,

(i) $f_R \in X_{\sigma,M}$,
(ii) $\mathrm{SOL}(f, \Psi) = \mathrm{SOL}(f_R, \Psi)$,
(iii) $\mathrm{CSL}(f, \Psi, \varepsilon) = \mathrm{CSL}(f_R, \Psi, \varepsilon)$.

Let, $A \subset \mathbb{R}^{1+n}$. Define $C_L(A) \subset C(A)$ as the set of all lipschitz in x continuous functions. Put

$$L_x[u] = \sup\left\{\frac{|u(t,x) - u(t,\bar{x})|}{|x - \bar{x}|}\colon\ (t,x), (t,\bar{x}) \in A, x \neq \bar{x}\right\}.$$

For a fixed $\tilde{L} \geqslant 0$ we write $C_L(A; \tilde{L}) = \{u \in C_L(A)\colon\ L_x[u] \leqslant \tilde{L}\}$.

In the following $C^{1+\alpha/2, 2+\alpha}(A)$, $C^{\alpha/2,\alpha}(A)$ stand for the Hölder spaces which are considered in the classical theory of parabolic equations (see [14]). We write $C^{\alpha/2,\alpha}(\mathbb{D}, q)$ for the ball in $C^{\alpha/2,\alpha}(\mathbb{D})$.

Assumption 2.1.

Let $\alpha \in (0, 1)$. Suppose that

1) $\|\Psi\|_\Gamma \leqslant M$, and there exists $\sigma \in O_M$ such that $f \in X_{\sigma,M}$;

2) there exists a nondecreasing function $\varrho\colon\ \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x,u,w,p)| \leqslant \varrho(\max{(|u|, \|w\|_{\mathbb{D}})}(1 + |p|^2)$$

in $\Theta \times \mathbb{R} \times C(\mathbb{D}) \times \mathbb{R}^n$;

3) for every $R, L \geqslant 0$ there exists a constant $C(R, L) \geqslant 0$ such that

$$|f(t,x,u,w,p) - f(t,x,\bar{u},\overline{w},\bar{p})| \leqslant C(R,L)(|u - \bar{u}|^\alpha + \|w - \overline{w}\|_{\mathbb{D}} + |p - \bar{p}|)$$

in $\Theta \times [-R, R] \times K(R) \times B(L)$;

4) for every $R, q, L \geqslant 0$ there exists a constant $H(R, q, L) \geqslant 0$ such that

$$|f(t,x,u,w,p) - f(\bar{t},\bar{x},u,w,p)| \leqslant H(R,q,L)(|t - \bar{t}|^{\alpha/2} + |x - \bar{x}|^\alpha)$$

in $\Theta \times [-R, R] \times C^{\alpha/2,\alpha}(\mathbb{D}, q) \times B(L)$;

5) there exists $\tilde{\Psi} \in C^{1+\alpha/2,2+\alpha}(\overline{\Theta}) \cap C^{\alpha/2,\alpha}(\overline{E})$ such that $\tilde{\Psi}_{|\Gamma} = \Psi$.

Definition 2.4. We will say that IBVP (9), (10) satisfies the compatibility condition if

$$D_t\Psi(0,x) - \varepsilon\Delta_x\Psi(0,x) = f(0,x,\Psi(0,x),\Psi_{(0,x)},D\Psi(0,x)) \tag{18}$$

for $x \in \partial\Omega$.

We base on

Theorem 2.1. *Assume that $\partial\Omega$ belongs to class $C^{2+\alpha}$, problem* (9), (10) *satisfies* (18) *and f, Ψ satisfy Assumption* 2.1. *Then* (9), (10) *has a solution $u \in C^{1+\alpha/2,2+\alpha}(\overline{\Theta}) \cap C^{\alpha/2,\alpha}(\overline{E})$.*

This is a simplified version of Theorem 2.1 [19].

To prove convergence of the vanishing viscosity method we need

Assumption 2.2. Suppose that,

1. $\|\Psi\|_\Gamma \leqslant M$ and there exists $\sigma \in O_M$ such that $f \in X_{\sigma,M}$. Put $R = R(\sigma, M)$.
2. There exists a *modulus* ω_R and a constant $C_R \geqslant 0$ such that
$$|f(t,x,u,w,p) - f(t,x,\bar{u},\overline{w},\bar{p})| \leqslant C_R \max(|u-\bar{u}|, \|w-\overline{w}\|_{\mathbb{D}}) + \omega_R(|p-\bar{p}|)$$
in $\Theta \times [-R,R] \times K(R) \times \mathbb{R}^n$.
3. There exists $\tilde{C}_R \geqslant 0$ such that
$$|f(t,x,u,w,p) - f(t,y,u,w,p)| \leqslant \tilde{C}_R(1 + \max(L_x[w], |p|))|x-y|$$
in $\Theta \times [-R,R] \times K(R) \cap C_L(\mathbb{D}) \times \mathbb{R}^n$.
4. There exists $L_0 \geqslant 0$ and $\tilde{\Psi} \in C_L(E, L_0)$ such that, $\tilde{\Psi}_{|\Gamma} = \Psi$.

The fact that we take the space $C_L(\mathbb{D})$ in 3) allows us to apply our results not only to differential-integral equations but to equations with a retarded and deviated argument as well (see the last paragraph). It would not be possible if we took in 3) $C(\mathbb{D})$ in place of $C_L(\mathbb{D})$ (without $L_x[w]$ on the right). Of course the assumption would be stronger in that case.

Remark 2.2. In view of Proposition 2.2 and Remark 2.1, without loss of generality we can assume that $C_R = C$, $\tilde{C}_R = \tilde{C}$, $\omega_R = \omega$ i.e. 2.) and 3.) of Assumption 2.2 are global.

Definition 2.5. We will say that $f \in Y(\sigma, M, C, \tilde{C})$ if f satisfies Assumption 2.2 in global sense (i.e. $C_R = C$, $\tilde{C}_R = \tilde{C}$, $\omega_R = \omega$).

Remark 2.3. Let $\|\Psi\|_\Gamma \leqslant M, \sigma \in O_M$. If f satisfies Assumption 2.2 then for every $R \geqslant R(\sigma, M)$, $f_R \in Y(\sigma, M, C, \tilde{C})$ where $C = 2C_R$, $\tilde{C} = \tilde{C}_R$.

Put $\gamma = \|f(\cdot,\cdot,0,0,0)\|_{\overline{\Theta}}$. Corollary 2.1 implies

Remark 2.4. Let f satisfy Assumption 2.2 and let $u \in CLS(f, \Psi, \varepsilon)$ ($u \in \mathrm{SOL}(f, \Psi)$). Then

$$\|u\|_{E_t} \leqslant e^{Ct}(\|\Psi\|_{\Gamma_t} + \gamma t), \quad t \in [0,T]. \tag{19}$$

To make our notation shorter we introduce

Definition 2.6. We will say that a family $(f_\varepsilon, \Psi_\varepsilon)_{\varepsilon>0}$ has the B-property if there exists $\tilde{L}_0 \geqslant 0$ independent of ε such that $\|Du_\varepsilon\|_{[0,T]\times\delta\Omega} \leqslant \tilde{L}_0$, for every $u_\varepsilon \in \mathrm{CSL}(f_\varepsilon, \Psi_\varepsilon, \varepsilon)$.

The following lemma says when the B-property is global. It is crucial in our method.

Lemma 2.1. *Let $\|\Psi_\varepsilon\|_\Gamma \leqslant M$ and let $u_\varepsilon \in \mathrm{CSL}(f_\varepsilon, \Psi_\varepsilon, \varepsilon)$, $\varepsilon > 0$. Suppose that*

(i) *there exist $C, \tilde{C}, M, L_0 \geqslant 0$, $\sigma \in O_M$ such that $f_\varepsilon \in Y(\sigma, M, C, \tilde{C})$, $\Psi_\varepsilon \in C_L(E, L_0)$ for every $\varepsilon > 0$;*

(ii) *$(f_\varepsilon, \Psi_\varepsilon)_{\varepsilon>0}$ has the B-property.*

Then there exists $L \geqslant 0$ independent of ε such that $\|Du_\varepsilon\|_{\overline{\Theta}} \leqslant L$.

Proof. Of course we can assume that $\|Du_\varepsilon\|_{[0,T]\times\delta\Omega} \leqslant \tilde{L}_0$, where $L_0 \leqslant \tilde{L}_0$. Define $\Omega(\delta) = \{x \in \Omega\colon \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta\}, \Theta(\delta) = (0,T) \times \Omega(\delta)$, $E(\delta) = \{(t,x) \in E\colon \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_{r_0}) > \delta\}$, $\Gamma(\delta) = E(\delta) \setminus \Theta(\delta)$, where Ω_{r_0} is defined in the first section.

Fix $\delta, \varepsilon > 0$ sufficiently small. Let $\xi \in R^n$ such that $|\xi| < \delta$. Put $\bar{u}_\varepsilon(t,x) = u_\varepsilon(t, x+\xi)$ for $(t,x) \in E(\delta)$ and $\overline{f}_\varepsilon(t,x,u,w,p) = f_\varepsilon(t, x+\xi, u, w, p)$. Write $L_t^\varepsilon = \max(\|Du_\varepsilon\|_{\overline{\Theta}_t}, \tilde{L}_0)$. It is easy to check that $\bar{u}_\varepsilon$ satisfies

$$D_t u(t,x) - \varepsilon\Delta_x u(t,x) = \overline{f}_\varepsilon(t,x,u(t,x),u_{(t,x)},Du(t,x)) = 0 \quad \text{in } \Theta(\delta).$$

Put

$$\begin{aligned} g(t,x,z,w,p) = {} & \overline{f}_\varepsilon(t,x,z+u_\varepsilon(t,x), w+u_{\varepsilon(t,x)}, p+Du_\varepsilon(t,x)) \\ & - f_\varepsilon(t,x,u_\varepsilon(t,x),u_{\varepsilon(t,x)},Du_\varepsilon(t,x)). \end{aligned}$$

Of course $\bar{u}_\varepsilon - u_\varepsilon$ satisfies

$$D_t u(t,x) - \varepsilon\Delta_x u(t,x) = g(t,x,u(t,x),u_{(t,x)},Du(t,x)) = 0 \quad \text{in } \Theta(\delta)$$

and

$$\begin{aligned} |g(t,x,z,w,0)| \leqslant {} & |f_\varepsilon(t,x+\xi,z+u_\varepsilon(t,x), w+u_{\varepsilon(t,x)}, Du_\varepsilon(t,x)) \\ & - f_\varepsilon(t,x+\xi,u_\varepsilon(t,x),u_{\varepsilon(t,x)},Du_\varepsilon(t,x))| \\ & + |f_\varepsilon(t,x+\xi,u_\varepsilon(t,x),u_{\varepsilon(t,x)},Du_\varepsilon(t,x)) \\ & - f_\varepsilon(t,x,u_\varepsilon(t,x),u_{\varepsilon(t,x)},Du_\varepsilon(t,x))| \\ \leqslant {} & C(\max(|u|, \|w\|_{\mathbb{D}})) + \tilde{C}(1+L_t^\varepsilon)|\xi|. \end{aligned}$$

Thus by Corollary 2.1 (for $E(\delta)$) we get

$$\|u_\varepsilon - \bar{u}_\varepsilon\|_{E_t(\delta)} \leqslant e^{Ct}[\|u-\bar{u}\|_{\Gamma_t(\delta)} + t\tilde{C}(1+L_t^\varepsilon)\,|\xi|]$$

and

$$\max(\|Du_\varepsilon\|_{\overline{\Theta(\delta)}_t}, \tilde{L}_0) \leqslant \mathrm{e}^{Ct}(\max(\|Du^\varepsilon\|_{\Gamma_\delta\cap\overline{\Theta}}, \tilde{L}_0) + \tilde{C}t(1+L_t^\varepsilon)].$$

Letting $\delta \to 0$ (note that $u_\varepsilon \in C_*^{1,2}(E)$) we conclude that

$$L_t^\varepsilon \leqslant \mathrm{e}^{Ct}[\tilde{L}_0 + \tilde{C}t(1+L_t^\varepsilon)].$$

Let $m \in \mathbb{N}$ be such that $1 - \tilde{C}h\mathrm{e}^{ch} \geqslant 0$ for $h = T/m, t_i = ih, i = 0,1,2,\dots m, L_i^\varepsilon = L_{t_i}^\varepsilon$. Aa analysis similar to the above shows that

$$L_i^\varepsilon \leqslant \mathrm{e}^{Ch}[L_{i-1}^\varepsilon + h\tilde{C}(1+L_i^\varepsilon)] \quad \text{for } i = 1,2,\dots m,$$

which yields

$$L_i^\varepsilon \leqslant \frac{\mathrm{e}^{Ch}}{1-\tilde{C}h\mathrm{e}^{Ch}}(L_{i-1}^\varepsilon + \tilde{C}h).$$

Writing $\alpha(h) = \mathrm{e}^{Ch}/(1-\tilde{C}h\mathrm{e}^{Ch})$ we get

$$L^\varepsilon = L_m^\varepsilon \leqslant \alpha^m(h)\tilde{L}_0 + \tilde{C}h\sum_{k=1}^{n}\alpha^k(h) \leqslant \alpha^m(h)(\tilde{L}_0 + \tilde{C}T). \tag{20}$$

Since $\alpha^m(h) \to \mathrm{e}^{(C+\tilde{C})T}$ as $h \to 0$ it is easily seen that

$$L^\varepsilon \leqslant \mathrm{e}^{(C+\tilde{C})T}(\tilde{L}_0 + \tilde{C}T) = L,$$

which proves the lemma. □

Now we will specify some conditions under which the B-property holds. For the sake of simplicity we consider a constant family $(f,0)$.

Let $\varrho(x) = \mathrm{dist}(x,\partial\Omega)$. Put $\Omega^\delta = \{x \in \Omega\colon\ \varrho(x) < \delta\}$ for $\delta > 0$. We define

$$\Theta^\delta,\ \Gamma^\delta,\ E^\delta,\ \partial_0\Omega^\delta,\ \Theta_0^\delta,\ \partial_0\Theta^\delta \quad \text{in the same way as } \Theta,\ \Gamma,\ E,\ \partial_0\Omega,\ \Theta_0,\ \partial_0\Theta.$$

We define the upper and lower (classical) solution of (9) in a standard way. We will need

Lemma 2.2. *Consider problem* (9), (10) *in the set* E^δ (E). *Suppose that* f *satisfies Assumption* 2.2 1) 2) *and is nondecreasing with respect to* w. *Let* u *be a lower solution and* $\bar{u}$ *an upper solution of* (9). *Then* $u \leqslant \bar{u}$ *in* Γ^δ (Γ) *implies* $u \leqslant \bar{u}$ *in* E^δ (E).

The proof is a simple application of Theorem 3 in [18].

Write $C^{1,2}_*(E^\delta) = C^{1,2}(\overline{\Theta}^\delta) \cap C(\overline{E}^\delta)$. Put

$$P_\varepsilon[z](t,x) = D_t z(t,x) - \varepsilon\Delta_x z(t,x) - f(t,x,z(t,x),z_{(t,x)},Dz(t,x)).$$

Lemma 2.3. *Suppose that*

(i) f *satisfies Assumption* 2.2 *and is nondecreasing with respect to* w,

(ii) $f(t,x,0,0,0) \geqslant 0$ *for* $(t,x) \in \Theta$,

(iii) *there exist* $\delta \geqslant 0$ *and* $\varphi \in C^{1,2}_*(E^\delta)$ *such that,* $\varphi = 0$ *in* $\partial_0\Theta^\delta \cap \partial_0\Theta$, $\varphi(t,x) \geqslant \gamma t e^{Ct}$ *in* $\partial_0\Theta^\delta \setminus \partial_0\Theta$, $\varphi \geqslant 0$ *in* Θ^δ_0 *and*

$$D_t\varphi(t,x) - f(t,x,\varphi(t,x),\varphi_{(t,x)},D\varphi(t,x)) > 0 \quad \text{in} \quad \Theta^\delta. \tag{21}$$

Then $(f,0)$ *has the* B*-property.*

Proof. Let $u^\varepsilon \in CLS(f,0,\varepsilon)$. From (i), (ii) and from Lemma 2.2 we have $u^\varepsilon \geqslant 0$ in Θ^δ (0 is a lower solution). On the other hand, $u^\varepsilon \leqslant \varphi$ in Γ^δ (see Remark 2.4)) and

$$P_\varepsilon[\varphi](t,x)) > \lambda_0 - \varepsilon\Delta_x\varphi > \lambda_0/2 \geqslant 0 \quad \text{in } \Theta^\delta$$

for some $\lambda_0 > 0$ and ε sufficiently small. Thus φ is an upper solution of our parabolic problem in E^δ. By Lemma 2.2 this gives $u^\varepsilon \leqslant \varphi$ in Θ^δ. Finally,

$$0 \leqslant u^\varepsilon(t,x) - u^\varepsilon(t,x_0) \leqslant \varphi(t,x) - \varphi(t,x_0) \quad \text{for } x_0 \in \partial\Omega,\ x \in \Omega^\delta$$

since $u^\varepsilon(t,x_0) = \varphi(t,x_0) = 0$ and $D\varphi$ is bounded. This completes the proof. □

To illustrate the above lemma we will give the following examples.

Example 2.1. Let $\partial\Omega \in C^2$. Then $\varrho \in C^2(\Omega^\delta)$ and $|D\varrho(x)| \geqslant \frac{1}{2}$ for some $\delta > 0$. Let $b > 0$ be such that $\delta e^{b(t+1)} > \gamma t e^{Ct}$. Then there exists $\varphi \in C^{1,2}_*(E^\delta)$ such that $\varphi \geqslant 0$, $\varphi = 0$ in $\partial_0\Theta$, $\varphi(t,x) = \varrho(x)e^{b(t+1)}$ in Θ^δ and $\gamma t e^{Ct} \leqslant \varphi(t,x) \leqslant 2\delta e^{b(t+1)}$ in $\partial_0\Theta^\delta \setminus \partial_0\Theta$. Assume that there exists $\lambda_0 > 0$ such that for $\bar{x} \in \partial\Omega$, $t \in [0,T]$

$$\limsup_{|p|\to\infty,k\to\infty} f(t,\bar{x},0,k,p) < -\lambda_0 < 0 \text{ if } r_0 > 0,$$
$$\limsup_{|p|\to\infty} f(t,\bar{x},0,0,p) < -\lambda_0 < 0 \text{ if } r_0 = 0$$

where k denotes any constant function. (Recall that $r_0 = 0$ means that there is no functional dependence in x in the equation). It is easy to check that φ satisfies (iii) in Lemma 2.3 for some b.

We will verify inequality (21). For $(t,x) \in \Theta^\delta$ and some $\bar{x} \in \partial\Omega$ we have

$$
\begin{aligned}
& b\varrho(x)\mathrm{e}^{b(t+1)} - f(t,x,\varrho(x)\mathrm{e}^{b(t+1)},\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant b\varrho(x)\mathrm{e}^{b(t+1)} - f(t,x,\varrho(x)\mathrm{e}^{b(t+1)},\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \quad + f(t,x,0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) - f(t,x,0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant b\varrho(x)\mathrm{e}^{b(t+1)} - C\varrho(x)\mathrm{e}^{b(t+1)} - f(t,x,0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant b\varrho(x)\mathrm{e}^{b(t+1)} - C\varrho(x)\mathrm{e}^{b(t+1)} + f(t,\bar{x},0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \quad - f(t,x,0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) - f(t,\bar{x},0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant b\varrho(x)\mathrm{e}^{b(t+1)} - C\varrho(x)\mathrm{e}^{b(t+1)} \\
& \quad - \tilde{C}(1+\|D\varrho\|_{\Omega^\delta}\mathrm{e}^{b(t+1)})|x-\bar{x}| - f(t,\bar{x},0,\varphi_{(t,x)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant b\varrho(x)\mathrm{e}^{b(t+1)} - C\varrho(x)\mathrm{e}^{b(t+1)} \\
& \quad - 2\tilde{C}(1+\mathrm{e}^{b(t+1)})\varrho(x) - f(t,\bar{x},0,2\delta\mathrm{e}^{b(T+1)},D\varrho(x)\mathrm{e}^{b(t+1)}) \\
& \geqslant\ - f(t,\bar{x},0,2\delta\mathrm{e}^{b(T+1)},D\varrho(x)\mathrm{e}^{b(t+1)}) > \lambda_0 > 0
\end{aligned}
$$

for b sufficiently large (notice that $\varphi_{(t,x)} \leqslant 2\delta\mathrm{e}^{b(T+1)}$). The case $r_0 = 0$ we treat in a similar way (see the next lemma).

Now we will present another set of assumptions which yields (ii) of Lemma 2.1

Lemma 2.4. *Suppose that $\partial\Omega$ is analytic, f satisfies Assumption 2.2, is nondecreasing in w and there exists $p_0 \geqslant 0$ such that for every $\bar{x} \in \partial\Omega$, $t \in [0,T]$ and a constant function $k \in C(\mathbb{D})$*

$$f(t,\bar{x},0,k,p) = 0 \quad \text{for } |p| \geqslant p_0, \ \text{if } r_0 > 0 \tag{24}$$

$$f(t,\bar{x},0,0,p) = 0 \quad \text{for } |p| \geqslant p_0, \ \text{if } r_0 = 0 \tag{25}$$

Then $(f,0)$ has the B-property.

Proof. Since $\partial\Omega$ is analytic, by the Cauchy-Kowalewska theorem there exist $\delta > 0$ and $\varphi \in C^2(\Omega^\delta)$ such that

$$\Delta\varphi = 0 \text{ in } \Omega^\delta, \quad \varphi = 0 \text{ in } \partial\Omega, \quad \partial_n\varphi = 1 \text{ in } \partial\Omega$$

where $\partial_n\varphi$ denotes the normal interior derivative of φ. Of course without loss of generality we can assume that $|D\varphi| \geqslant \frac{1}{2}$ in Ω^δ. Moreover, we can also assume that $\varphi > 0$ in $\Omega^\delta \setminus \Omega$ and consequently

$$\varphi(x) = |\varphi(x)| = |\varphi(x) - \varphi(\bar{x})| \geqslant \frac{1}{2}|x-\bar{x}| \tag{26}$$

for some $\bar{x} \in \partial\Omega$ and $x \in \Omega^\delta$.

Put $\delta_0 = \|\varphi\|_{\partial\Omega^\delta} > 0$ and $\delta_1 = \min\limits_{\partial\Omega^\delta\setminus\partial\Omega} \varphi > 0$. The maximum principle implies that $\varphi \leqslant \delta_0$ in Ω^δ. Let $\psi \in C_*^{1,2}(E^\delta)$ be such that $\psi \geqslant 0$, $\psi = 0$ in $\partial_0\Theta^\delta \cap \partial_0\Theta$, $\psi(t,x) = \varphi(x)\mathrm{e}^{b(t+1)}$ in Θ^δ and $\gamma t\mathrm{e}^{Ct} \leqslant \psi(t,x) \leqslant 2\delta_0\mathrm{e}^{b(t+1)}$ in $\partial_0\Theta^\delta \setminus \partial_0\Theta$ where b is such that $\gamma t\mathrm{e}^{Ct} < \delta_1\mathrm{e}^{b(t+1)}$.

We will show that ψ is an upper solution of the problem

$$D_t u(t,x) - \varepsilon\Delta_x u(t,x) = f_\varepsilon(t,x,u(t,x),u_{(t,x)},Du(t,x)) \quad \text{in } \Theta^\delta, \tag{27}$$

$$u(t,x) = 0 \text{ in } \Gamma^\delta. \tag{28}$$

Indeed, since φ is harmonic , by Assumption 2.2 we have

$$\begin{aligned}
P_\varepsilon[\psi](t,x) &= b\varphi(x)\mathrm{e}^{b(t+1)} - \varepsilon\mathrm{e}^{b(t+1)}\Delta\varphi(x) - f(t,x,\varphi(x)\mathrm{e}^{b(t+1)},\psi_{(t,x)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&= b\varphi(x)\mathrm{e}^{b(t+1)} - f(t,x,\varphi(x)\mathrm{e}^{b(t+1)},\psi_{(t,x)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&\quad + f(t,x,0,\psi_{(t,x)},D\varphi(x)\mathrm{e}^{b(t+1)}) - f(t,x,0,\psi_{(t,x)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&\geqslant b\varphi(x)\mathrm{e}^{b(t+1)} - C\varphi(x)\mathrm{e}^{b(t+1)} - f(t,x,0,\psi_{(t,x)},D\varphi(x)\mathrm{e}^{b(t+1)})
\end{aligned}$$

and by the monotonicity of f

$$\begin{aligned}
P_\varepsilon[\psi](t,x) &\geqslant b\varphi(x)\mathrm{e}^{b(t+1)} - C\varphi(x)\mathrm{e}^{b(t+1)} - f(t,x,0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&\geqslant b\varphi(x)\mathrm{e}^{b(t+1)} - C\varphi(x)\mathrm{e}^{b(t+1)} - f(t,x,0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&\quad + f(t,\bar{x},0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)}) - f(t,\bar{x},0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)})
\end{aligned}$$

for some $\bar{x} \in \partial\Omega$ satisfying (26). This implies in view of Assumption 2.2 3.)

$$\begin{aligned}
P_\varepsilon[\psi](t,x) &\geqslant b\varphi(x)\mathrm{e}^{b(t+1)} - C\varphi(x)\mathrm{e}^{b(t+1)} \\
&\qquad - \tilde{C}(1 + |\mathrm{e}^{b(t+1)}D\varphi|)|x-\bar{x}| - f(t,\bar{x},0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)})
\end{aligned}$$

and by (26) we obtain

$$\begin{aligned}
P_\varepsilon[\psi](t,x) &\geqslant b\varphi(x)\mathrm{e}^{b(t+1)} - C\varphi(x)\mathrm{e}^{b(t+1)} - 2\tilde{C}(1 + \mathrm{e}^{b(t+1)}\|D\varphi\|_{\Omega^\delta})\varphi(x) \\
&\quad - f(t,\bar{x},0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)}) \\
&\geqslant (b-A)\varphi(x)\mathrm{e}^{b(t+1)} - f(t,\bar{x},0,2\delta_0\mathrm{e}^{b(t+1)},D\varphi(x)\mathrm{e}^{b(t+1)}),
\end{aligned}$$

where $A = 2\tilde{C}(1 + \|D\varphi\|_{\Omega^\delta})$. This yields (see (24))

$$P_\varepsilon[\psi](t,x) \geqslant 0 \quad \text{for } b \text{ sufficiently large.}$$

In a similar way we can show that $-\psi$ is a lower solution of problem (27), (28) for b large.

In view of Lemma 2.2 we have $-\psi(t,x) \leqslant u^\varepsilon(t,x) \leqslant \psi(t,x)$ for $(t,x) \in \Theta$, for some b and since $\psi(t,x_0) = u^\varepsilon(t,x_0) = 0$ for $x_0 \in \partial\Omega$,we have

$$|u^\varepsilon(t,x) - u^\varepsilon(t,x_0)| \leqslant |\psi(t,x) - \psi(t,x_0)| \quad \text{for } x_0 \in \partial\Omega,\ x \in \Omega^\delta,$$

which gives the desired conclusion.

The proof for $r_0 = 0$ is similar. The only difference is that in this case $\|\psi_{(t,x)}\|_{\mathbb{D}} \leqslant \varphi(x)\mathrm{e}^{b(t+1)}$ (this is not necessarily true for $r_0 > 0$) and we can treat the functional and nonfunctional arguments similarly. □

3. The vanishing viscosity method

In the proof of the next lemma we will apply the following property of viscosity inequalities,

Proposition 3.1. *Let $a > 0$ and $h, H \in C([0,a])$. Assume that h is a viscosity solution of $h' \leqslant H$ (i.e. h is a viscosity subsolution of $h' = H$) in $(0,a)$. Then*

$$h(t) \leqslant h(s) + \int_s^t H(\tau)\,\mathrm{d}\tau \quad \text{for } 0 \leqslant s \leqslant t \leqslant a.$$

The proof can be found in [8], p. 12.

Lemma 3.1. *Suppose that the hypotheses of Lemma 2.1 are satisfied. Then there exist $K \geqslant 0$ and R_0, L independent of ε and $\varepsilon_0 > 0$ such that*

$$\|u_\varepsilon - u_\kappa\|_E \leqslant K(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A) \tag{29}$$

for $\varepsilon, \kappa \geqslant \varepsilon_0 > 0$ where $A = \Theta \times [-R_0, R_0] \times K(R_0) \times B(L)$.

Proof. Fix $\varepsilon, \kappa \geqslant 0$. Put for simplicity $u = u_\varepsilon, v = u_\kappa$. Define

$$m(t) = \sup\{|u(\tau,x) - v(\tau,x)|\colon\ 0 \leqslant \tau \leqslant t, x \in \overline{\Omega}\},\ t \in [0,T].$$

Put $R_0 = \tilde{R} = R(\sigma, M)$ and let L be defined as in Lemma 2.1. Then of course $\|u\|_E, \|v\|_E \leqslant R_0$ and L is a lipschitz constant in x for u, v.

Put $R = 5(R_0+1)$, $h = \sqrt[4]{\varepsilon} + \sqrt[4]{\kappa}$.

Set $\beta_h(z) = \beta(z/h)$, $\beta \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant \beta \leqslant 1$, $\beta(z) = 0$ for $|z| > 1$, $\beta(z) = 1 - |z|^2$ for $|z| \leqslant \sqrt{2}/2$, $\beta(z) < 1/2$ for $|z| > \sqrt{2}/2$.

Define

$$\begin{aligned} M(t) = \sup\{&|u(t-s,x) - v(t-s,y)| \\ &+ R\mathrm{e}^{Ct}\beta_h(x-y)\colon\ 0 \leqslant s \leqslant t,\ |x-y| \leqslant Lh^2,\ x,y \in \overline{\Omega}\}. \end{aligned} \tag{30}$$

□

It is important that m and M are not defined here in the same way as in the classical theory of viscosity solutions. We must take the supremum also for the past. This makes the proof much more complicated.

We see at once that

$$M(t) \geqslant \sup\{|u(t-s,x) - v(t-s,x)| + R\mathrm{e}^{Ct} \colon\ 0 \leqslant s \leqslant t,\ x \in \overline{\Omega},\} \tag{31}$$

which yields

$$M(t) \geqslant m(t) + R\mathrm{e}^{Ct}. \tag{32}$$

Let $\tilde{K} = 2L^2$. If

$$M(T) \leqslant \tilde{K}(\sqrt{\varepsilon} + \sqrt{\kappa}) + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{CT}$$

then by (32)

$$m(T) \leqslant \tilde{K}(\sqrt{\varepsilon} + \sqrt{\kappa}) + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma,$$

and the proof is complete.

Suppose that

$$M(T) > \tilde{K}(\sqrt{\varepsilon} + \sqrt{\kappa}) + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{CT}.$$

Since $M(0) \leqslant (Lh)^2 + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R$ there exists t_* such that

$$M(t_*) = \tilde{K}(\sqrt{\varepsilon} + \sqrt{\kappa}) + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{Ct_*} \tag{33}$$

and

$$M(t) > \tilde{K}(\sqrt{\varepsilon} + \sqrt{\kappa}) + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{Ct} \quad \text{for } t \in (t_*, T]. \tag{34}$$

We will show that there exists $K_1 \geqslant 0$ such that

$$M'(t) \leqslant CM(t) + K_1(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A) \tag{35}$$

in a viscosity sense for $t \in (t_*, T)$.

Let $\eta \in C^1((t^*, T))$ and suppose that $M - \eta$ has a local maximum at $\tilde{t} \in (t_*, T)$. Of course we can assume that $\eta'(\tilde{t}) > 0$. It follows from Lemma 1.4 of [8] that we can find a function $\overline{\eta} \in C^1([t^*, T])$ such that $\overline{\eta}'(\tilde{t}) = \eta'(\tilde{t})$ and $(M - \overline{\eta})(t_0) > (M - \overline{\eta})(t)$ for $t \neq t_0$. To simplify notation we continue to write η for $\overline{\eta}$.

Set $\Delta = \{(t,s)\colon\ t \in [t^*, T],\ 0 \leqslant s \leqslant t\}$. Let $\Phi\colon\ \Delta \times \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be given by

$$\Phi(t,s,x,y) = |u(t-s,x) - v(t-s,y)| + R\mathrm{e}^{Ct}\beta_h(x-y) - \eta(t). \tag{36}$$

Let $(t_0, s_0, x_0, y_0) \in \Delta \times \overline{\Omega} \times \overline{\Omega}$ be such that

$$\Phi(t_0,s_0,x_0,y_0) = \sup\{\Phi(t,s,x,y)\colon\ (t,s,x,y) \in \Delta \times \overline{\Omega} \times \overline{\Omega}\}. \tag{37}$$

We proceed to show that

$$|x_0 - y_0| \leqslant Lh^2. \tag{38}$$

We first prove that

$$|x_0 - y_0| \leqslant \frac{\sqrt{2}}{2}h. \tag{39}$$

Indeed, since $\Phi(t_0,s_0,x_0,y_0) \geqslant \Phi(t_0,s,x,x)$ we get

$$2P + R\mathrm{e}^{Ct_0}\beta_h(x_0-y_0) \geqslant R\mathrm{e}^{Ct_0}$$

which yields

$$\beta_h(x_0-y_0) \geqslant (3P+5)/(5P+5) > 1/2$$

and finally leads to (39).

We will show that

$$|u(t_0-s_0,x_0) - v(t_0-s_0,y_0)| > \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma. \tag{40}$$

On the contrary, suppose that $|u(t_0-s_0,x_0) - v(t_0-s_0,y_0)| \leqslant \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma$.

Since $\Phi(t_0,s_0,x_0,y_0) \geqslant \Phi(t_0,s,x,y)$ we obtain for $s \leqslant t_0,\ |x-y| \leqslant Lh^2$,

$$\|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{Ct_0}\beta_h(x_0-y_0) \geqslant |u(t_0-s,x) - v(t_0-s,y)| + R\mathrm{e}^{Ct_0}\beta_h(x-y)$$

and consequently, $\|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + R\mathrm{e}^{Ct_0} \geqslant M(t_0)$ which contradicts (34).

Thus we can assume that $u(t_0-s_0,x_0) - v(t_0-s_0,y_0) > \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma$. The case $u(t_0-s_0,x_0) - v(t_0-s_0,y_0) < -\|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma$ we treat analogously. Since

$$\Phi(t_0,s_0,x_0,y_0) \geqslant \Phi(t_0,s_0,x,y_0) \tag{41}$$

it follows that

$$\begin{aligned} &u(t_0-s_0,x_0) - v(t_0-s_0,y_0) + R\mathrm{e}^{Ct_0}\beta_h(x_0-y_0) - \eta(t_0) \\ &\quad \geqslant u(t_0-s_0,x) - v(t_0-s_0,y_0) + R\mathrm{e}^{Ct_0}\beta_h(x-y_0) - \eta(t_0), \end{aligned}$$

hence

$$Re^{Ct_0}(\beta_h(x-y_0)-\beta_h(x_0-y_0)) \leqslant u(t_0-s_0,x_0)-u(t_0-s_0,x) \leqslant L|x-x_0|.$$

This together with (39) gives (38).

Our next claim is that $\tilde{t}=t_0$. Indeed, it follows from (37) that

$$\begin{aligned} u(t_0-s_0,x_0)-v(t_0-s_0,y_0)+Re^{Ct_0}\beta_h(x_0-y_0)-\eta(t_0) \\ \geqslant u(t-s,x)-v(t-s,y)+Re^{Ct}\beta_h(x-y)-\eta(t) \end{aligned}$$

where $0 \leqslant s \leqslant t,\ \|x-y\| \leqslant Lh^2$.

This in view of (30) and (38) gives $M(t_0)-\eta(t_0) \geqslant M(t)-\eta(t)$. Putting $t=\tilde{t}$ we obtain by the property of η that $\tilde{t}=t_0$.

Observe now that we may assume $x_0, y_0 \in \Omega$. Indeed, suppose that $x_0 \in \delta\Omega$. An analysis similar to the above (for $t=t_0$) and the inequality

$$|u(t_0-s_0,x_0)-v(t_0-s_0,x_0)| \leqslant \|\Psi_\varepsilon-\Psi_\kappa\|_\Gamma$$

imply

$$\|\Psi_\varepsilon-\Psi_\kappa\|_\Gamma+L|x_0-y_0|+Re^{Ct_0} \geqslant M(t_0)$$

which leads to

$$\|\Psi_\varepsilon-\Psi_\kappa\|_\Gamma+2L^2(\sqrt{\varepsilon}+\sqrt{\eta})+Re^{Ct_0} \geqslant M(t_0).$$

This contradicts (34).

Similarly we can show that $s_0 \neq t_0$. The equality $s_0=t_0$ implies

$$|u(0,x_0)-v(0,y_0)|+Re^{Ct_0}\beta_h(x-y)-\eta(t_0) \leqslant M(t_0)-\eta(t_0)$$

and the same reasoning leads to a contradiction.

Notice that the function

$$t \to (u(t-s_0,x_0)-v(t-s_0,y_0))+Re^{Ct}\beta_h(x_0-y_0)-\eta(t)$$

for $\max(t_*,s_0)<t<T$ attains its maximum at t_0. Thus

$$D_tu(t_0-s_0,x_0)-D_tv(t_0-s_0,y_0))+CRe^{Ct}\beta_h(x_0-y_0)=\eta'(t_0).$$

Moreover,

$$\begin{aligned} &x \to u(t_0-s_0,x)+Re^{Ct_0}\beta_h(x-y_0) \quad \text{has a maximum at } x_0\text{, and} \\ &y \to v(t_0-s_0,y)-Re^{Ct_0}\beta_h(x_0-y) \quad \text{has a minimum at } y_0. \end{aligned}$$

This gives

$$Du(t_0 - s_0, x_0) = Dv(t_0 - s_0, y_0) = p_0$$

and

$$\Delta_x u(t_0 - s_0, x_0) + R\mathrm{e}^{Ct_0}\Delta\beta_h(x_0 - y_0) \leqslant 0,$$
$$\Delta_x v(t_0 - s_0, y_0) - R\mathrm{e}^{Ct_0}\Delta\beta_h(x_0 - y_0) \geqslant 0.$$

By assumption,

$$D_t u(\tau_0, x_0) - \varepsilon\Delta u(\tau_0, x_0) = f_\varepsilon(\tau_0, x_0, u(\tau_0, x_0), u_{(\tau_0,x_0)}, Du(\tau_0, x_0)),$$
$$D_t v(\tau_0, y_0) - \kappa\Delta v(\tau_0, y_0) = f_\kappa(\tau_0, y_0, v(\tau_0, x_0), v_{(\tau_0,y_0)}, Dv(\tau_0, y_0))$$

where $\tau_0 = t_0 - s_0$. This implies

$$\begin{aligned}
&\eta'(t_0) - CR\mathrm{e}^{Ct_0}\beta_h(x_0 - y_0) + (\varepsilon + \kappa)R\mathrm{e}^{Ct_0}\Delta\beta_h(x_0 - y_0) \\
&\leqslant f_\varepsilon(\tau_0, x_0, u(\tau_0, x_0), u_{(\tau_0,x_0)}, p_0) - f_\kappa(\tau_0, y_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0) \\
&\leqslant f_\varepsilon(\tau_0, x_0, u(\tau_0, x_0), u_{(\tau_0,x_0)}, p_0) - f_\varepsilon(\tau_0, x_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0) \\
&\quad + f_\varepsilon(\tau_0, x_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0) - f_\varepsilon(\tau_0, y_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0) \\
&\quad + f_\varepsilon(\tau_0, y_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0) - f_\kappa(\tau_0, y_0, v(\tau_0, y_0), v_{(\tau_0,y_0)}, p_0)
\end{aligned}$$

and by Assumption 2.1

$$\eta'(t_0) - CR\mathrm{e}^{Ct_0} \leqslant C\|u_{(\tau_0,x_0)} - v_{(\tau_0,y_0)}\|_{\mathbb{D}} + C_P(1 + L)|x_0 - y_0| + \|f_\varepsilon - f_\kappa\|_A.$$

Since

$$\|u_{(\tau_0,x_0)} - v_{(\tau_0,y_0)}\|_{\mathbb{D}} \leqslant \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + m(t_0) + L|x_0 - y_0|$$

and $\Delta\beta_h(x_0 - y_0) = 2nh^{-2} \leqslant 2n(\sqrt{\varepsilon} + \sqrt{\kappa}$ we conclude (see (38), (32)) that

$$\eta'(t_0) \leqslant CM(t_0) + K_1(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A)$$

where $K_1 \geqslant 0$ is a constant independent of ε.

Thus (35) is proved. Applying Proposition 3.1 (for $H(t) = CM(t) + K_1(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A)$) we get (see (33))

$$M(t) \leqslant K_2(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A) + R\mathrm{e}^{Ct_*} + \int_{t_*}^{t} CM(s)\,\mathrm{d}s$$

for $t \in [t_*, T]$, $K_2 \geqslant 0$.

Hence Gronwall's inequality and (32) yield

$$m(t) \leqslant K(\sqrt{\varepsilon} + \sqrt{\kappa} + \|\Psi_\varepsilon - \Psi_\kappa\|_\Gamma + \|f_\varepsilon - f_\kappa\|_A)$$

in $[t_*, T]$ for some $K \geqslant 0$, and the proof is complete.

In view of the above result we can state

Theorem 3.1. *Suppose that there exist $f_\varepsilon \in Y(\sigma, M, C, \tilde{C}), \Psi_\varepsilon \in C_L(E, L_0)$ for $\varepsilon > 0$ and $\sigma, M, C, \tilde{C}, L_0$ independent of ε such that $(f_\varepsilon, \Psi_\varepsilon)_{\varepsilon>0}$ has the B-property, $(f_\varepsilon, \Psi_\varepsilon) \to (f, \Psi)$ as $\varepsilon \to 0$ and $\mathrm{CSL}(f_\varepsilon, \Psi_\varepsilon, \varepsilon) \neq 0$. Then there exists $u \in \mathrm{SOL}(f, \Psi)$ such that $u = lim_{\varepsilon\to 0} u_\varepsilon$ where $u_\varepsilon \in \mathrm{CSL}(f_\varepsilon, \Psi_\varepsilon, \varepsilon)$*

The formulation of this theorem is very general. It shows the method rather then any particular existence result.

Now we will present one of these results.

Suppose for simplicity that $\psi \equiv 0$. Assume that Assumption 2.2 holds and

$$f(0, \bar{x}, 0, 0, 0,) = 0 \quad \text{for} \quad \bar{x} \in \partial\Omega. \tag{42}$$

Let $\eta_\varepsilon \in C^\infty(\mathbb{R})$ be such that $\eta_\varepsilon(s) = 0$ for $|s| \leqslant \varepsilon$, $\eta_\varepsilon(s) = 1$ for $|s| > 2\varepsilon$ and $0 \leqslant \eta_\varepsilon \leqslant 1$. Let $R \geqslant R(\sigma, M)$. Put

$$f_{R,L}(t, x, u, w, p) = f(t, x, I_R(u), I_R(w), I_L(p)).$$

Define

$$\tilde{f}_\varepsilon(t, x, u, w, p) = \eta_\varepsilon(t)\eta_\varepsilon(\varrho(x))\eta_\varepsilon(|p|) f_{R,1/\varepsilon}(t, x, u, w, p)$$

and

$$f_\varepsilon(t, x, u, w, p) = \omega_\varepsilon^t * \omega_\varepsilon^x * \omega_\varepsilon^p * \tilde{f}_\varepsilon(\cdot, \cdot, u, w, \cdot)(t, x, p)$$

where $\omega_\varepsilon^z \in C_0^\infty(X)$, $\int \omega_\varepsilon^z = 1$, $\mathrm{supp}\omega_\varepsilon^z \subset K(0, \varepsilon)$, $X = \mathbb{R}, \mathbb{R}^n$, $z = t, x, p$. We can verify that $f_\varepsilon \in Y(\sigma_1, M_1, C, \tilde{C})$ for some $M_1, C, \tilde{C} \geqslant 0$ and $\sigma_1 \in O_{M_1}$, and $f_\varepsilon(0, x, 0, 0, 0,) = 0$ in $\partial\Omega$ for ε sufficiently large. Moreover, for every $L > 0$ we have $f_\varepsilon \to f$ in $\Theta \times [-R, R] \times K(R) \times B(L)$. Since f_ε is bounded for each ε and lipschitz continuous (global in all variable) the assumptions of Theorem 2.1 are satisfied. Thus $CLS(f_\varepsilon, 0, \varepsilon) \neq 0$.

The problem we face here is how to learn whether the family $(f_\varepsilon, 0)_{ep>0}$ has the B-property, or whether there exists a family satisfying the above condition which has the B-property. We do not know the answer to this question in general, i.e. without additional assumptions about f (in comparison to the Cauchy problem (see [11], [20]) where B-property is trivial). Such conditions are proposed in Example 2.1 and Lemma 2.4. We are aware of the fact that they may not be optimal. Nevertheless we can formulate an existence theorem,

Theorem 3.2. *Suppose that $\partial\Omega \in C^{2+\alpha}$ for $\alpha \in (0,1)$, f is nondecreasing in w and satisfies Assumption 2.2 and (42). Assume also that one of the following conditions holds:*

(i) *$f(t,x,0,0,0) \geqslant 0$ in Θ and f satisfies* (22) *(or* (23) *if $r_0 = 0$); or*

(ii) *$\partial\Omega$ is analytic and f satisfies* (24) *(or* (25) *if $r_0 = 0$)*

Then there exists a viscosity solution of (1), (2), *obtained by the vanishing viscosity method.*

P r o o f. In view of the above consideration it is enough to notice that f_ε satisfies (22), (23), (24), (25) with λ_0, p_0 independent of ε. □

4. First order IBVP with a deviated argument

In this section we will look more closely at Example 1.1. The only restriction is that now μ does not depend on x, i.e. $\mu(t,x) \equiv \mu(t)$. Recall some notation.

Let $g\colon\ \Theta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $\mu\colon\ [0,T] \to \mathbb{R}$, $\nu\colon\ \Theta \to \mathbb{R}^n$ and Ψ be given function. Suppose that condition (5) is satisfied. In this section we will consider problem (6), (2). To have the "vanishing viscosity method" working in this case we need

Assumption 4.1. *Let $\|\Psi\|_\Gamma \leqslant M$. Suppose that*

1) *there exists $\sigma \in O_M$ such that*

$$g(t,x,u,r,0)\operatorname{sgn}(u) \leqslant \sigma(t, \max(|u|,|r|)) \quad \text{in} \quad \Theta \times \mathbb{R} \times \mathbb{R};$$

2) *for every $R > 0$ there exists a modulus ω_R and positive constants C_R, $\tilde{C}_R \geqslant 0$ such that*

$$\begin{aligned} |g(t,x,u,r,p) - g(t,x,u,r,\bar{p})| &\leqslant \omega_R(|p-\bar{p}|),\\ |g(t,x,u,r,p) - g(t,x,\bar{u},\bar{r},p)| &\leqslant C_R \max(|u-\bar{u}|, |r-\bar{r}|),\\ |g(t,x,u,r,p) - g(t,y,u,r,p)| &\leqslant \tilde{C}_R(1+|p|)|x-y| \end{aligned}$$

in $\Theta \times [-R,R] \times [-R,R] \times \mathbb{R}^n$;

3) *$\nu \in C_L(\Theta, R)$;*

4) *there ares $L_0 \geqslant 0$ and $\tilde{\Psi} \in C_L(E, L_0)$ such that, $\tilde{\Psi}_{|\Gamma} = \Psi$.*

We will say that IBVP (6), (2) satisfies the compatibility condition if

$$D_t\Psi(0,x) - \varepsilon\Delta_x\Psi(0,x) = g(0,x,\Psi(0,x),\Psi(\mu(0,x),\nu(0,x)),D\Psi(0,x)) \tag{43}$$

for $x \in \partial\Omega$.

In view of the result obtained in Section 2 and recalling Example 1.1 we can formulate

Theorem 4.1. *Suppose that $\partial\Omega \in C^{2+\alpha}$, Assumption 4.1 is satisfied, g is nondecreasing in r, $g(0,x,0,0,0)=0$ in $\partial\Omega$. Assume that one of the following conditions holds:*

(a) *$g(t,x,0,0,0) \geqslant 0$ in Θ and*

$$\limsup_{|p|\to\infty, k\to\infty} g(t,x,0,k,p) < -\lambda_0 < 0 \quad \text{if } r_0 > 0,$$
$$\limsup_{|p|\to\infty} g(t,x,0,0,p) < -\lambda_0 < 0 \quad \text{if } r_0 = 0$$

or

(b) *$\partial\Omega$ is analytic and*

$$g(t,\bar{x},0,k,p) = 0 \quad \text{for } |p| \geqslant p_0, \quad \text{if } r_0 > 0, \tag{46}$$
$$g(t,\bar{x},0,0,p) = 0 \quad \text{for } |p| \geqslant p_0, \quad \text{if } r_0 = 0. \tag{47}$$

Then problem (6), (2) ($\psi \equiv 0$) has a solution.

Proof. In view of Theorem 3.2 it suffices to show that f given by (7) satisfies its hypothesis. We will show only Assumption 2.2 3), which is not only the most complicated but also closely related to the functional dependence in the equation. Let $w \in C_L(\mathbb{D})$, $|u|, \|w\|_{\mathbb{D}} \leqslant R$. Recalling formula 7 we can write

$$\begin{aligned}
&|f(t,x,u,w,p) - f(t,\bar{x},u,w,p)| \\
&\quad \leqslant g\big(t,x,u,w(\mu(t)-t,\nu(t,x)-x),p\big) - g\big(t,\bar{x},u,w(\mu(t)-t,\nu(t,\bar{x})-\bar{x}),p\big) \\
&\qquad g\big(t,x,u,w(\mu(t)-t,\nu(t,x)-x),p\big) - g\big(t,\bar{x},u,w(\mu(t)-t,\nu(t,x)-x),p\big) \\
&\qquad + g\big(t,\bar{x},u,w(\mu(t)-t,\nu(t,x)-x),p\big) - g\big(t,\bar{x},u,w(\mu(t)-t,\nu(t,\bar{x})-\bar{x}),p\big) \\
&\quad \leqslant \tilde{C}_R(1+|p|)|x-\bar{x}| + C_R|w(\mu(t)-t,\nu(t,x)-x) - w(\mu(t)-t,\nu(t,\bar{x})-\bar{x})| \\
&\quad \leqslant \tilde{C}_R(1+|p|)|x-\bar{x}| + C_R L_x(w) L_x(\nu)|x-\bar{x}|
\end{aligned}$$

which gives 3) of Assumption 2.2. The other items are easy to demonstrate, so we will not present them here.

Using similar argument it is also possible to obtain a theorem on existence and uniqueness of viscosity solutions for the differential-integral problem. We will not present it here, as it seems to be easier to transform this problem into (1), (2) than to the problem with a deviated argument. In Assumption 2.2 3) for instance we don't need to use the space $C_L(\mathbb{D})$.

It must be mentioned also that Assumption 2.2 guarantees the uniqueness of the viscosity solution for problems considered. See [17]. □

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