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TOPOLOGICAL INVARIANTS OF ISOLATED COMPLETE INTERSECTION CURVE SINGULARITIES

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Abstract. In this paper we present some formulae for topological invariants of projective complete intersection curves with isolated singularities in terms of the Milnor number, the Euler characteristic and the topological genus. We also present some conditions, involving the Milnor number and the degree of the curve, for the irreducibility of complete intersection curves.

Keywords: topological invariants, genus, Euler characteristic, irreducibility criterion

MSC 2010: 32S50, 58K65

1. Introduction

One interesting and important problem in singularity theory and algebraic geometry is to obtain topological information about a curve in $\mathbb{P}^n(\mathbb{C})$ using the equations which define this curve.

Several topological invariants are known, for example, the Milnor number, the Euler characteristic and the topological genus. But, computing theses invariants is a hard problem because, in general, we need to consider some data which is not contained explicitly in the equations that define the curve. In particular cases, we have simple formulae. For example, for the topological genus $g_T(C)$ of a non-singular irreducible plane curve C with degree d we have the classical Plücker's relation $g_T(C) = \frac{1}{2}(d-1)(d-2)$. However, for the general case we do not have a formula for this invariant. We present here a formula for computing topological genus of an irreducible complete intersection curve with isolated singularities.

We also rewrite Plücker's formula obtained by Kleiman [8] for complete intersection curves with isolated singularities in terms of the Milnor number and use this formula

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to establish a criterion for the irreducibility of such curves. In particular, we obtain an estimate for the Milnor number of a singular point in terms of the degree of the curve.

2. Affine and projective curves

Let $x_0, \dots, x_n$ be homogeneous coordinates in $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$, let

$$U_0 = \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n;\ x_0 \neq 0\}$$

be the affine chart and let $\varphi\colon U_0 \to \mathbb{C}^n$ be defined by $(x_0 : x_1 : \dots : x_n) \mapsto (x_1/x_0, \dots, x_n/x_0)$.

Let $X \subset \mathbb{C}^n$ be an affine variety. The closure of $\varphi^{-1}(X)$ in $\mathbb{P}^n$ with respect to the Zariski topology is called the *projective closure* of X in $\mathbb{P}^n$ and denoted by $\overline{X}$. We denote by X_∞ the part at infinity of X, that is, $X_\infty = \overline{X} \cap H_\infty$ where $H_\infty = V(x_0) := \{(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n;\ x_0 = 0\}$ is the *hyperplane at infinity.*

Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal and define $I^h := \langle f^h;\ f \in I\rangle \subset \mathbb{C}[x_0, \dots, x_n]$, where f^h is the homogenization of f with respect to x_0. The ideal I^h is called the *homogenization* of I. For $g \in \mathbb{C}[x_0, \dots, x_n]$, we set $g|_{H_\infty} := g(0, x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ and denote $I_\infty := \langle g|_{H_\infty};\ g \in I^h\rangle \subset \mathbb{C}[x_1, \dots, x_n]$.

If $X \subset \mathbb{P}^n$ is a projective variety then $X^a = \varphi_0(X \cap U_0) \subset \mathbb{C}^n$ is called the *affine part* of X and if $J \subset \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous ideal, then the ideal

$$J^a := \langle f^a(x_1, \dots, x_n) := f(1, x_1, \dots, x_n);\ f \in J\rangle$$

is called the *affinization* of J.

We observe that it is not sufficient to homogenize an arbitrary set of generators of I in order to obtain a set of generators of I^h. For example, if we consider the ideal $I = \langle x^3 + z^2, x^3 + y^2\rangle \subset \mathbb{C}[x, y, z]$, then $I^h \ni z^2 - y^2 \notin J = \langle x^3 + z^2u, x^3 + y^2u\rangle$, where u is the homogenizing coordinate. Indeed, if we consider the curve $X = V(I)$ in $\mathbb{C}^3$, then its projective closure $\overline{X} = V(I^h)$ should meet H_∞ only at finitely many points, while $V(J)$ contains the line $\{u = x = 0\}$ as a component in H_∞.

However, if $f_1, \dots, f_k$ is a Gröbner basis of I with respect to a degree ordering, then $I^h = \langle f_1^h, \dots, f_k^h\rangle$ and we have $\overline{V(I)} = V(f_1^h, \dots, f_k^h)$ and $V(I)_\infty$ is $V(f_1^h|_{H_\infty}, \dots, f_k^h|_{H_\infty})$.

Theorem 2.1 (Mumford [11]). *Let $\mathscr{O}_{X,x}$ be the local ring of a variety X of dimension d at a point $x \in X$, $\mathscr{M}$ the maximal ideal of $\mathscr{O}_{X,x}$ and $I \subset \mathscr{O}_{X,x}$ any ideal such that $I \supset \mathscr{M}^k$ for some $k > 0$. Then $\mathscr{O}_{X,x}/\mathscr{O}_{X,x} \cdot I^l$ is a finitely generated $\mathscr{O}_{X,x}/\mathscr{M}^{lk}$-module, hence it has a finite dimension as a complex vector space. Then there is a polynomial $P_{\mathrm{HS}}(l)$ of degree at most d called the Hilbert-Samuel polynomial such that:*

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_{X,x}}{\mathscr{O}_{X,x} \cdot I^l} = P_{\mathrm{HS}}(l) \quad \text{if } l \gg 0.$$

If

$$P_{\mathrm{HS}}(t) = e \cdot \frac{t^d}{d!} + \{\text{lower order terms}\},$$

then $e = e_x(I; \mathscr{O}_{X,x})$, or shorter $e_x(I)$, is called the multiplicity of $\mathscr{O}_{X,x}$ with respect to I.

Let X be any variety and $x \in X$. The *multiplicity* of X at x is the multiplicity of the local ring $\mathscr{O}_{X,x}$ with respect to the maximal ideal $\mathscr{M} \subset \mathscr{O}_{X,x}$, that is, $e_x(\mathscr{M})$, and is denoted by $m_x(X)$.

Geometrically, we may interpret $m_x(X)$ as follows (see [11]). Let $X \subset \mathbb{C}^n$ be a variety of dimension d defined by an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$, $x \in X$, and let $C_x(X)$ be the tangent cone of X at x, that is, $C_x(X) = V(In_x(I))$, where $In_x(I)$ denotes the ideal generated by initial forms of all $f \in I$.

We define

$$m_x(X) = \min\{(X, H)_x \colon H \text{ linear space such that } \{x\} = X \cap H\}$$

where $(X, H)_x$ denotes the intersection multiplicity of X and H at x where H is a linear space of dimension $n - d$ through x, which is transversal to $C_x(X)$.

Example 2.2. Let X be the cuspidal cubic curve given by $f = x^3 - y^2 = 0$. The tangent cone is the x-axis $\{y = 0\}$ and a line that is transversal to the tangent cone is $H = \{x + by = 0\}$ with $b \in \mathbb{C}$. Then $m_0(X) = 2$.

Let $X \subset \mathbb{C}^n$ (or $\mathbb{P}^n$) be an affine (or projective) variety of dimension d. We define the *degree* $\deg(X)$ of X to be d times the leading coefficient of the Hilbert polynomial P_H of X or $\mathscr{O}/I$ (see [6]). A geometric interpretation of the degree is given in [11]. If $X \subset \mathbb{C}^n$ (or $\mathbb{P}^n$) is an affine (or projective) variety of dimension d, then

$$\deg(X) = \sharp(X \cap H),$$

where $\sharp$ denotes the number of elements counting multiplicity and $H \subset \mathbb{C}^n$ (or $\mathbb{P}^n$) is a sufficiently general affine (or projective) hyperplane of dimension $n - d$.

If $X = \mathbb{C}^n$ then we denote the local ring at x by $\mathscr{O}_{n,x}$. We also denote by (X, x) a germ at $x \in \mathbb{C}^n$ of a complete intersection curve with isolated singularities X in $\mathbb{C}^n$ defined by equations $f_1 = 0, \ldots, f_{n-1} = 0$ with $f_i \in \mathscr{O}_{n,x}$, that is, X is the affine curve

$$X = V(f_1, \ldots, f_{n-1}) = \{P \in \mathbb{C}^n;\ f_i(P) = 0, \forall\, i = 1, \ldots, n-1\}.$$

We choose a generic linear projection $p\colon\ \mathbb{C}^n \to \mathbb{C}$ and define the space

$$X_1 = V(f_1, \ldots, f_{n-1}, p)$$

which is an isolated complete intersection singularity (ICIS) (see [4]).

We let $J(f_1, \ldots, f_{n-1}, p)$ be the variety defined by the determinant of the Jacobian matrix and denote by $\mu_x(X)$ the Milnor number of X at x, that is, the codimension of the Jacobian ideal of X in $\mathscr{O}_{n,x}$. Applying the theorem of Greuel in [4] we have

$$\mu_x(X_1) + \mu_x(X) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n,x}}{(f_1, \ldots, f_{n-1}, J(f_1, \ldots, f_{n-1}, p))}$$

or

$$\mu_x^{n-1}(X) + \mu_x^{n}(X) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n,x}}{(f_1, \ldots, f_{n-1}, J(f_1, \ldots, f_{n-1}, p))}$$

where $\mu^i(X)$ denotes the Milnor number of X restricted to a linear space of dimension i in $\mathbb{C}^n$.

Because p is generic we have that X_1 is a zero dimensional space and

$$\mu_x(X_1) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n,x}}{(f_1, \ldots, f_{n-1}, p)} - 1 = (X, H)_x - 1,$$

where $H = p^{-1}(x)$ (see [10]). Hence,

$$\mu_x(X) - (X, H)_x + 1 = (X, J(f_1, \ldots, f_{n-1}, p))_x. \tag{1}$$

We consider now $C \subset \mathbb{P}^n$ to be a projective complete intersection curve given by the ideal $I = (F_1, \ldots, F_{n-1})$, where $F_i \in \mathbb{C}[X_0, \ldots, X_n]$ is a homogeneous polynomial of degree $d_i > 0$. We can obtain a projective counterpart to formula (1). Indeed, let $P(C)$ be the relative polar curve defined by $J(F_1, \ldots, F_{n-1}, p) = 0$ where p is as above. Then we have the following result.

Lemma 2.3. *For any $x \in C$ we have*

$$(C, P(C))_x = \mu_x(C) + (C, H)_x - 1,$$

where H is a line passing through x and a different point y.

P r o o f. It is sufficient to choose coordinates of $\mathbb{P}^n$ such that $x = (0 : \ldots : 0 : 1) \in \mathbb{P}^n$ and $y = (0 : \ldots : 0 : 1 : 0) \in \mathbb{P}^n$. In this way, if X is the affine curve associated to C, then the lemma reduces to the affine case, that is, $\mu_x(X) + (X, H)_x - 1 = (X, P(X))_x$. $\square$

We consider as above a complete intersection curve C with isolated singularities in $\mathbb{P}^n$ and the affine curve X associated to C given by $f\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ with $f = (f_1, \ldots, f_{n-1})$, that is, $X = f^{-1}(0)$, where $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ is a polynomial of degree d_i. Using Bézout's Theorem we have that $d = \prod_{i=1}^{n-1} d_i$ is the degree of X.

Remark 2.4. If X is a complete intersection curve as above, then applying Lemma 2.3 and Bézout's Theorem we obtain

$$\sum_{x \in \operatorname{Sing}(X)} ((X, H)_x - 1) = d \sum_{i=1}^{n-1} (d_i - 1) - \sum_{x \in \operatorname{Sing}(X)} \mu_x(X)$$

where H is a linear space transverse to X at x and $\operatorname{Sing}(X)$ is the set of the singular points of X.

Using Lemma 2.3 we rewrite Plücker's formula for complete intersection curves.

Theorem 2.5 (Plücker's Formula). *Let $C \subset \mathbb{P}^n$ be a complete intersection curve with isolated singularities defined by the ideal $(F_1, \ldots, F_{n-1}) \subset \mathbb{C}[X_0, \ldots, X_n]$ where each F_i is homogeneous of degree d_i. Then the number $\check{d}$ of hyperplanes through a fixed general codimension 2-plane in $\mathbb{P}^n$ and tangent to C is given by*

$$\prod_{i=1}^{n-1} d_i \sum_{i=1}^{n-1} (d_i - 1) - \sum_{x \in \operatorname{Sing}(C)} (\mu_x^n(C) + \mu_x^{n-1}(C)).$$

P r o o f. By Bézout's Theorem, we have

$$\begin{aligned} (C, P(C))_{\mathbb{P}^n} &= \sum_{x \in C \cap P(C)} (C, P(C))_x \\ &= \sum_{x \in (C - \operatorname{Sing}(C)) \cap P(C)} (C, P(C))_x + \sum_{x \in \operatorname{Sing}(C) \cap P(C)} (C, P(C))_x, \end{aligned}$$

where $P(C)$ is the polar curve (see above).

By the above observations we have

$$(C, P(C))_x = \mu_x^n(C) + \mu_x^{n-1}(C)$$

for singular points of C. Using the geometric interpretation of $\check{d}$ given by Kleiman [8], we have $\check{d} = \sum_{x \in (C - \mathrm{Sing}(C)) \cap P(C)} (C, P(C))_x$ and

$$\prod_{i=1}^{n-1} d_i \sum_{i=1}^{n-1} (d_i - 1) = \check{d} + \sum_{x \in \mathrm{Sing}(C)} \mu_x^n(C) + \mu_x^{n-1}(C).$$

□

Remark 2.6. Notice that if C is a plane curve, then the number of tangents $\check{d}$ through a fixed general point is given by

$$\begin{aligned} \check{d} &= d(d-1) - \Sigma_{x \in \mathrm{Sing}(C)} (\mu_x^2(C) + \mu_x^1(C)) \\ &= d(d-1) - \Sigma_{x \in \mathrm{Sing}(C)} (\mu_x(C) + m_x(C) - 1), \end{aligned}$$

which is the classical Plücker's formula.

Our next result is an extension of the classical formula of Max Noether for complete intersection curves. Before starting it we require the following result.

Theorem 2.7 (Riemann-Hurwitz [9])**.** *Let X and Y be compact topological spaces, Y triangulable and let $\pi\colon X \to Y$ be a continuous open surjective mapping with finite fibers. Suppose there is a finite subset $A \subset Y$ such that $\pi\colon X \setminus \pi^{-1}(A) \to Y \setminus A$ is a covering with d sheets. Then X is triangulable and*

$$\chi(X) = d \cdot \chi(Y) + \sum_{y \in A} (\sharp \pi^{-1}(y) - d)$$

where $\chi(Z)$ denotes the Euler characteristic of Z.

P r o o f. See [9]. □

Let X and Y be algebraic projective curves, Y smooth and let $\pi\colon X \to Y$ be a regular surjective mapping with finite fibers. Then the assumptions of Theorem 2.7 are satisfied with $d = \deg \pi$ (the geometric degree of π) and for every $x \in X$ the multiplicity $\mathrm{mult}_x \pi$ is defined in such a way that $\sum_{x \in \pi^{-1}(y)} \mathrm{mult}_x \pi = \deg(\pi)$, and for generic $y \in Y$ we have $\operatorname{mult}_x \pi = 1$ if $x \in \pi^{-1}(y)$ (see the definition of $\operatorname{mult}_x(\pi)$ in [11]). With this notation, we have

$$\chi(X) = \deg(\pi) \cdot \chi(Y) - \sum_{x \in X} (\operatorname*{mult}_x(\pi) - 1).$$

If $X \subset \mathbb{P}^n$ is a complete intersection curve with isolated singularities and $Y = \mathbb{P}(\mathbb{C})$, then by [11, p. 121, Theorem A.10] we have $\operatorname{mult}_x \pi = e_x(\mathscr{M}_{Y,y} \cdot \mathscr{O}_{X,x}) = (X, H)_x$, where $\mathscr{M}_{Y,y}$ is the maximal ideal of $\mathscr{O}_{Y,y}$ and H is a hyperplane that intercepts transversely X at x.

We have the following theorem.

Theorem 2.8 (Max Noether's Formula). *Let $C \subset \mathbb{P}^n$ be a complete intersection curve with isolated singularities defined by $F_1, \dots, F_{n-1} \in \mathbb{C}[X_0, \dots, X_n]$ where each F_i is a homogeneous polynomial of degree d_i. Then*

$$\chi(C) = d\left(2 - \sum_{i=1}^{n-1}(d_i - 1)\right) + \sum_{x \in C} \mu_x(C),$$

where d is the degree of C, that is, $d = \prod_{i=1}^{n-1} d_i$.

Proof. We consider a generic projection $\pi\colon C \to \mathbb{P}(\mathbb{C})$ such that $\pi^{-1}(0:1) = H$ is a hyperplane which intercepts transversely C at x and $\deg(\pi) = d$. Then using the fact that $\chi(\mathbb{P}(\mathbb{C})) = 2$, Remark 2.4 and Theorem 2.7, we have

$$\begin{aligned}\chi(C) &= d \cdot \chi(\mathbb{P}(\mathbb{C})) - \Sigma_{x \in C}((C, H)_x - 1)\\ &= 2d - d\sum_{i=1}^{n-1}(d_i - 1) + \sum_{x \in C} \mu_x(C)\\ &= d\left(2 - \sum_{i=1}^{n-1}(d_i - 1)\right) + \sum_{x \in C} \mu_x(C).\end{aligned}$$

□

Remark 2.9.

1. We observe that in the case when C is a projective plane curve of degree d, we have
$$\chi(C) = \sum_{x \in C} \mu_x(C) - d(d - 3).$$

2. When the curve $C \subset \mathbb{P}^n$ is smooth and given by $(F_1, \dots, F_{n-1})$ where each F_i is a homogeneous polynomial of degree d_i, we have

$$\chi(C) = \left(\prod_{i=1}^{n-1} d_i\right)\left(1 + n - \sum_{i=1}^{n-1} d_i\right).$$

Using Theorem 2.8, we have a formula for the topological genus of an irreducible complete intersection curve. Note that there exist formulae for computing the arithmetic genus of projective complete intersections, see for example [1].

Proposition 2.10. *Let $N\colon \widetilde{C} \to C$ be the normalization of the complete intersection curve $C \subset \mathbb{P}^n$ given by $(F_1, \dots, F_{n-1}) \subset \mathbb{C}[X_0, \dots, X_n]$ with F_i a homogeneous polynomial of degree d_i and let $r_x(C)$ be the number of irreducible components of C at $x \in C$.*

We have

$$\chi(\widetilde{C}) = \chi(C) + \sum_{x \in C} (r_x(C) - 1)$$

and if C is irreducible then the topological genus of C, $g_T(C)$, is given by

$$g_T(C) = 1 + \frac{1}{2} d\left(\sum_{i=1}^{n-1} d_i - n - 1\right) - \frac{1}{2}\sum_{x \in C} (\mu_x(C) + r_x(C) - 1),$$

where $d = \prod_{i=1}^{n-1} d_i$.

Proof. The normalization N has degree 1 and $\sharp N^{-1}(x) = r_x(C)$. Then by the Riemann-Hurwitz theorem and Milnor's formula $\mu_x(C) = (C, P(C))_x - r_x(C) + 1$ we have the first equality. Now using Theorem 2.8 we obtain

$$\chi(\widetilde{C}) = d\left(2 - \sum_{i=1}^{n-1} (d_i - 1)\right) + \sum_{x \in C} (\mu_x(C) + r_x(C) - 1).$$

If C is an irreducible curve then we have the formula of the topological genus $g_T(C)$ using the relation $\chi(\widetilde{C}) = 2 - 2g_T(C)$ (see [3]). $\square$

3. Examples and applications

In this section we present particular cases of the previous results and we deduce some well known relations.

Example 3.1. Let the irreducible plane cuspidal curve C be given by $x^2z+y^3=0$. We have $d=3$ and $\mu=2$. Using Remark 2.9 and Proposition 2.10 we obtain

$$g_T(C)=0 \quad \text{and} \quad \chi(C)=2.$$

Example 3.2. Using Proposition 2.10, we may obtain some classical formulae for the genus. For example, if C is an irreducible and smooth curve, then we have

$$g_T(C)=1+\frac{1}{2}d\Big(\sum_{i=1}^{n-1} d_i-(n+1)\Big).$$

In particular, if C is a curve in $\mathbb{P}^3$ obtained by the intersection of two hypersurfaces with degree d_1 and d_2 (see [12]), we have

$$g_T(C)=1+\frac{1}{2}d_1d_2(d_1+d_2-4).$$

The last formula is a generalization of the classical case of a plane curve. In fact, if C is a smooth irreducible plane curve with degree d, then we have

$$g_T(C)=1+\frac{1}{2}d(d-3)=\frac{(d-1)(d-2)}{2}.$$

Example 3.3. If X is a smooth curve in $\mathbb{P}^n$ with $g_T(X)=2$, then X is not a complete intersection. In fact, if X is irreducible then using the previous example, we have $2=d\Big(\sum\limits_{i=1}^{n-1} d_i-n-1\Big)$.

We have two possibilities:

- $d=1$ and $\sum\limits_{i=1}^{n-1} d_i-n-1=2$ implies $n=-2$, which is absurd.
- $d=2$ and $\sum\limits_{i=1}^{n-1} d_i-n-1=1$, which is also absurd.

Example 3.4. If X is a smooth projective complete intersection curve in $\mathbb{P}^n$ defined by $(F_1,\dots,F_{n-1})$, where F_i is a homogeneous polynomial with degree d_i, then X is diffeomorphic to a torus if and only if $(d_1,\dots,d_{n-1})$ is a partition of $n+1$.

In fact, Harris in [5] showed that two oriented 2-manifolds are diffeomorphic if and only if they have the same genus. As the torus has genus 1, then it is sufficient to show that the curves with the listed properties have genus 1.

Using the formula given in Example 3.2 and the fact that $(d_1, \ldots, d_{n-1})$ is a partition of $n+1$, that is, $\sum\limits_{i=1}^{n-1} d_i = n+1$, we have

$$g_T(X) = 1 + \frac{1}{2} d \left(\sum_{i=1}^{n-1} d_i - n - 1 \right) = 1,$$

proving the result.

4. Irreducibility conditions

In this section we present some conditions involving topological invariants for a projective complete intersection curve to be irreducible.

Proposition 4.1. *Let $C \subset \mathbb{P}^n$ be a projective complete intersection curve of degree $d = \prod\limits_{i=1}^{n-1} d_i$ with isolated singularities. Suppose that C has m irreducible components at a point $x \in C$. Then*

$$\mu_x(C) \leqslant 2d^2 + 2m - (n+2)d.$$

Proof. Let $C_1, \ldots, C_m$ be the irreducible components of C at x. Using Proposition 2.10 for irreducible curves we have

$$\mu_x(C_j) \leqslant (d^j - 1) \left(\sum_{i=1}^{n-1} d_i^j - 2 \right) + \sum_{i=1}^{n-1} (d_i^j - d^j)$$

where $d^j = \prod\limits_{i=1}^{n-1} d_i^j$, d_i^j is the degree of the equations which define C_j for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, m$. Also, using the expressions for the Milnor numbers [2], we obtain

$$\mu_x(X) - 1 = \sum_{i=1}^{r} (\mu_x(X_i) - 1) + 2 \sum_{i=1}^{r-1} (X_i, X_j)_x.$$

We have

$$\begin{aligned}\mu_x(C) + m - 1 &= \mu_x \left(\bigcup_{j=1}^{m} C_j \right) + m - 1 = \sum_{j=1}^{m} \mu_x(C_j) + 2 \sum_{1 \leqslant j < k \leqslant m} (C_j, C_k)_x \\ &\leqslant \sum_{j=1}^{m} \left((d^j - 1) \left(\sum_{i=1}^{n-1} d_i^j - 2 \right) + \sum_{i=1}^{n-1} (d_i^j - d^j) \right) + 2 \sum_{1 \leqslant j < k \leqslant m} d^j d^k\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m}\Big(d^j\sum_{i=1}^{n-1}d_i^j\Big) - 2\sum_{j=1}^{m}d^j + 2m - (n-1)\sum_{j=1}^{m}d^j + 2\sum_{1\leqslant j<k\leqslant m}d^jd^k\\
&= \sum_{j=1}^{m}\Big(d^j\sum_{i=1}^{n-1}d_i^j\Big) - 2\sum_{j=1}^{m}d^j + 2m - (n-1)\sum_{j=1}^{m}d^j + \Big(\sum_{j=1}^{m}d^j\Big)^2 - \sum_{j=1}^{m}d^j\\
&\leqslant \sum_{j=1}^{m}d^j\sum_{j=1}^{m}\Big(\sum_{i=1}^{n-1}d_i^j\Big) - 3\sum_{j=1}^{m}d^j + 2m - (n-1)\sum_{i=j}^{m}d^j + \Big(\sum_{j=1}^{m}d^j\Big)^2\\
&\leqslant d^2 - 3d + 2m - (n-1)\sum_{j=1}^{m}d^j + d^2\\
&= 2d^2 - 3d + 2m - (n-1)d
\end{aligned}
$$

and this concludes the proof. □

Let X be a projective complete intersection curve given by an ideal $(F_1,\ldots,F_{n-1})\subset\mathbb{C}[X_0,\ldots,X_n]$ where each F_i is homogeneous. We assume that X is ICIS with multidegree $(d_1,\ldots,d_{n-1})$, that is, the degree of F_i is d_i and X has degree $d=\prod_{i=1}^{n-1}d_i$.

Definition 4.2. Let $X\subset\mathbb{P}^n$ be an ICIS reducible projective curve with degree d. If $\mathscr{O}_{X,x}$ is the local ring of X at $x\in X$ and $\overline{\mathscr{O}}_{X,x}$ is the normalization of $\mathscr{O}_{X,x}$, then we define the δ_x-*invariant* of X at x by

$$
\delta_x(X) = \dim_{\mathbb{C}}\frac{\overline{\mathscr{O}}_{X,x}}{\mathscr{O}_{X,x}}.
$$

If we assume that $X=\bigcup_{i=1}^{r}X_i$ is such that $X_i\cap X_j=\{x\}$ for $i\neq j$ then we have

$$
\delta_x(X) = \sum_{i=1}^{r}\delta_x(X_i) + \sum_{1\leqslant i<j\leqslant r}(X_i,X_j)_x \tag{2}
$$

(see [7]) and

$$
\mu_x(X) - 1 = \sum_{i=1}^{r}(\mu_x(X_i)-1) + 2\sum_{1\leqslant i<j\leqslant r}(X_i,X_j)_x
$$

(see [2]).

With this notation we have the following irreducibility criteria.

Proposition 4.3. *Let $X \subset \mathbb{P}^n$ be an ICIS projective curve given by the ideal $(F_1, \dots, F_{n-1})$, where each F_i is a homogeneous polynomial of degree d_i and $d = \prod_{i=1}^{n-1} d_i$.*

1. *If $\sum_{x \in X} \delta_x < d - 1$, then X is an irreducible curve.*
2. *If X has no linear factor and $\sum_{x \in X} \delta_x < 2d - 4$, then X is an irreducible curve.*

Proof. 1. Suppose that X has two components X_1 and X_2 such that $X = X_1 \cup X_2$. If $\deg(X_1) = e$, then $\deg(X_2) = d - e$ (see [3]). By Bézout's theorem we have $\sum_{x \in X} (X_1, X_2)_x = (d - e)e$ and using Hironaka's formula 2, we have $\sum_{x \in X} \delta_x \geqslant \sum_{x \in X} (X_1, X_2)_x = (d - e)e \geqslant d - 1$, which contradits $\sum_{x \in X} \delta_x < d - 1$.

2. We have, as in the previous item, $\sum_{x \in X} (X_1, X_2)_x = (d - e)e$. But if X has no linear factor, then $e > 1$ and we have $(d - e)e \geqslant 2(d - 2)$. This is a contradiction with our assumption. □

In particular, with the previous notation, we have the following corollary.

Corollary 4.4. *If $\mu(X) < 2(d - 1) - \sharp \operatorname{Sing}(X)$, then X is an irreducible curve.*

Proof. In fact, denoting $X = X_1 \cup X_2$ as in the proof of the previous proposition, we have, using the Hironaka's Theorem (see [2]),

$$\mu_x(X) + 1 = \sum_{i=1}^{2} \mu_x(X_i) + 2(X_1, X_2)_x,$$

hence

$$\sum_{x \in \operatorname{Sing}(X)} (\mu_x(X) + 1) = \sum_{x \in \operatorname{Sing}(X)} \sum_{i=1}^{2} \mu_x(X_i) + 2 \sum_{x \in \operatorname{Sing}(X)} (X_1, X_2)_x.$$

However, we have $\mu(X) = \sum_{x \in \operatorname{Sing}(X)} \mu_x(X)$, therefore

$$\mu(X) + \sharp \operatorname{Sing}(X) = \sum_{x \in \operatorname{Sing}(X)} \sum_{i=1}^{2} \mu_x(X_i) + 2 \sum_{x \in \operatorname{Sing}(X)} (X_1, X_2)_x.$$

From this equality we have

$$\mu(X) + \sharp \operatorname{Sing}(X) \geqslant 2 \sum_{x \in \operatorname{Sing}(X)} (X_1, X_2)_x = 2 \sum_{x \in X} (X_1, X_2)_x = 2(d - e)e \geqslant 2(d - 1).$$

Therefore $\mu(X) \geqslant 2(d - 1) - \sharp \operatorname{Sing}(X)$, which is impossible by the assumption. □

Example 4.5. a) The cuspidal curve C given by $x^2z + y^3 = 0$ has a single singular point. We have $\mu(C) = 2$ and the degree is $d = 3$. Then we have $2 = \mu(C) < 2(d-1) - \sharp\,\mathrm{Sing}(X) = 3$, that is, the curve C is irreducible as is well known.

b) Let C be the complete intersection curve given by $V(xy - zt, x^2 + 2z^2 - ty) \subset \mathbb{P}^3$. Since $\sharp\,\mathrm{Sing}(C) = 0$, we have $0 = \mu(C) < 2(d-1) - \sharp\,\mathrm{Sing}(C) = 6$, hence C is an irreducible curve.

c) We observe that the converse of the previous results is false. For example, if we consider the irreducible curve $C \subset \mathbb{P}^3$ defined by the equations

$$xy - wz = 0, \quad z^6w^9 + x^{15} + y^{10}w^5 = 0,$$

then $d = 30$, $\mu(C) = 126$, $\sharp\,\mathrm{Sing}(C) = 1$ and $\mu(C) > 2(d-1) - \sharp\,\mathrm{Sing}(C)$.

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