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A REVISED CLOSED GRAPH THEOREM FOR QUASI-SUSLIN SPACES

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Dedicated to the memory of Professor Walter Roelcke

Abstract. Some results about the continuity of special linear maps between F-spaces recently obtained by Drewnowski have motivated us to revise a closed graph theorem for quasi-Suslin spaces due to Valdivia. We extend Valdivia's theorem by showing that a linear map with closed graph from a Baire tvs into a tvs admitting a relatively countably compact resolution is continuous. This also applies to extend a result of De Wilde and Sunyach. A topological space X is said to have a (relatively countably) compact resolution if X admits a covering $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of (relatively countably) compact sets such that $A_\alpha \subseteq A_\beta$ for $\alpha \leqslant \beta$. Some applications and two open questions are provided.

Keywords: K-analytic space, web space, quasi-Suslin space

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1. Introduction

A set X is said to have a *resolution* if X is covered by a family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets such that $A_\alpha \subseteq A_\beta$ for $\alpha \leqslant \beta$. A (topological) space X has a (closed, relatively countably) compact resolution if X has a resolution consisting of (respectively closed, relatively countably) compact sets. Each quasi-Suslin space admits a countably compact resolution [1, Theorem 2] and any K-analytic space has a compact resolution [16]. Hence metrizable complete separable topological vector

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spaces (tvs) and their countable inductive limits admit compact resolutions [17, Theorem 2.1]. Very recently Drewnowski [6] has used the notion of resolution to prove some nice results about certain continuous maps between Fréchet and Banach spaces. Drewnowski's paper [6] motivated us to look again at Valdivia's classic closed graph theorem for quasi-Suslin spaces [18, I.4.2 (11)], which states that a linear mapping with closed graph from a metrizable Baire locally convex space (lcs) E into a quasi-Suslin lcs F is continuous. So we use some techniques of [18] and of our own to get a closed graph theorem (Theorem 1) which extends Valdivia's [18, I.4.2 (11)] as well as Drewnowski's [6, Corollary 4.10, Corollary 4.11]. In doing so we work with topological vector spaces rather than locally convex spaces since we think this is the natural setting for this kind of closed graph theorems (this has been done before, see for instance [12], [13]). We need only F to be a tvs with a relatively countably compact resolution (RCC resolution shortly), whereas E is assumed (only) to be a Baire tvs. Theorem 1 applies to get an analytic graph type result stating that a linear map from a Baire tvs into a tvs whose graph admits a RCC resolution is continuous, and will be used to extend and provide another (and shorter) proofs of some recent results obtained by Drewnowski [6] about the continuity of special linear maps between F-spaces.

For a resolution $\{A_\alpha\colon\ \alpha\in\mathbb{N}^{\mathbb{N}}\}$ set

$$D_{n_1,n_2,\ldots,n_p} := \bigcup\{A_\alpha\colon\ \alpha\in\mathbb{N}^{\mathbb{N}},\ \alpha(i)=n_i,\ 1\leqslant i\leqslant p\}$$

for $(n_i)_i$ in $\mathbb{N}$. If $(k_p)_p$ is contained in $\mathbb{N}$ and $x_p\in D_{k_1k_2,\ldots,k_p}$, then:

(A) There exists $\alpha\in\mathbb{N}^{\mathbb{N}}$ such that $x_p\in A_\alpha$ for each $p\in\mathbb{N}$, see [9].

For a tvs X with a resolution $\{A_\alpha\colon\ \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of bounded sets (the notion of bounded resolution was first used in [1, Theorem 5]) the following holds:

(B) For $(n_i)_i$ contained in $\mathbb{N}$, the filter base $\{p^{-1}D_{n_1,n_2,\ldots,n_p}\colon\ p\in\mathbb{N}\}$ converges to zero in X.

For a subset B of a space X let $D(B)$ be the set of points x of X such that every neighborhood of x intersects B in a set of second category. If $O(B)$ is the interior of $D(B)$, then B has the *Baire property* iff $O(B)\setminus B$ is of first category in X, [18, I.1.1 (9)]. If (B_n) is a sequence of subsets of X whose union is B, then $D(B)\setminus\bigcup\limits_{n\in\mathbb{N}}O(B_n)$ is rare [18, I.1.1 (7)].

(C) If B is covered by a web $\{B_{n_1,n_2,\ldots,n_p}\colon\ p,n_1,n_2,\ldots,n_p\in\mathbb{N}\}$, in the sense that $B=\bigcup\limits_{m=1}^{\infty}B_m$ and $B_{n_1,n_2,\ldots,n_p}=\bigcup\limits_{m=1}^{\infty}B_{n_1,n_2,\ldots,n_p,m}$ for every $p\in\mathbb{N}$, and if $\bigcap\limits_{p=1}^{\infty}O(B_{n_1,n_2,\ldots,n_p})\subseteq B$ for each sequence $(n_p)_p$ in $\mathbb{N}$ then B has the Baire property.

In fact, $C := O(B) \setminus \bigcup_{m\in\mathbb{N}} O(B_m)$ and

$$C_{n_1,n_2,\dots,n_p} := O(B_{n_1,n_2,\dots,n_p}) \setminus \bigcup_{m\in\mathbb{N}} O(B_{n_1,n_2,\dots,n_p,m}),$$

where $p, n_1, n_2, \dots, n_p \in \mathbb{N}$, are rare sets. By the hypothesis the first category set

$$C \cup \Bigl\{\bigcup\{C_{n_1,n_2,\dots,n_p}\colon\ p, n_1, n_2, \dots, n_p \in \mathbb{N}\}\Bigr\}$$

contains $O(B) \setminus B$. Therefore B has the Baire property.

(D) Let X be a space admitting a weaker first countable topology τ. If B is a subset of X with a τ-closed resolution $\{A_\alpha\colon\ \alpha \in \mathbb{N}^{\mathbb{N}}\}$, then B has the Baire property.

Indeed, choose $(n_p)_p$ in $\mathbb{N}$ and $x \in O(D_{n_1,n_2,\dots,n_p})$, $p \in \mathbb{N}$. Let $(U_p)_p$ be a τ-neighborhood basis of x and select $x_p \in U_p \cap D_{n_1,n_2,\dots,n_p}$. By (A) there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $x_p \in A_\alpha$ for $p \in \mathbb{N}$. From the τ-closedness condition it follows that $x \in A_\alpha \subseteq B$ and (C) applies.

2. A closed graph theorem

The following theorem extends Valdivia's [18, I.4.2 (11)] and its special case (when $E = F$) extends a result of De Wilde and Sunyach [18, I.4.3 (21)]. For a tvs E we denote by $\mathfrak{F}(E)$ the set of all balanced neighborhoods of zero in E.

Theorem 1. *Let E and F be tvs such that E is Baire and F admits a RCC resolution $\{A_\alpha\colon\ \alpha \in \mathbb{N}^{\mathbb{N}}\}$. If $f\colon\ E \to F$ is a linear map with closed graph, then f is continuous and there exist $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that* $\operatorname{Int} f^{-1}(\overline{D_{n_1,\dots,n_k}})$ *is non-empty for each $k \in \mathbb{N}$ and a sequence $(U_n)_n$ in $\mathfrak{F}(E)$ such that for every $V \in \mathfrak{F}(F)$ there exists $m \in \mathbb{N}$ with $m^{-1}U_m \subseteq f^{-1}(V)$. If $E = F$, then E is a separable F-space.*

Proof. Since E is a Baire space and $\{f^{-1}(D_{n_1,n_2,\dots,n_p})\colon\ p, n_1, n_2, \dots, n_p \in \mathbb{N}\}$ is a web of subsets of E covering E, there is a sequence $(r_p)_p$ in $\mathbb{N}$ such that $\operatorname{Int} \overline{f^{-1}(D_{r_1,r_2,\dots,r_p})} \neq \emptyset$ for every $p \in \mathbb{N}$. Hence $\overline{f^{-1}(H_p)} - \overline{f^{-1}(H_p)}$ is a neighborhood of the origin for each $p \in \mathbb{N}$, where $H_p := D_{r_1,r_2,\dots,r_p}$. Let $(U_p)_p$ be a sequence of balanced neighborhoods of the origin in E such that $U_{p+1} + U_{p+1} \subseteq U_p$ and $U_p \subseteq \overline{f^{-1}(H_p)} - \overline{f^{-1}(H_p)}$, $p \in \mathbb{N}$. Let τ be the semi-metrizable translation invariant vector topology on E defined by the basis $(p^{-1}U_p)_p$ of neighborhoods of zero. Since the graph of f is closed, there is a coarser linear topology ϱ on F such that the map $f\colon\ E \to (F, \varrho)$ is continuous [11, Lemma 3.1]. We claim that $f\colon\ (E, \tau) \to (F, \varrho)$ is continuous as well. In fact, if V is a closed neighborhood of zero in (F, ϱ) then, by (B), there exists $q \in \mathbb{N}$ such that $q^{-1}(H_q - H_q) \subseteq V$. As $f^{-1}(V)$ is a closed subset of E then $q^{-1}U_q \subseteq q^{-1}(\overline{f^{-1}(H_q)} - \overline{f^{-1}(H_q)}) \subseteq f^{-1}(V)$. This proves the claim. If $W \in \mathfrak{F}(F)$ is closed and $B_\alpha := f^{-1}(A_\alpha \cap W)$, then (using the previous claim and the fact that $\{A_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is RCC) one gets $\overline{B_\alpha}^{\tau} \subseteq f^{-1}(W)$. Thus $\{\overline{B_\alpha}^{\tau} \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a τ-closed resolution for $f^{-1}(W)$ and $f^{-1}(W)$ has the Baire property by (D). Since $f^{-1}(W)$ is of second category, Banach's difference theorem [10, Chapter 3, Sec. 10, Theorem 10.4] ensures that $f^{-1}(W) - f^{-1}(W)$ is a neighborhood of zero in E. Hence $f\colon E \to F$ is continuous.

If $V \in \mathfrak{F}(F)$ is closed there is (by (B)) $m \in \mathbb{N}$ such that $m^{-1}\overline{H_m} - m^{-1}\overline{H_m} \subseteq V$, so $m^{-1}U_m \subseteq f^{-1}(V)$. If $E = F$, then E is metrizable with a compact resolution and [2, Theorem 15] applies to get that E is analytic. But according to [3, Theorem 5.4] any two-sidely invariant metric generating the topology of E is a complete metric, hence E is a separable F-space. □

Theorem 1 applies to get the Grothendieck-Floret factorization theorem: let E be a Baire tvs and F the inductive limit of a sequence $(F_n)_n$ of tvs such that the inclusion $F_n \hookrightarrow F_{n+1}$ is compact for each $n \in \mathbb{N}$, i.e. there is $U_n \in \mathfrak{F}(F_n)$ whose closure $\overline{U_n}^{n+1}$ is compact in F_{n+1}. If $f\colon E \to F$ is a linear map with closed graph, then f is continuous and there is $n \in \mathbb{N}$ such that $f(E) \subseteq F_n$. Indeed, we may choose $(U_n)_n$ increasing. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha = n_1\overline{U_{n_1}}^{n_1+1}$. Then $\{A_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in F and $D_{n_1,\dots,n_k} = n_1\overline{U_{n_1}}^{n_1+1}$ is compact in F for each $k \in \mathbb{N}$. Apply Theorem 1.

Theorem 1 fails for topological groups in general: if a compact group of Ulam-measurable cardinality is either Abelian or connected, then it admits a strictly finer countably compact group topology [4]. Corollary 2 below provides an analytic graph type result which extends the second part of [3, Theorem 5.2] for the case of tvs and [6, Corollary 4.11], see also [15, Theorem 2.10.1].

Corollary 2. *Every linear map f from a Baire tvs E into a tvs F whose graph G admits a RCC resolution is continuous. Hence every linear map from an F-space into a separable metrizable tvs whose graph admits a complete resolution is continuous.*

P r o o f. The projection $P(x, f(x)) = x$ of G onto E is continuous, so P^{-1} is continuous by Theorem 1. But $f = Q \circ P^{-1}$, where $Q\colon G \to F$ is the projection. To complete the proof we show that f is continuous on every closed separable vector subspace E_0 of E. Let $\{A_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a complete resolution in G. The sets $A_\alpha \cap (E_0 \times F)$ form a complete resolution on $G \cap (E_0 \times F)$. But every metrizable and separable tvs with a complete resolution is analytic [6, Corollary 3.2] and the first part applies. □

We conjecture that the last part of Corollary 2 fails for non separable metrizable tvs F.

Problem 3. *Find a Fréchet space E which admits a strictly finer metrizable vector topology ξ such that E is covered by a sequence of ξ-complete sets.* Since every metrizable tvs admits a bounded resolution we may assume that (E, ξ) admits a ξ-bounded and complete resolution. Note that ξ cannot be separable, otherwise (E, ξ) would be analytic, being covered by a sequence of analytic subsets, and then Theorem 1 would apply. Nevertheless, using Valdivia's closed graph theorem [19] one gets:

Proposition 4.

(a) *A linear map between a metrizable Baire lcs and a metrizable lcs whose graph G admits a complete resolution of absolutely convex sets is continuous.*

(b) *A linear map with closed graph from a Baire lcs into a metrizable lcs with a complete resolution of absolutely convex sets is continuous.*

Proof. (a) Let $(U_n)_n$ be a decreasing basis of absolutely convex closed neighborhoods of zero in G. Set $A_\alpha := \bigcap_{k=1}^{\infty} n_k U_k$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and let $\{G_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a complete absolutely convex resolution in G. Then the sets $K_\alpha := A_\alpha \cap G_\alpha$ compose a complete and bounded absolutely convex resolution in G. Therefore the sets K_α are Banach discs, so G is a quasi-(LB)-space (in the sense of Valdivia [19]). Since every linear map from a Baire lcs into a quasi-(LB)-space with closed graph is continuous [19, Corollary 1.5], the map P^{-1} (from proof of Corollary 2) is continuous. Part (b) follows similarly. $\square$

The next example (motivated by Problem 3 and Proposition 4) also shows that Valdivia's result stating that a barrelled space covered by an increasing sequence $(A_n)_n$ of complete absolutely convex sets is complete, see [14, Proposition 8.2.28], fails if the above sets are not absolutely convex.

Example 5. There exist separable metrizable non-complete unordered Baire-like lcs covered by a sequence (resolution) of complete (complete and bounded) sets.

Proof. Drewnowski and Labuda [8] constructed an F-space λ_0 with a basis $(U_n)_n$ of balanced neighborhoods of zero closed in $\mathbb{R}^{\mathbb{N}}$ for which the weak topology $\sigma(\lambda_0, \lambda_0')$ is inherited from the topology of $\mathbb{R}^{\mathbb{N}}$: λ_0 is the space of all sequences $x = (\varrho_n)$ of real numbers such that $\|tx\| \to 0$ as $t \to 0$, where $\|x\| := \sup \|x\|_n$, $\|x\|_n := n^{-1} \sum_{j=1}^{n} \min(1, |\varrho_j|)$. Set

$$U_n := \{x \in \lambda_0 : \|x\| \leqslant n^{-1}\}$$

for $n \in \mathbb{N}$. Then λ_0 with the F-norm $\|\cdot\|$ is metrizable, complete and the sets U_n are closed in $\mathbb{R}^{\mathbb{N}}$, see [8]. For fixed $m \in \mathbb{N}$ one has $\lambda_0 = \bigcup\limits_{n=1}^{\infty} nU_m$ and each nU_m is $\sigma(\lambda_0, \lambda_0')$-complete but $\sigma(\lambda_0, \lambda_0')$ is metrizable non-complete. Since $\sigma(\lambda_0, \lambda_0')$ is metrizable, it is the finest locally convex topology on λ_0 weaker than the original topology ξ of λ_0, so $\sigma(\lambda_0, \lambda_0')$ is the Mackey envelope of ξ (in the sense of [5]). This yields that $\sigma(\lambda_0, \lambda_0')$ is unordered Baire-like, i.e. $\sigma(\lambda_0, \lambda_0')$ cannot be written as a sequence of nowhere dense absolutely convex sets, see [5, Theorem 1]. The other case goes similarly: For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha := \bigcap\limits_{k=1}^{\infty} n_k U_k$. Then $\{A_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is as required. $\square$

Corollary 6. *Let f be a linear functional on an F-space E. The following conditions are equivalent:*

(i) *E admits a resolution $\{A_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that f is continuous on each A_α.*

(ii) *f is continuous on E.*

(iii) *The kernel $N := \{x \in E \colon f(x) = 0\}$ has a complete resolution.*

Proof. (i) $\Rightarrow$ (ii): By [6, Proposition 4.1] the map f is continuous on the closure of each A_α, so we may assume that A_α are closed. We may also assume that E is separable. Let $\{K_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a compact resolution of E. Then $D_\alpha := A_\alpha \cap K_\alpha$ form a compact resolution in E and f is continuous on each D_α. Suppose that f is discontinuous and let H be its (dense) kernel. Clearly $\{H \cap D_\alpha \colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution on H. Also H admits a strictly weaker metrizable and complete vector topology. Indeed, if D is an algebraic complement to H in E, the restriction of the quotient map $q|_H \colon H \to E/D$ generates on H such a topology. Theorem 1 applies to reach a contradiction. A similar argument as in (i) $\Rightarrow$ (ii) applies to get (iii) $\Rightarrow$ (i). $\square$

Corollary 6 has been proved in [6, Corollary 4.5, Corollary 4.6] in a different way. Note also that the assumption on E to be metrizable and complete cannot be dropped even if A_α are compact, see [6, Remark 4.4 (ii)]. R. Pol proved, see [7], using Mycielski's theorem about independent functions, that an analytic vector subspace of a separable F-space has codimension either finite or $2^{\aleph_0}$. Theorem 1 yields easily the same fact but under (CH).

Corollary 7. *An algebraic direct sum of vector subspaces F and G of a separable F-space (E, ξ) is topological iff F and G admit a complete resolution. Consequently, if F is analytic then (under (CH)) the codimension of F is either finite or $2^{\aleph_0}$.*

Proof. We may assume that F and G have a compact resolution. Therefore F and G are analytic and E endowed with the direct sum topology $\tau := \xi|_F \oplus \xi|_G$

(stronger than ξ) is analytic. But Theorem 1 applies to get $\xi = \tau$, which proves the first part. Assume that F is analytic and that the dimension of G is countable. Since G is a countable union of finite dimensional subspaces G_n and each G_n (as metrizable complete and separable) is analytic, so G is analytic. The previous part applies and hence F is closed in E and the Baire category theorem yields that the codimension of F is finite. □

This shows that any algebraic complement of a closed non topologically complemented subspace in a separable F-space is "highly" non-closed, i.e. does not admit a complete resolution.

Problem 8. *Does there exist an F-space which is an algebraic direct sum of two non-closed subspaces admitting a complete resolution?* See also [6, Problem 4.21].

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