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GOING DOWN IN (SEMI)LATTICES OF FINITE MOORE FAMILIES
AND CONVEX GEOMETRIESBORDALO GABRIELA, Lisboa, CASPARD NATHALIE, Paris, and
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Abstract. In this paper we first study what changes occur in the posets of irreducible elements when one goes from an arbitrary Moore family (respectively, a convex geometry) to one of its lower covers in the lattice of all Moore families (respectively, in the semilattice of all convex geometries) defined on a finite set. Then we study the set of all convex geometries which have the same poset of join-irreducible elements. We show that this set—ordered by set inclusion—is a ranked join-semilattice and we characterize its cover relation. We prove that the lattice of all ideals of a given poset P is the only convex geometry having a poset of join-irreducible elements isomorphic to P if and only if the width of P is less than 3. Finally, we give an algorithm for computing all convex geometries having the same poset of join-irreducible elements.

Keywords: closure system, Moore family, convex geometry, (semi)lattice, algorithm

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1. INTRODUCTION

Finite Moore families (also called closure systems) are fundamental mathematical objects. For instance, they are the set representations of finite lattices. The set \mathbb{M} of all these families defined on a set P and ordered by set inclusion is a lattice studied by many authors (see Caspard and Monjardet [7], 2003). The set \mathbb{M}_P of all Moore families whose poset of join-irreducible elements is isomorphic to a poset (P, \leq) (where \leq is any given partial order defined on P) is a lattice studied in Bordalo and Monjardet [4], 2002. Convex geometries are a significant class of Moore families appearing in many domains. They are the set representations of the so-called lower locally distributive (or meet-distributive) lattices studied as early as 1940 by Dilworth [11], and they also are the families of closed sets of antiexchange closure

operators (see, for instance, Edelman and Jamison [12], 1985 and Monjardet [15], 1990). Moreover, they are in a one-to-one correspondence with the so-called path-independent choice functions of the theory of choice functions in microeconomics (see Monjardet and Raderanirina [16], 2001). The set \mathbb{G} of all convex geometries defined on a set P and ordered by set inclusion is a semilattice studied in Edelman and Jamison [12], 1985 and Caspard and Monjardet [8], 2004. One of the aims of the present paper is to study the poset \mathbb{G}_P of all convex geometries defined on a finite set P and whose poset of join-irreducible elements is isomorphic to a given poset (P, \leq) . For instance, if (P, \leq) is an antichain, \mathbb{G}_P is the semilattice of all atomistic convex geometries with $|P|$ atoms (the number of such convex geometries is 87 for $|P| = 4$ and 16686 for $|P| = 5$, Nourine [19], 2003). Obviously, \mathbb{G}_P is contained in the lattice \mathbb{M}_P , but it will appear that their behaviour is quite different. For instance, \mathbb{G}_P is a semilattice which is generally not a lattice.

In the above (semi)lattices \mathbb{M} , \mathbb{M}_P , \mathbb{G} and \mathbb{G}_P , one goes from an element \mathcal{G}' of the semilattice to one of its lower covers \mathcal{G} by deleting one (meet-irreducible) element of \mathcal{G}' . In this paper, we make precise the characterization of those meet-irreducible elements which can be deleted and study what changes occur between the irreducible elements of \mathcal{G}' and those of \mathcal{G} .

In Section 2, we recall some generalities on posets, Moore families and convex geometries.

In Section 3, we recall and complete some results on (semi)lattices of Moore families. For instance, Propositions 1, 2 and Corollary 1 describe the changes occurring in the irreducible elements of a Moore family belonging to \mathbb{M} , \mathbb{M}_P or \mathbb{G} , when one considers one of its lower covers in the corresponding semilattices.

Section 4 considers the particular case where the Moore family is the convex geometry—and a distributive lattice— $\mathcal{O}(P)$ of all order ideals of a poset (P, \leq) . We study the lower covers of $\mathcal{O}(P)$ in \mathbb{M} , \mathbb{M}_P , \mathbb{G} or \mathbb{G}_P . We point out the connection with the study of the maximal sublattices of a distributive lattice made by many authors (see Schmid [23], 2002 and the references there).

Section 5 is devoted to the study of \mathbb{G}_P . We first show that \mathbb{G}_P is a join-subsemilattice of the semilattice \mathbb{G} of all convex geometries (its maximum is $\mathcal{O}(P)$). Then, we study the cover relation of \mathbb{G}_P by characterizing the so-called \mathbb{G}_P -deletable elements of a convex geometry of \mathbb{G}_P . In general, \mathbb{G}_P is not a lattice and we study its coatoms and its minimal elements. We then show that $|\mathbb{G}_P| = 1$ (i.e. that the lattice $\mathcal{O}(P)$ of order ideals of (P, \leq) is the unique convex geometry having a poset of join-irreducible elements isomorphic to (P, \leq)) if and only if the width of P is less than 3. At last, we formulate several open problems and give an algorithm for computing all convex geometries having the same poset of join-irreducible elements.

The symbol $+$ denotes the disjoint union set, $P + x$ stands for $P + \{x\}$. $P \oplus Q$ denotes the ordinal sum of two posets P and Q (where $x < y$ for all x in P and y in Q).

2. PRELIMINARIES

In this section we recall or introduce some definitions, notation and results on posets, Moore families and convex geometries.

In this paper, P always denotes a finite set. When there is no ambiguity, P will also denote the poset (P, \leq) , where \leq is a partial order defined on P .

When x is covered by y —denoted by $x \prec y$ —in the poset P , we say that x (respectively, y) is a *lower cover* (respectively, an *upper cover*) of y (respectively, of x).

An element of a poset P is *join-irreducible* or *meet-irreducible* respectively if it is not the join or the meet, respectively, of elements different from itself. An element is doubly-irreducible if it is join- and meet-irreducible. $J(P)$ and $M(P)$ denote respectively the sets of join-irreducible and of meet-irreducible elements of the poset P . If P is a lattice, then x^- denotes the unique element covered by the join-irreducible element x , while x^+ denotes the unique element covering the meet-irreducible element x .

An element x of a poset P is a *node* if x is comparable to every element of P .

An (*order*) *ideal* of a poset P is a subset I of P such that $x \in I$ and $y < x$ imply $y \in I$. One defines dually the notion of an (*order*) *filter*. We denote by $\mathcal{O}(P)$ the distributive lattice of all ideals of P (ordered by set inclusion). For an element x of a poset P , $(x] = \{y \in P : y \leq x\}$ and $[x) = \{y \in P : x \leq y\}$ denote the principal ideal and the principal filter, respectively, defined by x . Moreover, $(x[$ denotes the ideal $(x] \setminus \{x\}$ (where \setminus is the symbol for set difference) and $]x)$ denotes the filter $[x) \setminus \{x\}$. We denote by \mathcal{P} the poset $(\{(x], x \in P\}, \subseteq)$. So, the posets P and \mathcal{P} are isomorphic. \mathcal{P} is the poset of all join-irreducible elements of the lattice $\mathcal{O}(P)$. The meet-irreducible elements of $\mathcal{O}(P)$ are the ideals $P \setminus [x)$ for $x \in P$.

Let \mathcal{G} denote a *Moore family* (also called a *closure system*) defined on a set P , i.e. a subset of the set 2^P of all subsets of P that contains P and is closed under set-intersection. Obviously, (\mathcal{G}, \subseteq) is a lattice and its elements are called *closed sets*. Closed sets are the fixed points of the closure operator canonically associated with \mathcal{G} and denoted by $\varphi_{\mathcal{G}}$. A preorder $R_{\mathcal{G}}$ is defined on P by setting $xR_{\mathcal{G}}y$ if $\varphi_{\mathcal{G}}(x) \subseteq \varphi_{\mathcal{G}}(y)$. Observe that a closed set G of \mathcal{G} is an ideal of $R_{\mathcal{G}}$ (i.e. $x \in G$ and $yR_{\mathcal{G}}x$ imply $y \in G$), and that the principal ideals of $R_{\mathcal{G}}$ are the closures by $\varphi_{\mathcal{G}}$ of the elements of P : $\{y \in P : yR_{\mathcal{G}}x\} = \varphi_{\mathcal{G}}(x)$. In 1943, Öre [20] has characterized

the join-irreducible elements of the lattice (\mathcal{G}, \subseteq) :

$$J(\mathcal{G}) = \{\varphi_{\mathcal{G}}(x) : \varphi_{\mathcal{G}}(x) \setminus \{y \in P : \varphi_{\mathcal{G}}(y) = \varphi_{\mathcal{G}}(x)\} \in \mathcal{G}\}.$$

Using this result, one easily proves

Lemma 1. *Let \mathcal{G} be a Moore family on P , $\varphi_{\mathcal{G}}$ the associated closure operator and $R_{\mathcal{G}}$ the associated preorder. The following two conditions are equivalent:*

- (1) *for every $x \in P$, $\varphi_{\mathcal{G}}(x) \setminus \{x\} \in \mathcal{G}$,*
- (2) *$J(\mathcal{G}) = \{\varphi_{\mathcal{G}}(x), x \in P\}$, and $\varphi_{\mathcal{G}}(x) = \varphi_{\mathcal{G}}(y)$ implies $x = y$.*

When these conditions are satisfied, $R_{\mathcal{G}}$ is an order denoted by $\leq_{\mathcal{G}}$ or, when no ambiguity can occur, by \leq . Then every element of \mathcal{G} is an ideal of the poset $(P, \leq_{\mathcal{G}})$ and the closed sets of the form $\varphi_{\mathcal{G}}(x)$ are the principal ideals of $(P, \leq_{\mathcal{G}})$ as well as the join-irreducible elements of \mathcal{G} .

Remark 1. The two parts in Condition (2) are independent. When \mathcal{G} is a Moore family such that $\varphi_{\mathcal{G}}(x) = \varphi_{\mathcal{G}}(y)$ implies $x = y$ (i.e. when $R_{\mathcal{G}}$ is an order), the above Condition (1) is not necessarily satisfied.

In this paper we will generally consider Moore families satisfying Condition (1) (or (2)) of Lemma 1. In this case, a Moore family \mathcal{G} on P will be viewed as a set of ideals of the associated poset $(P, \leq_{\mathcal{G}})$ and, since the set $J(\mathcal{G})$ of its join-irreducible elements is the poset \mathcal{P} defined by the principal ideals of $(P, \leq_{\mathcal{G}})$, $J(\mathcal{G})$ is isomorphic to $(P, \leq_{\mathcal{G}})$.

Observe that, for any given poset (P, \leq) , there exists at least one Moore family \mathcal{G} such that $J(\mathcal{G})$ is isomorphic to (P, \leq) , namely the set $\mathcal{O}(P)$ of all ideals of (P, \leq) , but there exist in general several such Moore families (the so-called *strict completions* of P , see Bordalo and Monjardet [4], 2002).

A Moore family \mathcal{G} defined on a set P is a *convex geometry* if it satisfies the following two properties:

- (1) the empty set \emptyset is a closed set,
- (2) for every closed set C different from P there exists $x \notin C$ such that $C + \{x\}$ is a closed set.

It is well-known that these properties imply that a convex geometry satisfies the above Condition (1) of Lemma 1, and so, any convex geometry \mathcal{G} on P will be viewed as a set of ideals of the associated poset $(P, \leq_{\mathcal{G}})$. The “abstract” lattices corresponding to convex geometries are the *lower locally distributive* (also called *meet-distributive*) lattices. In particular, they are lower semimodular (see Edelman and Jamison [12], 1985 or Monjardet [15], 1990 for a number of characterizations of convex geometries or lower locally distributive lattices). We will often use some

properties of lower locally distributive lattices recalled in the following lemma. The first three are well-known characterizations of lower locally distributive lattices. The two others are true for lower semimodular lattices, since they are the duals of the two properties of upper semimodular lattices proved in Bordalo and Monjardet [3], 1996, Lemmas 7 and 8).

Lemma 2. *Let L be a lower locally distributive lattice. Then*

- (1) *the set of lower covers of any non minimum element of L is the set of coatoms of a Boolean sublattice of L ,*
- (2) *for any meet-irreducible x of L there exists a unique join-irreducible element y such that $x \vee y = x^+$,*
- (3) *L is lower semimodular and every modular sublattice of L is distributive,*
- (4) *L does not contain a sublattice isomorphic to the lattice N_5 where the element y of the chain $0 \prec x \prec y \prec 1$ of N_5 is join-irreducible in L ,*
- (5) *if t, y and z are three different elements of L such that $z \prec y$ and $z \prec t$, then there exist two elements l, m such that $z \prec m \prec l$ and $y \prec l$. In particular, when z has only two upper covers, this implies $m = t$ and, when t is meet-irreducible, this implies $l = t^+$.*

3. GOING DOWN IN MOORE FAMILIES: THE CHANGES IN IRREDUCIBLE ELEMENTS

In this section we describe what changes occur in the irreducible elements of a Moore family belonging to a poset of Moore families when one considers one of its lower covers in this poset. We consider the following four posets of Moore families, where the order is the set inclusion between families (observe that the Moore families of the last three posets satisfy Condition (1) of Lemma 1):

- \mathbb{M} denotes the poset of all Moore families defined on a set P . This poset is a lattice which has been studied by several authors (see Caspard and Monjardet [7], 2003). For $\mathcal{G}, \mathcal{G}' \in \mathbb{M}$, $\mathcal{G} \prec \mathcal{G}'$ if and only if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, with I a meet-irreducible element of \mathcal{G}' . Moreover, $\mathcal{G} \wedge \mathcal{G}' = \mathcal{G} \cap \mathcal{G}'$ and $\mathcal{G} \vee \mathcal{G}' = \{G \cap G', G \in \mathcal{G}, G' \in \mathcal{G}'\}$.
- \mathbb{M}_P denotes the poset of all Moore families \mathcal{G} defined on a set P and such that $J(\mathcal{G}) = \mathcal{P}$ is isomorphic to a given poset (P, \leq) . The poset \mathbb{M}_P is a lattice which was studied in Bordalo and Monjardet [4], 2002. Its maximum is the lattice $\mathcal{O}(P)$ of all ideals of P . Its minimum is the lattice $\mathcal{D}(P)^*$ which is a meet-subsemilattice of the lattice $\mathcal{O}(P)$ consisting of the ideals

$$\mathcal{D}(P)^* = \{P\} \cup \{(x], x \in P\} \cup \left\{ \bigcap (x_i], x_i \in S \subseteq P \right\} \cup \{(x[, x \in P\}.$$

In fact, \mathbb{M}_P is exactly the interval $[\mathcal{D}(P)^*, \mathcal{O}(P)]$ of \mathbb{M} consisting of all meet-subsemilattices of $\mathcal{O}(P)$ containing $\mathcal{D}(P)^*$.

- \mathbb{G} denotes the join-semilattice of all convex geometries defined on a set P . This semilattice has been studied in Edelman and Jamison [12], 1985 and Caspard and Monjardet [8], 2004. In particular, \mathbb{G} is ranked.
- \mathbb{G}_P denotes the poset of all convex geometries \mathcal{G} defined on a set P and such that $J(\mathcal{G}) = \mathcal{P}$ is isomorphic to a given poset (P, \leq) . So $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$. We shall study this poset in Section 5.

Observe that \mathbb{M}_P is a sublattice of the lattice \mathbb{M} of all Moore families, whereas \mathbb{G} is only a join-subsemilattice of \mathbb{M} . In \mathbb{G} , $\mathcal{G} \wedge \mathcal{G}'$ (when it exists) can be strictly contained in $\mathcal{G} \cap \mathcal{G}'$. Observe also that the covering relation of \mathbb{M}_P or \mathbb{G} is the restriction of the covering relation of \mathbb{M} to these posets. In fact, the following definitions and results allow us to make more precise the covering relation of \mathbb{M}_P and \mathbb{G} (the notions of forced and \mathbb{M}_P -deletable ideals are in Bordalo and Monjardet [4], 2002). In these definitions and results, \mathcal{G}' (rather than \mathcal{G}) will denote an arbitrary Moore family belonging to one of our four posets of Moore families, and \mathcal{G} will always denote a Moore family covered by \mathcal{G}' in this poset.

Definition 1. Let P be a poset and I an ideal of P .

- (1) I is *forced* if $I \in \mathcal{D}(P)^*$, i.e. if either $I = P$, or $I = (x]$ or $(x[$ for some x in P , or I is the intersection of a family of principal ideals of P . Observe that the empty set is a forced ideal. An ideal which is not forced is called *unforced*.
- (2) In a Moore family $\mathcal{G}' \in \mathbb{M}_P$ containing I , I is *\mathbb{M}_P -deletable* (w.r.t. \mathcal{G}') if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{M}_P$.
- (3) In a Moore family $\mathcal{G}' \in \mathbb{G}_P$ containing I , I is *\mathbb{G} -deletable* (w.r.t. \mathcal{G}') if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}$.
- (4) In a convex geometry $\mathcal{G}' \in \mathbb{G}_P$ containing I , I is *\mathbb{G}_P -deletable* (w.r.t. \mathcal{G}') if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}_P$.

Notation. When no ambiguity occurs, we will denote by J and M the set $J(\mathcal{G})$ of join-irreducible elements and the set $M(\mathcal{G})$ of meet-irreducible elements, respectively, of a Moore family \mathcal{G} . Similarly for J' and M' (instead of $J(\mathcal{G}')$ and $M(\mathcal{G}')$) for a Moore family \mathcal{G}' . Using the notation given above for irreducible elements of a lattice, I^- denotes the unique element of \mathcal{G} covered by a join-irreducible element I of \mathcal{G} , while I^+ denotes the unique element of \mathcal{G} covering a meet-irreducible element of \mathcal{G} .

The following theorem recalls and completes some useful results on the deletable elements of a Moore family \mathcal{G}' belonging to \mathbb{M}_P or to \mathbb{G} .

Theorem 3.1. *Let P be a poset.*

- (1) *Let \mathcal{G}' be a Moore family in \mathbb{M}_P and $I \in \mathcal{G}'$ an ideal of P . The following three properties are equivalent:*
- a. *I is \mathbb{M}_P -deletable w.r.t. \mathcal{G}' ,*
 - b. *I is an unforced ideal of P and $I \in M'$,*
 - c. *$I \in M'$ and $I, I^+ \notin J'$.*
- (2) *Let $\mathcal{G}' \in \mathbb{G}_P$ and let $I \in \mathcal{G}'$ be an ideal of P . Properties (a) and (b) below are equivalent and imply (c):*
- a. *I is \mathbb{G} -deletable w.r.t. \mathcal{G}' ,*
 - b. *$I \in M'$, $I \neq \emptyset$ and I does not cover any meet-irreducible element of \mathcal{G}' ,*
 - c. *$I \in M'$ and $I^+ \notin J'$ (or, equivalently, $I \in M'$ and it is not a node of \mathcal{G}').*
- Moreover, if $I \in J' \cap M'$, the following four properties are equivalent:*
- d. *I is \mathbb{G} -deletable w.r.t. \mathcal{G}' ,*
 - e. *$I^- \notin M'$,*
 - f. *$I^+ \notin J'$,*
 - g. *I is not a node of \mathcal{G}' ,*

and, in this case, I^+ has a unique other lower cover I' and $I^- \prec I'$.

Proof. (1) is proved in Bordalo and Monjardet [4], 2002.

(2) (a) implies (b): it is obvious that $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}$ implies $I \in M'$ and $I \neq \emptyset$. If I covers $G \in M'$, then in \mathcal{G} , G is covered by I^+ with $|I^+| = |G| + 2$, a contradiction with \mathcal{G} being a convex geometry.

(b) implies (a): we have to show that $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ contains the empty set and that, for every element K of \mathcal{G} (different from P), there exists $x \notin K$ such that $K + x$ belongs to \mathcal{G} . But this is obvious from the hypotheses made on I .

(b) implies (c): in order to show that $I^+ \notin J'$, apply Point (5) of Lemma 2 (with $l = I^+$, $z = G \prec I = y$ and $G \prec H = t$), which shows that there exists H' (different or not from H) with $H' \prec I^+$. We now prove that when $I \in M'$, $I^+ \notin J'$ if and only if I is not a node of \mathcal{G}' . If $I^+ \notin J'$, then $I^+ = I \vee K$ and so I is not a node of \mathcal{G}' . Conversely, assume that $I \in M'$ and it is not a node of \mathcal{G}' . So, there exists $K \in \mathcal{G}'$ incomparable with I . If $I^+ \in J'$, then $H = I \vee K = I^+ \vee K \supset I^+$ and $G = I^+ \wedge K = I \wedge K \subset I$. Then $\{G, I, K, I^+, H\}$ is a sublattice of \mathcal{G}' isomorphic to the lattice N_5 and such that I^+ is join-irreducible in \mathcal{G}' . But this is a contradiction with Point (4) of Lemma 2.

Assume now that $I \in J' \cap M'$ (which implies $I \neq \emptyset$). Then (a) is equivalent to (b) implies (d) is equivalent to (e). Since $I \in M'$, (f) and (g) are equivalent (by (c)) and, since (a) implies (c), (d) implies (f). In order to prove the equivalence between the four conditions (d), (e), (f) and (g), it remains to show that (f) implies (d). Let $I' (\neq I)$ be covered by I^+ in \mathcal{G}' . Since the lattice \mathcal{G}' is a convex geometry, it is lower

semimodular, and, since $I \in J'$, one has $I^- \prec I'$. Now, for the same reason as in the above proof of [(b) implies (a)], $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ is a convex geometry.

Since $I^+ \notin J'$, there exists $I' (\neq I)$ covered by I^+ . Assume that there exists $I'' (\neq I', I)$ covered by I^+ . As above, one would have $I^- \prec I''$. Then, $\{I^-, I, I', I'', I^+\}$ would be a sublattice of J' isomorphic to the modular lattice M_5 , a contradiction with J' being a lower locally distributive lattice (by Point (3) of Lemma 2). \square

Remark 2. 1. Some of the above characterizations of an ideal \mathbb{G} -deletable w.r.t. a convex geometry are already in Caspard and Monjardet [8], 2004 with some other ones using the notion of extreme elements of a closed set. 2. Since Property (4) of Lemma 2 holds in a lower semimodular lattice, the above proof shows that in such a lattice, a meet-irreducible element m is a node if and only if m^+ is join-irreducible.

We now examine what changes occur in the posets of irreducible elements when one goes from a Moore family \mathcal{G}' to one of its lower covers \mathcal{G} in the posets \mathbb{M} , \mathbb{M}_P or \mathbb{G} . The following propositions and corollaries answer completely this question.

In Proposition 1 we consider a lower cover $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ of an arbitrary Moore family \mathcal{G}' (with I a meet-irreducible element of \mathcal{G}') and determine its poset of join-irreducible elements. The result depends on four disjoint possibilities for I and I^+ : (a) $I, I^+ \notin J'$; (b) $I \notin J'$ and $I^+ \in J'$; (c) $I \in J'$, this last case leading to two different situations (c1) and (c2) according to different behaviours of I^+ .

Proposition 1. *Let \mathcal{G}' be a Moore family in \mathbb{M} and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is a meet-irreducible element of \mathcal{G}' . Then*

1. $J = J' \iff I, I^+ \notin J'$,
2. $J = J' \setminus \{I^+\} \iff I \notin J'$ and $I^+ \in J'$,
3. $J = J' \setminus \{I\} \iff I \in J'$ and either $I^+ \in J'$, or (I^+ has a unique other lower cover I' in \mathcal{G}' and I' satisfies $I^- \not\prec I'$), or I^+ has at least three lower covers,
4. $J = (J' \setminus \{I\}) + \{I^+\} \iff I \in J'$, I^+ has a unique other lower cover I' in \mathcal{G}' and this unique lower cover satisfies $I^- \subset I'$.

In the first three cases, the order on J is the order induced by J' ; in the last one, the order on $J = (J' \setminus \{I\}) + \{I^+\}$ is obtained by replacing I by I^+ in all the ordered pairs (K, I) and (I, L) of J' , and by adding the ordered pairs (K, I^+) for $K \in J'$ and satisfying $K \subset I'$ and $K \not\subset I^-$.

Proof. Items (1) and (2) are straightforward.

For Item (3), if $J = J' \setminus \{I\}$ then $I \in J'$ and either $I^+ \in J'$ and it remains join-irreducible in \mathcal{G} (since, in \mathcal{G} , I^- is its unique lower cover), or $I^+ \notin J'$ and it does not become join-irreducible in \mathcal{G} , which is clearly equivalent to the condition [I^+ has a unique other lower cover I' and $I^- \not\prec I'$, or I^+ has at least three lower covers]. The converse implication is obvious.

For Item (4), if $J = (J' \setminus \{I\}) + \{I^+\}$ then $I \in J'$ and, since I^+ becomes join-irreducible in \mathcal{G} , it has a unique other lower cover I' in \mathcal{G}' and $I^- \subset I'$ (otherwise, I^- and I' are lower covers of I^+ in \mathcal{G}). The converse implication is obvious.

Observing that, in this last case, the only elements K of J' such that $K \subset I^+$ and $(K, I) \notin J'$ are the elements K contained in I' and not contained in I^- , one gets the description of the order of $J = (J' \setminus \{I\}) + \{I^+\}$. \square

In the next proposition, we consider a lower cover $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ of an arbitrary Moore family \mathcal{G}' (with I a meet-irreducible of \mathcal{G}') and determine its poset of meet-irreducible elements. For $I \in M'$ we use the following notation:

$$\mathcal{J}^-(I) = \{G \in \mathcal{G}' : G \prec I, G \text{ has a unique other upper cover } I', \text{ and } I' \prec I^+\}.$$

Proposition 2. *Let \mathcal{G}' be a Moore family in \mathbb{M} and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is a meet-irreducible element of \mathcal{G}' . Then $M = (M' \setminus \{I\}) + \mathcal{J}^-(I)$.*

The order on M is obtained by adding to the induced order on $M' \setminus \{I\}$ the ordered pairs (G, H) and (L, G) , where $G \in \mathcal{J}^-(I)$, $H, L \in M'$, $I' \subseteq H$ and $L \subset G$.

The proof is straightforward.

Remark 3. One can observe that, when one goes from a Moore family to one of its lower covers in \mathbb{M} , the number of join-irreducible elements decreases from at most one and never increases, whereas the number of meet-irreducible elements can decrease from at most one but can also increase.

Corollary 1. *Let P be a poset.*

- (1) *Let $\mathcal{G}' \in \mathbb{M}_P$ and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is an ideal of P which belongs to \mathcal{G}' and is \mathbb{M}_P -deletable w.r.t. \mathcal{G}' . Then*
 - a. $J = J'$,
 - b. $M = (M' \setminus \{I\}) + \mathcal{J}^-(I)$.
- (2) *Let $\mathcal{G}' \in \mathbb{G}_P$ and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is an ideal of P which belongs to \mathcal{G}' and is \mathbb{G} -deletable w.r.t. \mathcal{G}' . Then*
 - a. $J = J' \iff I \notin J'$,
 $J = (J' \setminus \{I\}) + \{I^+\} \iff I \in J' \iff I \in J' \cap M'$.

When $I \in J'$, I^+ has a unique other lower cover I' in \mathcal{G}' and I' satisfies $I^- \subset I'$.

The order on $J = (J' \setminus \{I\}) + \{I^+\}$ is obtained by replacing I by I^+ in all ordered pairs (K, I) or (I, L) of J' and by adding the unique ordered pair (K, I^+) where $K \in J'$, $K \subset I'$ and $K \not\subseteq I^-$.

- b. $M = (M' \setminus \{I\}) + \{G \in \mathcal{G}' : G \prec I \text{ and } G \text{ has a unique other upper cover } I'\}$.

In particular, when $I \in J' \cap M'$, either I^- has a unique other upper cover I' in \mathcal{G}' and $M = (M' \setminus \{I\}) + \{I^-\}$, or I^- has at least two other upper covers in \mathcal{G}' , and $M = M' \setminus \{I\}$.

Proof. (1) is obvious by definition of \mathbb{M}_P (observe also that if I is \mathbb{M}_P -deletable, it is unforced and so $I, I^+ \notin J'$).

(2)(a) results from Cases (1) and (4) of Proposition 1. Indeed, if I is a \mathbb{G} -deletable ideal, one has $I^+ \notin J'$ (according to the implication of (2)(c) by (2)(a) in Theorem 3.1). So, Case (1) of Proposition 1 corresponds to $I \notin J'$, and Case (2) of Proposition 1 does not occur. Moreover, when $I \in J' \cap M'$, I^+ has a unique other lower cover I' with $I^- \subset I'$ (by Point (2) of Theorem 3.1). So, Case (3) of Proposition 1 does not occur.

The result on the order on J is the same as in Proposition 1 except that now there exists a unique ordered pair (K, I^+) with $K \in J', K \subset I'$ and $K \not\subseteq I^-$. Indeed, the last two conditions are equivalent to $K \vee I = I^+$ and, since a convex geometry is a lower locally distributive lattice, for any meet-irreducible I there exists a unique $K \in J'$ such that $K \vee I = I^+$ (Point (2) of Lemma 2).

In order to prove (2)(b) and according to Proposition 2, we have to prove that, if $G \prec I$ and if G has a unique other upper cover I' in \mathcal{G}' , then $I' \prec I^+$. But, since \mathcal{G}' is a lower locally distributive lattice, this results from Point (5) of Lemma 2. The final assertion results from Point (2) of Theorem 3.1. Indeed, when I is \mathbb{G} -deletable w.r.t. \mathcal{G}' and $I \in J' \cap M'$, I^- cannot belong to M' . \square

Remark 4. 1. When \mathcal{G}' is a convex geometry, we have just shown that $\mathcal{S}^-(I) = \{G \in \mathcal{G}' : G \prec I \text{ and } G \text{ has a unique other upper cover } I'\}$. This is also true if \mathcal{G}' is a Moore family without a sublattice isomorphic to the lattice N_5 .

2. When \mathcal{G}' is a convex geometry and $\mathcal{G} = \mathcal{G}' \setminus \{I\} \prec \mathcal{G}'$ are two convex geometries such that $J = J'$, it is not difficult to see that, even if $|M| = |M'|$, M cannot be isomorphic to M' .

4. THE LOWER COVERS OF A DISTRIBUTIVE LATTICE

Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P , $x \in P$ and $I = P \setminus [x]$ a meet-irreducible element of $\mathcal{O}(P)$. The Moore family $\mathcal{O}(P) \setminus I$ may or may not belong to \mathbb{M}_P , \mathbb{G} or \mathbb{G}_P . In this section we characterize the elements x of P corresponding to these different cases. We need to recall or define the following notions:

- x is *join-prime* if there exists $y \in P$ such that $P = [x] + (y)$. In this case, we also say that x is the *conjugate element of y* (Berman and Bordalo [2], 1998).

- x is *strongly meet-irreducible* if x is covered by a unique element, which we denote by x^+ . Then x is a meet-irreducible element of P .

One defines dually the notions of a *meet-prime element* and of a *strongly join-irreducible element*.

Let x, y be two elements of P . The ordered pair (x, y) is a *critical pair* or a *weak critical pair* respectively for P if x and y are incomparable or $x \not\leq y$ respectively and, for all $z, t \in P$, $z < x$ implies $z \leq y$, and $y < t$ implies $x \leq t$. It is clear that (x, y) is a critical pair for P if and only if $P + (x, y)$ is a poset (cf. for instance Barbut and Monjardet [1], 1970, II, p. 54, or Rabinovitch and Rival [21], 1979) and that, if (x, y) is a weak critical pair, then y is covered by x in P .

The following lemma gives some known or not known (but anyway obvious or easy to prove) relations between the above notions.

Lemma 3. *Let x be an element of a poset P . Then*

- if x is a node covering an element y , then y is a strong meet-irreducible element of P ,
- x is a join-prime element of P if and only if $P \setminus [x]$ is a doubly-irreducible element of $\mathcal{O}(P)$.

Let x be a join-prime element of a poset P , the conjugate of the element y of P . Then

- x is join-irreducible, y is meet-prime and (x, y) is a weak critical pair for P ,
- x is a node of P if and only if x covers a strong meet-irreducible element—which is equal to y —if and only if $P \setminus [x] = (y]$ is a doubly-irreducible element and a node of $\mathcal{O}(P)$. In this case, $P = (y] \oplus [x]$, and so y is covered by x in P ;
- if x is a node, then (x, y) is a critical pair for P .

In the next theorem, we characterize the elements x of P for which $\mathcal{O}(P) \setminus (P \setminus [x])$ belongs to \mathbb{M}_P , \mathbb{G} or \mathbb{G}_P .

Theorem 4.1. *Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P and $I = P \setminus [x]$ a meet-irreducible element of $\mathcal{O}(P)$. Then*

- (1) I is \mathbb{M}_P -deletable w.r.t. $\mathcal{O}(P)$ if and only if x is neither join-prime nor a node of P ,
- (2) I is \mathbb{G} -deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover any strong meet-irreducible element of P and x is not the minimum of P (if this minimum exists),
- (3) I is \mathbb{G}_P -deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists).

Proof. (1) is proved in Bordalo and Monjardet [4], 2002.

(2) By Point (2) of Theorem 3.1, $I = P \setminus [x]$ is \mathbb{G} -deletable w.r.t. $\mathcal{O}(P)$ if and only if [$I \neq \emptyset$ and I does not cover any meet-irreducible of $\mathcal{O}(P)$]. The ideal I is empty if and only if P has a minimum, which is then x . There exists $G \in M(\mathcal{O}(P))$ with $G \prec I$ if and only if $G = P \setminus [y] \prec P \setminus [x] = I$, if and only if $P \setminus [x] = (P \setminus [y]) + \{t\}$ for some t in P , if and only if $[x] =]y)$, if and only if $y = t$ is a strong meet-irreducible element covered by x .

(3) Since $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$, we put together Conditions (1) and (2) and observe that, since x does not cover a strong meet-irreducible element of P and is not the minimum of P (when this minimum exists), it cannot be a node. \square

An immediate consequence of this theorem (and of Lemma 3, for Item (3) below) is

Corollary 2. *Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P and $I = P \setminus [x]$ a meet-irreducible element of $\mathcal{O}(P)$. Then:*

- (1) *I is \mathbb{M} -deletable but not \mathbb{M}_P -deletable w.r.t. $\mathcal{O}(P)$ if and only if x is join-prime or a node of P ,*
- (2) *I is \mathbb{M} -deletable but not \mathbb{G} -deletable w.r.t. $\mathcal{O}(P)$ if and only if x covers a strong meet-irreducible element of P or is the minimum of P (when this minimum exists),*
- (3) *I is \mathbb{G} -deletable but not \mathbb{G}_P -deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover any strong meet-irreducible element of P and is join-prime.*

Considering as before $I = P \setminus [x]$ and using Lemma 3, it is easy to check that the above results induce a partition of the poset P into the following—some possibly empty—four classes P_i , $i = 1, 2, 3, 4$, where the last three cover the case when x is not a node:

- $P_1 = \{x \in P : x \text{ is a node}\} = \{x \in P : \mathcal{O}(P) \setminus I \text{ belongs neither to } \mathbb{M}_P \text{ nor to } \mathbb{G}\}$,
- $P_2 = \{x \in P : x \text{ covers a strong meet-irreducible element and is not a node}\} = \{x \in P : \mathcal{O}(P) \setminus I \text{ belongs to } \mathbb{M}_P \setminus \mathbb{G} \text{ (and to } \mathbb{M}_P \setminus \mathbb{G}_P)\}$,
- $P_3 = \{x \in P : x \text{ does not cover a strong meet-irreducible element of } P \text{ and is join-prime}\} = \{x \in P : \mathcal{O}(P) \setminus I \text{ belongs to } \mathbb{G} \setminus \mathbb{G}_P\}$,
- $P_4 = \{x \in P : x \text{ does not cover a strong meet-irreducible element of } P \text{ and is neither join-prime nor the minimum of } P \text{ (if this minimum exists)}\} = \{x \in P : \mathcal{O}(P) \setminus I \text{ belongs to } \mathbb{G}_P\}$.

We illustrate this partition on the poset P represented in Figure 1. For this poset, $P_1 = \emptyset$, $P_2 = \{5\}$, $P_3 = \{1, 2\}$, and $P_4 = \{3, 4\}$. So the five lower covers of $\mathcal{O}(P)$ in \mathbb{M} are $\mathcal{O}(P) \setminus \{1234\}$, $\mathcal{O}(P) \setminus \{4\}$, $\mathcal{O}(P) \setminus \{1345\}$, $\mathcal{O}(P) \setminus \{124\}$, $\mathcal{O}(P) \setminus \{123\}$.

The first belongs to $\mathbb{M}_P \setminus \mathbb{G}_P$, the second and the third to $\mathbb{G} \setminus \mathbb{G}_P$ and the last two to \mathbb{G}_P .

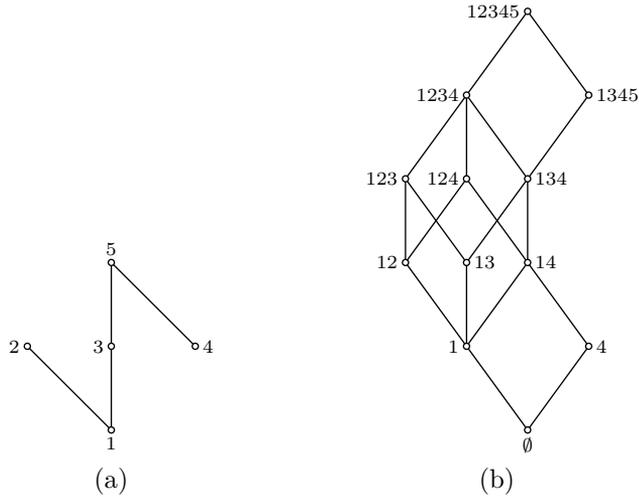


Figure 1. (a) P and (b) $\mathcal{O}(P)$

We now aim at giving an additional result for the case when $\mathcal{O}(P) \setminus I$ belongs to $\mathbb{G} \setminus \mathbb{G}_P$. To do so, we first prove the following lemma:

Lemma 4. *Let P be a poset and (x, y) a critical pair for P . Then $\mathcal{O}(P + (x, y))$ is a sublattice of $\mathcal{O}(P)$ obtained from $\mathcal{O}(P)$ by deleting the ideals $I \in \mathcal{O}(P)$ such that $y \in I$ and $x \notin I$.*

Proof. By the well-known Birkhoff's duality between posets and distributive lattices we have that $\mathcal{O}(P + (x, y))$ is a sublattice of $\mathcal{O}(P)$. Now we have $(y \in I$ and $x \notin I)$ if and only if $I \notin \mathcal{O}(P + (x, y))$. \square

Proposition 3. *Let P be a poset and x a join-prime of P which does not cover any strong meet-irreducible element of P . Then $P = [x] + (y)$, (x, y) is a critical pair for P , $I = P \setminus [x] = (y)$ is a doubly-irreducible element of $\mathcal{O}(P)$ and $\mathcal{O}(P) \setminus I = \mathcal{O}(P + (x, y))$ is a (maximal) sublattice of $\mathcal{O}(P)$ which belongs to $\mathbb{G} \setminus \mathbb{G}_P$. Moreover, $J(\mathcal{O}(P) \setminus I) = (J(\mathcal{O}(P)) \setminus I) + \{I^+\}$ and $M(\mathcal{O}(P) \setminus I) = (M(\mathcal{O}(P)) \setminus I) + \{I^-\}$.*

Proof. If x is join-prime, then $P \setminus [x]$ is a doubly-irreducible element of $\mathcal{O}(P)$ and (x, y) is a critical pair for P (by Lemma 3). Moreover, (y) is the unique ideal of P which contains y and not x . So, by Lemma 4, $\mathcal{O}(P) \setminus I = \mathcal{O}(P + (x, y))$ is a maximal sublattice of $\mathcal{O}(P)$. Since x does not cover any strong meet-irreducible element of P , it belongs to $\mathbb{G} \setminus \mathbb{G}_P$ (by Item (3) of Corollary 2). By Item (2)(a) of Corollary 1, $J(\mathcal{O}(P) \setminus I) = (J(\mathcal{O}(P)) \setminus I) + \{I^+\}$. Using the fact that in the distributive lattice $\mathcal{O}(P)$, I^- being the lower cover of a meet-irreducible element can

have only one other upper cover, Item (2)(b) of Corollary 1 implies that $M(\mathcal{O}(P) \setminus I) = (M(\mathcal{O}(P)) \setminus I) + \{I^-\}$. \square

By Lemma 3 we know that a join-prime element x of a poset P covers a strong meet-irreducible element of P if and only if $P \setminus [x] = (y)$ is a doubly-irreducible element which is a node of $\mathcal{O}(P)$. Thus we obtain

Corollary 3. *There is a bijection between the doubly-irreducible elements which are not nodes of a distributive lattice $L = \mathcal{O}(P)$ and the family of convex geometries covered by $\mathcal{O}(P)$ in $\mathbb{G} \setminus \mathbb{G}_P$. Moreover, these convex geometries are maximal (distributive) sublattices of L .*

Remark 5. Some of the above results can be compared with known results on the maximal sublattices of a distributive lattice (see Schmid [23], 2002 and the references there). The simplest way to get a maximal sublattice of a distributive lattice $L = \mathcal{O}(P)$ is to delete a doubly-irreducible element $I = P \setminus [x] = (y)$ of L (when such an element exists). In our study, this case splits into two sub-cases (see Lemma 3):

- a. I is a node of $\mathcal{O}(P)$ (or, equivalently, by Lemma 3, x is a node of P , i.e. $x \in P_1$). Then, $\mathcal{O}(P) \setminus I$ belongs to $\mathbb{M} \setminus (\mathbb{M}_P \cup \mathbb{G})$ (but remains a distributive lattice).
- b. I is not a node of $\mathcal{O}(P)$ or, equivalently, x is join-prime and does not cover a strong meet-irreducible element of P (i.e. $x \in P_3$). Then $\mathcal{O}(P) \setminus I$ is a maximal sublattice of $\mathcal{O}(P)$ which is also a convex geometry.

Using the Galois connection between families of subsets of a set X and binary relations on X (Lorrain [14], 1969, Chacron [9], 1971, Barbut and Monjardet [1], 1970), Schmid [23], 2002 has made a systematic study of the lattice of the (0-1) sublattices of a (finite or infinite) distributive lattice. The lattice of the finite \cup -stable Moore families was studied in Caspard and Monjardet [8], 2004.

Previously we have defined a partition of a poset P into—some possibly empty—four classes P_i , ($i = 1, 2, 3, 4$). We consider two cases when only one of these classes is non-empty.

First case:

Proposition 4. *For a poset P , the following properties are equivalent:*

- (1) *for every x in P , x is join-prime and does not cover a strong meet-irreducible element of P (i.e. $x \in P_3$),*
- (2) *for every x in P , x is join-prime and is not a node,*
- (3) *for all x in P and $I = P \setminus [x]$, $\mathcal{O}(P) \setminus I$ is a (maximal) sublattice of $\mathcal{O}(P)$,*
- (4) *P is an ordinal sum of 2-elements antichains.*

Remark 6. The class of posets which are ordinal sums of 2-elements antichains is a subclass of the class of posets P such that $|\mathbb{M}_P| = 1$ (see Proposition 16 in Bordalo and Monjardet [4], 2002). It is also a very special subclass of the class of *distributive posets* studied by several authors (Erné [13], 1991, Niederle [18], 1995 and Reading [22], 2002).

Second case: We consider a poset P such that, for all x in P and $I = P \setminus \{x\}$, $\mathcal{O}(P) \setminus I \in \mathbb{G}_P$ (i.e. $x \in P_4$). This is equivalent to say that, for every x in P , x does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists).

Observe that this property is also equivalent to the equality $|\{\text{coatoms of } \mathbb{G}_P\}| = |P|$ (and this implies that $|\{\text{coatoms of } \mathbb{M}_P\}| = |P|$).

It is easy to get examples of such posets. For instance, any complete bipartite poset $B_{p,q}$ (i.e. the ordinal sum of two antichains of sizes p and q) is such a poset if and only if $p, q \geq 3$. We give below another class of such posets. Let us say that a poset is *smi-free* if it has no strong meet-irreducible element. The following result shows that almost all non connected smi-free posets possess this property.

Proposition 5. *Let P be a smi-free poset. If P is not connected, every meet-irreducible element of $\mathcal{O}(P)$ is \mathbb{G}_P -deletable unless P is the (cardinal) sum of a singleton and a poset with a minimum.*

Proof. First observe that P is a smi-free poset if and only if every non maximal element of P has at least two upper covers. But then $|P| = 1$ or P has no maximum. Now, when P is not connected, no element of P is join-prime unless P is the disjoint union of a singleton and of a poset with a minimum (which is then the unique join-prime element of P). □

Remark 7.

- The simplest example of a poset satisfying the conditions of Proposition 5 is the disjoint union of a singleton and of the complete bipartite poset $B_{2,2}$.
- The complete bipartite poset $B_{p,q}$ (with $p, q \geq 3$) is an example of a connected smi-free poset satisfying the conditions of Proposition 5.

5. THE SEMILATTICE \mathcal{G}_P OF CONVEX GEOMETRIES WITH THE SAME POSET P OF JOIN-IRREDUCIBLE ELEMENTS

We begin this section with an additional result on the changes in the poset of join-irreducible elements of a convex geometry when one goes to one of its lower covers in the semilattice \mathbb{G} of convex geometries.

Proposition 6. *Let $\mathcal{G}' \in \mathbb{G}_P$ be a convex geometry, \mathcal{G} one of its lower covers in the semilattice \mathbb{G} of all convex geometries and J the poset of join-irreducible elements of \mathcal{G} . Then either $J = J'$ and $\mathcal{G} \in \mathbb{G}_P$, or $J = (J' \setminus \{I\}) + \{I^+\}$ and $\mathcal{G} \in \mathbb{G}_{P+(x,y)}$, where $x = I^+ \setminus I$, $y = I \setminus I^-$ and (x, y) is a critical pair for P .*

Proof. By Item (2)(a) of Corollary 1, if $J \neq J'$, we have $J = J' \setminus \{I\} + \{I^+\}$ with $I \in J' \cap M'$ and I^+ having a unique other lower cover I' in \mathcal{G}' ; moreover, I^- is covered by I' in \mathcal{G}' . The order on J is the union of two orders. The first is the order obtained by replacing I by I^+ in all ordered pairs (K, I) and (I, L) of J' ; this order is isomorphic to the order of J' and so to the poset P . The second order is the unique ordered pair (G, I^+) with $G \in J'$ and $G \vee I = I^+$. Then, in the convex geometries \mathcal{G}' and \mathcal{G} , $G = \varphi_{\mathcal{G}'}(x) = \varphi_{\mathcal{G}}(x)$ where $x = I^+ \setminus I$ (see, for instance, Monjardet [15], 1990). Now, since I is join-irreducible in \mathcal{G}' , $I = \varphi_{\mathcal{G}'}(y)$, where $y = I \setminus I^- (= I^+ \setminus I')$, and we have $I^+ = \varphi_{\mathcal{G}}(y)$ in \mathcal{G} . Then, adding the covering ordered pair $(G, I^+) = (\varphi_{\mathcal{G}}(x), \varphi_{\mathcal{G}}(y))$ to J' corresponds to adding the covering pair (x, y) to the isomorphic poset P . Moreover, since $P + (x, y)$ is a poset, (x, y) is a critical pair for P . \square

Remark 8. The above proposition shows that in the semilattice \mathbb{G} of all convex geometries one goes from a convex geometry $\mathcal{G}' \in \mathbb{G}_P$ to one of its lower covers $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ either without changing its (up to isomorphism) poset P of join-irreducible elements or by adding to the cover relation of this poset a unique ordered pair $x \prec y$. The case when $\mathcal{G}' = \mathcal{O}(P)$ and $\mathcal{G} = \mathcal{O}(P) \setminus \{I\} \in \mathbb{G}_{P+(x,y)}$ has been already considered in Proposition 3.

Proposition 6 has two consequences:

Corollary 4. *Let P be a poset.*

- (1) *Let $\mathcal{G}' \in \mathbb{G}_P$ be a convex geometry with J' isomorphic to P . The number of critical pairs for P is an upper bound for the number of convex geometries \mathcal{G} covered by \mathcal{G}' in \mathbb{G} and satisfying $J \neq J'$.*
- (2) *Let $\mathcal{G}, \mathcal{G}'$ be two convex geometries in \mathbb{G} with $\mathcal{G} \subset \mathcal{G}'$ and $J = J'$. Then for any convex geometry \mathcal{G}'' with $\mathcal{G} \subset \mathcal{G}'' \subset \mathcal{G}'$ we have $J = J'' = J'$.*

Proof. (1) is obvious and (2) is immediate by considering in \mathbb{G} a maximal chain between \mathcal{G} and \mathcal{G}' which contains \mathcal{G}'' . \square

Proposition 7. *Let P be a poset.*

- (1) *The poset \mathbb{G}_P is a convex join-subsemilattice of the join-semilattice \mathbb{G} and a join-subsemilattice of the lattice \mathbb{M}_P .*

- (2) For $\mathcal{G}, \mathcal{G}' \in \mathbb{G}_P$, we have $\mathcal{G} \prec \mathcal{G}'$ in \mathbb{G}_P if and only if $\mathcal{G} \prec \mathcal{G}'$ in \mathbb{G} , if and only if $\mathcal{G} \prec \mathcal{G}'$ in \mathbb{M}_P . Then, if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, we have $J = J'$ and $M = (M' \setminus \{I\}) + \{G \in \mathcal{G}' : G \prec I \text{ and } G \text{ has a unique other upper cover}\}$.
- (3) \mathbb{G}_P is a ranked semilattice.

Proof. (1) It is clear that the poset \mathbb{G}_P has a greatest element which is $\mathcal{O}(P)$. Let \mathcal{G} and \mathcal{G}' be two convex geometries with $J = J' = \mathcal{P}$. We must show that $J(\mathcal{G} \vee \mathcal{G}') = J$. The elements of \mathcal{G} and \mathcal{G}' are ideals of P as well as of their intersections. So the join $\mathcal{G} \vee \mathcal{G}'$ of \mathcal{G} and \mathcal{G}' in \mathbb{G} is contained in $\mathcal{O}(P)$. Now since $J(\mathcal{O}(P)) = \mathcal{P} = J$ and $\mathcal{G} \subseteq \mathcal{G} \vee \mathcal{G}' \subseteq \mathcal{O}(P)$, Item (2) of Corollary 4 shows that $J(\mathcal{G} \vee \mathcal{G}') = J$. The same corollary says that \mathbb{G}_P is convex in \mathbb{G} .

(2) The first part is obvious since, for the covering relation \prec in \mathbb{M}_P , \mathbb{G} or \mathbb{G}_P , $\mathcal{G} \prec \mathcal{G}'$ if and only if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$. The second part is an immediate consequence of Corollary 1.

(3) Since \mathbb{G}_P is a convex subposet of the ranked semilattice \mathbb{G} , it is ranked. \square

Remark 9. The semilattice \mathbb{G}_P is generally not convex in \mathbb{M}_P . Consider, for instance, the atomistic convex geometry defined on the antichain $P = \{1, 2, 3, 4\}$ by taking all intervals of the linear order $1 > 2 > 3 > 4$. Then $\mathcal{G} = \{\emptyset, 1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$ (where, for instance, 134 stands for $\{1, 3, 4\}$) and $\mathcal{G} + \{14\} \in \mathbb{M}_P \setminus \mathbb{G}_P$.

We are now going to characterize the ideals I of a poset P which are \mathbb{G}_P -deletable in a convex geometry $\mathcal{G}' \in \mathbb{G}_P$. In fact, since $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ must belong to $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$, Items (1)(c) and (2)(b) of Theorem 3.1 show that I is \mathbb{G}_P -deletable if and only if $I \in M' \setminus J'$, $I^+ \notin J'$ (which implies $I \neq \emptyset$) and I does not cover any meet-irreducible element of \mathcal{G}' . The following result gives another interesting characterization.

Theorem 5.1. *Let P be a poset and $\mathcal{G}' \in \mathbb{G}_P$. An ideal I of P is \mathbb{G}_P -deletable w.r.t. \mathcal{G}' if and only if it satisfies the following three conditions:*

- (1) $I \in M'$,
- (2) $G \notin M'$ for every $G \in \mathcal{G}'$ with $G \prec I$,
- (3) I is a coatom of a Boolean sublattice $\underline{2}^3$ of \mathcal{G}' .

Proof. We have just observed above that Conditions (1) and (2) together with $I, I^+ \notin J'$ are equivalent to $\mathcal{G}' \setminus \{I\} \in \mathbb{G}_P$. Since Condition (3) obviously implies $I, I^+ \notin J'$, we have to show that Conditions (1) and (2) together with $I, I^+ \notin J'$ imply Condition (3). Since I is not join-irreducible, there exist at least G_1 and G_2 distinct and covered by I . Since these elements are not meet-irreducible (by Condition (2)), there exist K_1 and K_2 (different from I) with $G_1 \prec K_1$ and $G_2 \prec K_2$.

It is impossible to have $K_1 = K_2$ (since one would have G_1 and G_2 covered by I and K_1). If K_1 and K_2 are covered by I^+ we are done since, in the convex geometry \mathcal{G}' , $\{K_1, K_2, I\}$ generate a Boolean sublattice $\underline{2}^3$ in which I is a coatom. Assume that K_1 is not covered by I^+ . By Point (5) of Lemma 2, $G_1 \prec I$ and $G_1 \prec K_1$ imply the existence of $M_1, L_1 \in \mathcal{G}'$ with $G_1 \prec M_1 \prec L_1$ and $I \prec L_1$. Since $I \in M'$, one has $L_1 = I^+$ and M_1 must be different from K_1 . If K_2 is covered by I^+ , we are done (using $\{M_1, K_2, I\}$). If not, for the same reasons as above there exists $M_2 (\neq K_2)$ with $G_2 \prec M_2 \prec L_2 = I^+$. But $M_2 = M_1$ is impossible (since one would have G_1 and G_2 covered by I and M_1). Hence, $\{M_1, M_2, I\}$ generate a Boolean sublattice $\underline{2}^3$ in which I is a coatom. \square

The ideals $I = P \setminus [x]$ such that $\mathcal{O}(P) \setminus I$ is a coatom of the semilattice \mathbb{G}_P have already been characterized by properties of x in Theorem 4.1 (Item (3)). Using the above theorem, we can also say that these ideals are the coatoms of a Boolean sublattice $\underline{2}^3$ of $\mathcal{O}(P)$ which are meet-irreducible in $\mathcal{O}(P)$ and which do not cover any meet-irreducible element of $\mathcal{O}(P)$. Moreover, such an ideal I covers $k = |\{\text{maximal elements of } I\}|$ ideals which all become new meet-irreducible elements in $\mathcal{O}(P) \setminus I$. Now, we focus on the minimal elements of \mathbb{G}_P . Using Theorem 3.1 and Corollary 1, one gets:

Corollary 5. *Let \mathcal{G} be a convex geometry of \mathbb{G}_P . The following conditions are equivalent:*

- (1) \mathcal{G} is a minimal element of \mathbb{G}_P ,
- (2) for every $I \in M$, either there exists $G \prec I$ with $G \in M$, or $I \in M \cap J$,
- (3) the set M is a union of chains of M of the type $I_1 \prec I_2 \prec \dots \prec I_k$, such that the minimum I_1 of each chain is doubly-irreducible.

Remark 10. Let \mathcal{G} and \mathcal{G}' be two minimal elements of the semilattice \mathbb{G}_P . Since $\mathcal{G} \cap \mathcal{G}' \in \mathbb{M}_P \setminus \mathbb{G}_P$, the meet in \mathbb{G} (when it exists) of \mathcal{G} and \mathcal{G}' satisfies $\mathcal{G} \wedge \mathcal{G}' \subseteq \mathcal{G} \cap \mathcal{G}'$. Observe also that \mathcal{G} and \mathcal{G}' are not necessarily isomorphic lattices. For instance, there are two types of minimal atomistic convex geometries on the set $P = \{1, 2, 3, 4\}$. The former is formed by all intervals of a linear order defined on P , so it is, for example, $\{\emptyset, 1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. The latter is, for example $\{\emptyset, 1, 2, 3, 4, 13, 34, 23, 134, 234, 1234\}$.

We now characterize the posets P such that $|\mathbb{G}_P| = 1$, i.e. such that $\mathcal{O}(P)$ is the unique convex geometry \mathcal{G} defined on P with $J(\mathcal{G}) = \mathcal{P}$. Recall that the *width* $w(P)$ of a poset P is the maximum size of an antichain in P .

Theorem 5.2. *For a poset P , the following conditions are equivalent:*

- (1) $|\mathbb{G}_P| = 1$,
- (2) *for each x in P , either x is a node, or x is join-prime, or x covers a strong meet-irreducible element,*
- (3) *P is a poset of width at most 2.*

Proof. Since $|\mathbb{G}_P| = 1$ if and only if \mathbb{G}_P has no coatoms, the equivalence of (1) and (2) is an obvious consequence of Item (3) in Theorem 4.1.

Since $|\mathbb{G}_P| = 1$ if and only if $\mathcal{O}(P)$ has no \mathbb{G}_P -deletable element, the implication of (1) by (3) is an obvious consequence of Item (3) in Theorem 5.1.

Finally, we prove that (2) implies (3). Let P be a poset satisfying (2) and assume that $w(P) \geq 3$. Consider an antichain A of size $k \geq 3$, minimal in the classical order defined between antichains ($B \leq C$ if, for every $x' \in C$, there exists $x \in B$ such that $x \leq x'$). An element x in A cannot be a node of P (since A is an antichain) and cannot cover a strong meet-irreducible element, since then A would not be a minimal antichain. So x must be join-prime in P , i.e. $P = [x] + (y)$ for some y in P . This implies that the elements of $A \setminus \{x\}$ are all smaller than or equal to y . The same being true for every element of A , all the elements y form an antichain A' of size k such that A is smaller than A' in the order between antichains (and $A \cap A' = \emptyset$). Iterating this procedure, we would get an infinite chain of antichains of P , a contradiction with the fact that P is finite. \square

Remark 11. The equivalence between (2) and (3) in this theorem gives a characterization of posets of width at most 2 that is new (to our knowledge).

We finally study the more general case when \mathbb{G}_P has a unique coatom. We have the following result:

Proposition 8. *For a poset P , the following Properties (1) and (2) are equivalent and imply Property (3):*

- (1) \mathbb{G}_P has a unique coatom,
- (2) *there exists a unique element in P which does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists),*
- (3) *P must contain at least one strong meet-irreducible element.*

Proof. The equivalence between Properties (1) and (2) comes from Point (3) of Theorem 4.1.

Assume that \mathbb{G}_P has a unique coatom and that P is smi-free. Then Property (2) becomes: “There exists a unique element in P which is neither a node nor join-prime”. By Corollary 18 in Bordalo and Monjardet [4], 2002, this implies $|\mathbb{M}_P| =$

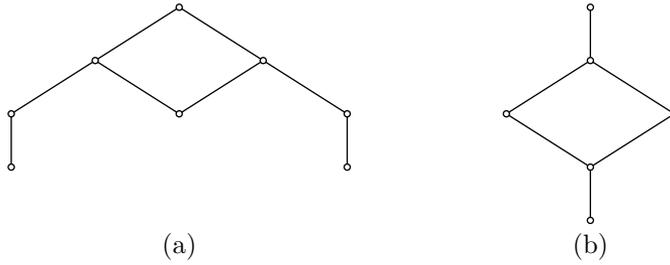


Figure 3.

- (1) What are the posets P such that \mathbb{G}_P has a unique coatom? In fact, we have given a characterization of these posets in Proposition 8, but is it possible to give a more precise description?
- (2) What are the posets P such that \mathbb{G}_P is a lattice?

Last but not least, another interesting problem can be considered: if \mathcal{G} and \mathcal{G}' are minimal elements of the semilattice \mathbb{G}_P , does this imply $|\mathcal{G}| = |\mathcal{G}'|$?

7. APPENDIX: ALGORITHM

Theorem 5.1 easily leads to the following algorithm for computing all convex geometries of \mathbb{G}_P .

Input: A poset P .

Output: The list CG_P of all convex geometries of \mathbb{G}_P .

```

begin
  1: Compute  $(\mathcal{O}(P), \subseteq)$ ;
      $CG_P := \{(\mathcal{O}(P), \subseteq)\}$ ;
  2: foreach  $L \in CG_P$ 
     Compute  $M(L)$ ;
     Compute  $J(L)$ ;
     Let  $M_L^* := M(L) \setminus J(L)$ ;
     foreach  $C \in M^*(L)$ 
       Compute  $\langle C \rangle := \{G \in L : G \prec C\}$ ;
       Compute  $\langle C^+ \rangle := \{H \in L \setminus C : H \prec C^+\}$ ;
       if  $(L \setminus C \notin CG_P)$  and  $(|\langle C \rangle \cap M(L)| = 0)$  and  $(|\langle C^+ \rangle| \neq 0)$  then
          $CG_P := CG_P + \{(L \setminus C, \subseteq)\}$ ;
end

```

Algorithm 1: Computing all convex geometries of \mathbb{G}_P .

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