Peer Christian Kunstmann

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NAVIER-STOKES EQUATIONS ON UNBOUNDED DOMAINS WITH ROUGH INITIAL DATA

Peer Christian Kunstmann, Karlsruhe

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Abstract. We consider the Navier-Stokes equations in unbounded domains $\Omega \subseteq \mathbb{R}^n$ of uniform $C^{1,1}$-type. We construct mild solutions for initial values in certain extrapolation spaces associated to the Stokes operator on these domains. Here we rely on recent results due to Farwig, Kozono and Sohr, the fact that the Stokes operator has a bounded H^∞-calculus on such domains, and use a general form of Kato's method. We also obtain information on the corresponding pressure term.

Keywords: Navier-Stokes equations, mild solutions, Stokes operator, extrapolation spaces, H^∞-functional calculus, general unbounded domains, pressure term

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1. Introduction and main results

In this paper we consider the Navier-Stokes equation

$$\left\{\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad (t>0),\\ \nabla\cdot u = 0,\\ u(0,\cdot) = u_0,\\ u|_{\partial\Omega} = 0,\end{aligned}\right. \tag{1.1}$$

in unbounded domains $\Omega \subseteq \mathbb{R}^n$ of uniform $C^{1,1}$-type for "rough" initial values u_0. Here, $u(t,x) \in \mathbb{R}^n$ denotes the unknown velocity field and $p(t,x) \in \mathbb{R}$ denotes the unknown pressure at the point $x \in \Omega$ and at time $t \geqslant 0$, $f = f(t,x) \in \mathbb{R}^n$ denotes an external force, and we have no motion at the boundary ("no slip"). We shall be concerned with the construction of mild solutions for initial values u_0 that are "rough" in the sense that they belong to suitable extrapolation spaces for the Stokes operator. On $\mathbb{R}^n$, half spaces or domains with compact boundary this has been done, e.g., in [12], [2], [1], [10]. The results were based on the L^q-theory for the Stokes operator on these domains, apart from [12] which, on $\mathbb{R}^n$, used Morrey spaces instead (cf. also [10, Sect. 4.3]). Here and in the following, the letter q is used to denote the integrability exponent for Lebesgue-spaces and their sums and intersections. When nothing else is said, we always understand that $q \in (1,\infty)$. It is well known that there is no L^q-theory for the Stokes operator in general unbounded domains Ω, even if they are smooth.

This lack has been overcome by Farwig, Kozono and Sohr ([4], [6]) who, instead of working in $L^q(\Omega)^n$ and $L^q_\sigma(\Omega)$, studied Helmholtz decomposition and Stokes operator for the following function spaces

$$\widetilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) \cap L^2(\Omega), & q \in [2,\infty), \\ L^q(\Omega) + L^2(\Omega), & q \in (1,2), \end{cases}$$
$$\widetilde{L}^q_\sigma(\Omega) := \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & q \in [2,\infty), \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & q \in (1,2), \end{cases}$$

where as usual $L^q_\sigma(\Omega)$, the space of solenoidal vector fields in L^q, is the closure in $L^q(\Omega)^n$ of $C^\infty_{c,\sigma}(\Omega) := \{\varphi \in C^\infty_c(\Omega)^n \colon \nabla \cdot \varphi = 0\}$. We denote by $D^q(\Omega) := W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ the domain of the Dirichlet Laplace operator Δ_q in $L^q(\Omega)$ (for a bounded domain we refer to [8, Sect. 2], the general case can be found in, e.g., [13]), and we write, for $q \in (1,\infty)$,

$$\widetilde{D}^q(\Omega) := \begin{cases} D^q(\Omega) \cap D^2(\Omega), & q \geqslant 2, \\ D^q(\Omega) + D^2(\Omega), & q < 2, \end{cases}$$
$$\widetilde{W}^{1,q}_0(\Omega) := \begin{cases} W^{1,q}_0(\Omega) \cap W^{1,2}_0(\Omega), & q \geqslant 2, \\ W^{1,q}_0(\Omega) + W^{1,2}_0(\Omega), & q < 2. \end{cases}$$

We recall some facts (for further details we refer to Section 2 below). The corresponding Helmholtz projection $\widetilde{\mathbb{P}}_q \colon \widetilde{L}^q(\Omega)^n \to \widetilde{L}^q_\sigma(\Omega)$ is bounded (cf. [5]), and the Stokes operator $\widetilde{A}_q$ in $\widetilde{L}^q_\sigma(\Omega)$ is defined by $\widetilde{A}_q := -\widetilde{\mathbb{P}}_q\Delta_q$ on $D(\widetilde{A}_q) := \widetilde{D}^q(\Omega)^n \cap \widetilde{L}^q_\sigma(\Omega)$ for $1 < q < \infty$ (cf. [6]). It was shown in [6] that $-\widetilde{A}_q$ generates an analytic semigroup in $\widetilde{L}^q_\sigma(\Omega)$ and that $\widetilde{A}_q$ has maximal L^r-regularity in these spaces for $r \in (1,\infty)$ (we cite these results as Theorems 2.2 and 2.3 below).

In [14] we showed that $\varepsilon + \widetilde{A}_q$ has a bounded H^∞-calculus in $\widetilde{L}^q_\sigma(\Omega)$ for any $\varepsilon > 0$. The latter result allows to identify domains of fractional powers of $\varepsilon + \widetilde{A}_q$ (for details we refer again to Section 2 below). Here we already fix the notations $\widetilde{W}^{1,q}_{0,\sigma}(\Omega) := \widetilde{W}^{1,q}_0(\Omega)^n \cap \widetilde{L}^q_\sigma(\Omega)$ and $\widetilde{W}^{-1,q}_\sigma(\Omega) := (\widetilde{W}^{1,q'}_{0,\sigma}(\Omega))'$ for the dual space. We shall use $\langle\cdot,\cdot\rangle$ to denote extensions of the usual L^2-duality throughout. We also recall the following notation from [10]: for a Banach space Z and $\alpha \in \mathbb{R}$, $p \in [1, \infty]$, $\tau \in (0, \infty]$ we write

$$\begin{aligned} L^p_\alpha(0,\tau;Z) &:= \{f : (0,\tau) \to Z \text{ measurable: } t \mapsto t^\alpha f(t) \in L^p(0,\tau;Z)\},\\ \|f\|_{L^p_\alpha(0,\tau;Z)} &:= \|t \mapsto t^\alpha f(t)\|_{L^p(0,\tau;Z)}. \end{aligned}$$

Coming back to (1.1) we start with the case $f = 0$ and use $\nabla \cdot u = 0$ to rewrite $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$. If u has values in $\widetilde{L}^q_\sigma(\Omega)$ we can apply $\widetilde{\mathbb{P}}_q$ to the equation to obtain

$$\text{(1.2)} \qquad \begin{cases} u' + \widetilde{A}_q u = -\widetilde{\mathbb{P}}_q \nabla \cdot (u \otimes u) \quad (t > 0),\\ \qquad u(0) = u_0. \end{cases}$$

In [4], Farwig, Kozono, Sohr applied their results to the construction of so-called "suitable weak solutions" of Navier-Stokes equations on unbounded domains.

In this paper we shall obtain solutions u for initial values u_0 belonging to some extrapolation space of $\widetilde{L}^q_\sigma(\Omega)$ with respect to $\widetilde{A}_q$. More precisely we let, for $q \in (n, \infty)$, $p \in [2, \infty]$,

$$\widetilde{X}^q_{p,\sigma}(\Omega) := (\widetilde{W}^{-1,q}_\sigma(\Omega), \widetilde{L}^q_\sigma(\Omega))_{n/q,p},$$

where $(\cdot,\cdot)_{\theta,p}$ denotes real interpolation. Essentially due to the results in [14], the Stokes semigroup $(\widetilde{T}_q(t))$ on $\widetilde{L}^q_\sigma(\Omega)$ extends to an analytic semigroup $(\widetilde{T}_{q,p}(t))$ on $\widetilde{X}^q_{p,\sigma}(\Omega)$ with negative generator $\widetilde{A}_{q,p}$ (cf. Corollary 2.6 below).

We look for *mild solutions of* (1.2), i.e. for continuous functions $u\colon [0,\tau) \to \widetilde{X}^q_{p,\sigma}(\Omega)$ satisfying the fixed point equation

$$\text{(1.3)} \qquad u(t) = T(t)u_0 - \int_0^t T(t-s)\widetilde{\mathbb{P}}\nabla \cdot (u(s) \otimes u(s))\,\mathrm{d}s, \quad t \in [0,\tau),$$

for some $\tau > 0$ where $T(\cdot) = \widetilde{T}_{q,p}(\cdot)$. Our main result reads as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of uniform $C^{1,1}$-type and $q \in (n, \infty)$. Fix $\alpha \geqslant 0$ and $p \in (2, \infty)$ satisfying $\alpha + 1/p = \frac{1}{2} - \frac{1}{2}n/q$. For any initial value $u_0 \in \widetilde{X}^q_{p,\sigma}(\Omega)$ there exists $\tau > 0$ depending on the norm $\|u_0\|_{\widetilde{X}^q_{p,\sigma}(\Omega)}$ such that* (1.3) *has a unique solution*

$$u \in C([0,\tau), \widetilde{X}^q_{p,\sigma}(\Omega)) \cap L^p_\alpha(0,\tau;\widetilde{L}^q_\sigma(\Omega)).$$

Remark 1.2. For $p = \infty$, the domain $D(\widetilde{A}_{q,\infty})$ of $\widetilde{A}_{q,\infty}$ is not dense in $\widetilde{X}^q_{\infty,\sigma}(\Omega)$, and we denote the closure of $D(\widetilde{A}_{q,\infty})$ in $\widetilde{X}^q_{\infty,\sigma}(\Omega)$ by $\widetilde{X}^{q,\flat}_{\infty,\sigma}(\Omega)$. For $q \in (n, \infty)$ and $\alpha \geqslant 0$ with $\alpha = \frac{1}{2} - \frac{1}{2}n/q$ the assertion of Theorem 1.1 holds for $p = \infty$ if we take $u_0 \in \widetilde{X}^{q,\flat}_{\infty,\sigma}(\Omega)$.

For the proof we use Kato's method and proceed as in [10]. We have to check that the necessary estimates, which are well known in an L^q-setting on bounded or exterior domains, also persist to the present $\widetilde{L}^q$-setting on unbounded domains of uniform $C^{1,1}$-type. To this end we shall use the main results of [14] (cited as Theorem 2.4, Corollary 2.5 below) on boundedness of H^∞-calculi and imaginary powers of the Stokes operator on unbounded domains which allow to identify domains of fractional powers of the Stokes operator. In particular, the space $\widetilde{X}^q_{p,\sigma}(\Omega)$ in Theorem 1.1 can thus be seen to be an extrapolation space associated to the Stokes operator in $\widetilde{L}^q_\sigma(\Omega)$.

Remark 1.3. An inspection of the proof of Theorem 1.1 and the results of [10] allow to obtain, under the assumptions of Theorem 1.1, time local solutions with the same regularity also for external forces $f \in L^{p/2}_{2\alpha}(0,\infty;\widetilde{W}^{-1,q/2}(\Omega)^n)$. The existence time τ then also depends on the norm $\|f\|_{L^{p/2}_{2\alpha}(\widetilde{W}^{-1,q/2}(\Omega)^n)}$. Instead of (1.3) one has to solve

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\widetilde{\mathbb{P}}(f(s) - \nabla\cdot(u(s)\otimes u(s)))\,\mathrm{d}s, \quad t\in[0,\tau). \tag{1.4}$$

Other assumptions on f are also possible when we split the second term on the right hand side of (1.4) and treat f and $\nabla\cdot(u\otimes u)$ separately. For $q\in(n,2n)$ it is thus possible to take $f\in L^r_\beta(0,\infty;\widetilde{L}^{q/2}(\Omega)^n)$ where $\beta\geqslant 0$ and $r\in(1,\infty]$ satisfy $\beta + 1/r = \frac{3}{2} - n/q$. We do not go into details here and refer to [10].

Remark 1.4. One might like to identify the space $\widetilde{X}^q_{p,\sigma}(\Omega)$, which is an extrapolation space for $\widetilde{L}^q_\sigma(\Omega)$, as a subspace of the extrapolation space $\widetilde{X}^q_p(\Omega)^n := (\widetilde{W}^{-1,q}(\Omega),\widetilde{L}^q(\Omega))^n_{n/q,p}$, in other words, one would like to have a Helmholtz decomposition of $\widetilde{X}^q_p(\Omega)^n$ as a direct sum of $\widetilde{X}^q_{p,\sigma}(\Omega)$ (representing divergence-free vector fields in $\widetilde{X}^q_p(\Omega)^n$) and the space of gradients in $\widetilde{X}^q_p(\Omega)^n$. This problem shall be studied in greater generality in another paper.

The next result studies the limit case $q = n$ (for $n\geqslant 3$) in the situation of Theorem 1.1. Observe that $q = n$ leads to $\alpha = 0$ and $p = \infty$. For $\Omega = \mathbb{R}^n$ and the L^q-scale in place of the $\widetilde{L}^q$-scale, the corresponding assertion has been proved by Y. Meyer ([17]). Here we use the notation

$$\widetilde{L}^n_{\infty,\sigma}(\Omega) := (\widetilde{L}^{q_0}_\sigma(\Omega),\widetilde{L}^{q_1}_\sigma(\Omega))_{\theta,\infty} \tag{1.5}$$

where $2 < q_0 < n < q_1 < \infty$ and $\theta\in(0,1)$ are such that $1/n = (1-\theta)/q_0 + \theta/q_1$. Again, we denote the closure of the domain of the Stokes operator $\widetilde{A}_{(n,\infty)}$ in $\widetilde{L}^n_{\infty,\sigma}(\Omega)$ by $\widetilde{L}^{n,\flat}_{\infty,\sigma}(\Omega)$.

Theorem 1.5. *Let $n \geqslant 3$ and $\Omega \subseteq \mathbb{R}^n$ be a domain of uniform $C^{1,1}$-type. For any initial value $u_0 \in \widetilde{L}^{n,\flat}_{\infty,\sigma}(\Omega)$ there exists $\tau > 0$ depending on the norm $\|u_0\|_{\widetilde{L}^n_{\infty,\sigma}(\Omega)}$ such that* (1.3) *has a unique solution*

$$u \in C([0,\tau), \widetilde{L}^n_{\infty,\sigma}(\Omega)).$$

The proof relies on a type of maximal L^∞- regularity and we shall check the sufficient condition of [10, Lem. 3.11].

Finally, we construct, in the situation of Theorem 1.1, solutions to the full equation (1.1) by recovering the corresponding pressure term.

Theorem 1.6. *Let Ω be an unbounded domain of uniform $C^{1,1}$-type and $q \in (n,\infty)$. For any initial value $u_0 \in \widetilde{X}^q_{p,\sigma}(\Omega)$ and any external force $f \in L^{p/2}_{2\alpha}(0,\infty;\widetilde{W}^{-1,q/2}(\Omega)^n)$ there exists a $\nabla p = \nabla p_1 + \partial_t \nabla \hat{p_2}$ satisfying $\nabla p_1 \in L^p_{\alpha+1}(0,\tau;\widetilde{L}^q(\Omega)^n)$ and $\nabla \hat{p_2} \in L^{p/2}_{2\alpha}(0,\tau;\widetilde{W}^{-1,q/2}(\Omega)^n)$ such that the local solution u of Theorem* 1.1 *and Remark* 1.3 *and ∇p satisfy* (1.1) *on* $(0,\tau)$.

The approach in the proof is inspired by [19, IV. Sec. 2.6]. We decompose $u = u_1 + u_2$ according to (1.4), and exploit the properties $u_1 \in L^p_{\alpha+1}(0,\tau;D(\widetilde{A}_q))$ and $u_2 \in L^{p/2}_{2\alpha}(0,\tau;\widetilde{W}^{1,q/2}_{0,\sigma}(\Omega))$. This shows how to give sense to the conditions $\nabla \cdot u = 0$ and $u|_{\partial\Omega} = 0$ in (1.1). For the interpretation of Δu we refer to Section 5.

The paper is organized as follows. In Section 2 we recall results on the Stokes operator from [6], [14] and we introduce certain interpolation and extrapolation spaces associated to the Stokes operator and the Dirichlet Laplacian. In Section 3 we prove Theorem 1.1 using the general approach presented in [10]. In Section 4 we prove Theorem 1.5. Finally, Theorem 1.6 is proved in Section 5.

In this paper, C denotes a generic constant, and dependence on parameters τ, ε, etc. is denoted by C_τ, C_ε, etc.

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2. The Stokes operator in unbounded domains

First we recall the precise definition of the class of domains Ω we shall work on (cf. [6, Def. 1.1]).

Definition 2.1. A domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is called of uniform $C^{1,1}$-type if there are constants $\alpha, \beta, K > 0$ such that, for each $x_0 \in \partial\Omega$, there is a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$ and a $C^{1,1}$-function h, defined on $\{y' \colon |y'| \leqslant \alpha\}$ and with $\|h\|_{C^{1,1}} \leqslant K$, such that, for the neighborhood

$$U_{\alpha,\beta,h}(x_0) = \{y = (y', y_n) \in \mathbb{R}^n \colon |y_n - h(y')| < \beta, |y'| < \alpha\}$$

of x_0 we have $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', h(y')) \colon |y'| < \alpha\}$ and

$$U_{\alpha,\beta,h}(x_0) \cap \Omega = \{(y', y_n) \colon h(y') - \beta < y_n < h(y'), |y'| < \alpha\}.$$

We refer to the definition of the spaces $\widetilde{L}^q(\Omega)$ and $\widetilde{L}^q_\sigma(\Omega)$ in the introduction. Denoting by $G^q(\Omega) := \{\nabla p \in L^q(\Omega)^n \colon p \in L^q_{\mathrm{loc}}(\Omega)\}$ the space of gradients in $L^q(\Omega)^n$, the space of gradients in $\widetilde{L}^q(\Omega)^n$ is defined by

$$\widetilde{G}^q(\Omega) := \begin{cases} G^q(\Omega) \cap G^2(\Omega), & q \in [2, \infty), \\ G^q(\Omega) + G^2(\Omega), & q \in (1, 2). \end{cases}$$

It has been shown in [5] that, for unbounded Ω of uniform $C^{1,1}$-type, the Helmholtz decomposition $\widetilde{L}^q(\Omega)^n = \widetilde{L}^q_\sigma(\Omega) \oplus \widetilde{G}^q(\Omega)$ is valid for all $q \in (1, \infty)$. The corresponding Helmholtz projection $\widetilde{\mathbb{P}}_q \colon \widetilde{L}^q(\Omega)^n \to \widetilde{L}^q_\sigma(\Omega)$ with kernel $\widetilde{G}^q(\Omega)$ is bounded, $C^\infty_{c,\sigma}(\Omega)$ is dense in $\widetilde{L}^q_\sigma(\Omega)$, and the duality relations

$$(\widetilde{L}^q_\sigma(\Omega))' = \widetilde{L}^{q'}_\sigma(\Omega), \qquad (\widetilde{\mathbb{P}}_q)' = \widetilde{\mathbb{P}}_{q'} \tag{2.1}$$

hold, where $q \in (1, \infty)$ and $1/q + 1/q' = 1$.

We refer to the introduction for the definition of the Stokes operator in $\widetilde{L}^q_\sigma(\Omega)$ and cite the following two results on properties of the Stokes operator in unbounded domains of uniform $C^{1,1}$-type.

Theorem 2.2 ([6] Thm. 1.3). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of uniform $C^{1,1}$-type. For $q \in (1, \infty)$ and $\varepsilon > 0$, the Stokes operator $\widetilde{A}_q$ is the negative generator of an analytic semigroup $(\widetilde{T}_q(t))_{t \geqslant 0}$ in $\widetilde{L}^q_\sigma(\Omega)$ satisfying*

$$\|\widetilde{T}_q(t) f\|_{\widetilde{L}^q} \leqslant M \mathrm{e}^{\varepsilon t} \|f\|_{\widetilde{L}^q}, \quad f \in \widetilde{L}^q_\sigma(\Omega),\ t > 0,$$

where $M = M(\varepsilon, q, \alpha, \beta, K)$ and α, β, K are the constants from Definition 2.1. Moreover, the operator $\varepsilon + \widetilde{A}_q$ is sectorial of type 0, and the duality relation $(\widetilde{A}_q)' = \widetilde{A}_{q'}$ holds.

Theorem 2.3 ([6] Thm. 1.4). *If $\Omega \subseteq \mathbb{R}^n$ is a domain of uniform $C^{1,1}$-type then, for $q, r \in (1,\infty)$ the operator $\widetilde{A}_q$ has maximal L^r-regularity on finite time intervals in $\widetilde{L}^q_\sigma(\Omega)$: for $T > 0$, $f \in L^r(0,T;\widetilde{L}^q_\sigma(\Omega)$ the mild solution to*

$$u'(t) + \widetilde{A}_q u(t) = f(t), \quad t \in [0,T], \qquad u(0) = 0,$$

satisfies

$$\|u'(t)\|_{L^r(0,T;\widetilde{L}^q_\sigma(\Omega))} + \|\widetilde{A}_q u(t)\|_{L^r(0,T;\widetilde{L}^q_\sigma(\Omega))} \leqslant C\|f\|_{L^r(0,T;\widetilde{L}^q_\sigma(\Omega))},$$

where $C = C(q,r,T,\alpha,\beta,K)$ and α, β, K are the constants from Definition 2.1.

As already mentioned, another technical ingredient is the main result from [14] on H^∞-calculi for the Stokes operator on unbounded domains. For the notion of a bounded H^∞-calculus, for further properties and for the relevance of bounded H^∞-calculi we refer to [15]. For bounded domains of $C^{1,1}$-type the corresponding result with $\varepsilon = 0$ has been proved in [11, Thm. 9.17].

Theorem 2.4 ([14] Thm. 1.1). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of uniform $C^{1,1}$-type. For $q \in (1,\infty)$ and any $\varepsilon > 0$, the operator $\varepsilon + \widetilde{A}_q$ has a bounded H^∞-calculus in $\widetilde{L}^q_\sigma(\Omega)$. In particular, $\varepsilon + \widetilde{A}_q$ has bounded imaginary powers in $\widetilde{L}^q_\sigma(\Omega)$.*

Since $\varepsilon + \widetilde{A}_q$ has bounded imaginary powers, the fractional domain spaces $D((\varepsilon + \widetilde{A}_q)^\theta)$, $\theta \in (0,1)$, can be obtained as complex interpolation spaces $[\widetilde{L}^q_\sigma(\Omega), D(\widetilde{A}_q)]_\theta$ (cf. [20]). It is this property that we shall exploit. The following consequence has been shown in [14].

Corollary 2.5 ([14] Cor. 1.2). *If $\Omega \subseteq \mathbb{R}^n$ is an unbounded domain of uniform $C^{1,1}$-type then $D((\varepsilon + \widetilde{A}_q)^{1/2}) = \widetilde{W}^{1,q}_{0,\sigma}(\Omega)$ for $q \in (1,\infty)$.*

We recall that $\widetilde{W}^{1,q}_{0,\sigma}(\Omega) = \widetilde{W}^{1,q}_0(\Omega)^n \cap \widetilde{L}^q_\sigma(\Omega)$. By Corollary 2.5, we can write

$$\widetilde{W}^{-1,q}_\sigma(\Omega) = (\widetilde{W}^{1,q'}_{0,\sigma}(\Omega))' = (D((\varepsilon + \widetilde{A}_{q'})^{1/2}))'.$$

For the usual $(\widetilde{L}^q, \widetilde{L}^{q'})$-duality, the dual space of $D((\varepsilon + \widetilde{A}_{q'})^{1/2})$ can easily be identified with

$$(\widetilde{L}^q_\sigma(\Omega), \|(\varepsilon + \widetilde{A}_q)^{-1/2} \cdot \|)^\sim,$$

if we recall $(\widetilde{A}_{q'})' = \widetilde{A}_q$ from Theorem 2.2 above (here $^\sim$ denotes completion). We thus obtain

Corollary 2.6. *For $q \in (1,\infty)$ we have*

$$\widetilde{W}_\sigma^{-1,q}(\Omega) = (\widetilde{L}^q_\sigma(\Omega), \|(\varepsilon + \widetilde{A}_q)^{-1/2}\cdot\|)^\sim.$$

The Stokes semigroup $\widetilde{T}_q(\cdot)$ on $\widetilde{L}^q_\sigma(\Omega)$ has an extension to an analytic semigroup $\widetilde{T}_q^{(-1)}(\cdot)$ on $\widetilde{W}_\sigma^{-1,q}(\Omega)$ satisfying

$$\|\widetilde{T}_q^{(-1)}(t)\| \leqslant C_\varepsilon \mathrm{e}^{\varepsilon t}, \quad t \geqslant 0,\ \varepsilon > 0.$$

Its restriction $\widetilde{T}_{q,p}(\cdot)$ to $\widetilde{X}^q_{p,\sigma}(\Omega)$ is an analytic semigroup on $\widetilde{X}^q_{p,\sigma}(\Omega)$ satisfying

$$\|\widetilde{T}_{q,p}(t)\| \leqslant C_\varepsilon \mathrm{e}^{\varepsilon t}, \quad t \geqslant 0,\ \varepsilon > 0.$$

For $p < \infty$, $\widetilde{X}^q_{p,\sigma}(\Omega)$ is reflexive and the negative generator $\widetilde{A}_{q,p}$ of $\widetilde{T}_{q,p}(\cdot)$ is densely defined.

P r o o f. The first assertion is clear from the arguments above. One can extend $J := (\varepsilon + \widetilde{A}_q)^{-1/2}$ to an isomorphism $\widetilde{J}\colon \widetilde{W}_\sigma^{-1,q}(\Omega) \to \widetilde{L}^q_\sigma(\Omega)$. Then $\widetilde{T}_q^{(-1)}(t) = \widetilde{J}^{-1}\widetilde{T}_q(t)\widetilde{J}$ defines the desired extension. For the remainig assertions we use real interpolation. □

We note a last consequence of Theorem 2.4, namely a Sobolev type embedding result.

Corollary 2.7. *Suppose that $\Omega \subset \mathbb{R}^n$ is an unbounded domain of uniform $C^{1,1}$-type. Let $\varepsilon > 0$, $q \in (1,\infty)$, $s \in (0,1)$ with $s < n/q$, and $r \in (1,\infty)$ such that $r^{-1} = q^{-1} - 2s/n$. Then*

$$D((\varepsilon + \widetilde{A}_q)^s) \hookrightarrow \widetilde{L}^r_\sigma(\Omega).$$

P r o o f. We start with an argument that has been used in the proof of [14, Cor. 1.2]. Since we have bounded imaginary powers, [7, Lem. 6] gives

$$D((\varepsilon + \widetilde{A}_q)^s) = D((\varepsilon - \widetilde{\Delta}_q)^s) \cap \widetilde{L}^q_\sigma(\Omega).$$

By consistency of the resolvents of the Laplacian we have

$$D((\varepsilon - \widetilde{\Delta}_q)^s) = \begin{cases} D((\varepsilon - \Delta_q)^s) + D((\varepsilon - \Delta_2)^s), & q \in (1,2), \\ D((\varepsilon - \Delta_q)^s) \cap D((\varepsilon - \Delta_2)^s), & q \in [2,\infty). \end{cases}$$

Since Ω is of uniform type $C^{1,1}$ and $s \in (0,1)$ we have the usual Sobolev embedding $D((\varepsilon - \Delta_q)^s) \hookrightarrow L^r(\Omega)$ (use a partition of unity similar to [5, pp. 242/243] and uniformity of the constants for the "parts"). We also have $D((\varepsilon-\Delta_2)^s)\hookrightarrow L^2(\Omega)$. From this the assertion follows in case that both $q,r\leqslant 2$ or that both $q,r\geqslant 2$.

If $q<2$ and $r>2$ we have to show in addition that

$$D((\varepsilon-\Delta_q)^s)\hookrightarrow L^2(\Omega)\quad\text{and}\quad D((\varepsilon-\Delta_2)^s)\hookrightarrow L^r(\Omega).$$

These embeddings hold by usual Sobolev embedding since $q^{-1}-2s/n=r^{-1}\leqslant 1/2$ and $1/2-2s/n\leqslant q^{-1}-2s/n=r^{-1}$. Now the assertion follows also for $q<2$ and $r>2$. □

3. Mild solutions and proof of theorem 1.1

We use the general approach from [10]. In an L^q-setting with $q\in(n,\infty)$ where $n\geqslant 2$ is the dimension, this approach is based on function spaces with exponents q and $\frac{1}{2}q$. Due to the structure of the spaces $\widetilde{L}^q(\Omega)$, which is different depending on whether $q\geqslant 2$ or $q<2$, we sometimes have to distinguish the cases $q\geqslant 4$ and $q\in(n,4)$. The latter case, of course, does only occur for $n\in\{2,3\}$.

First we fix the domain $\Omega\subset\mathbb{R}^n$ of uniform $C^{1,1}$-type and $q\in(n,\infty)$. For our approach via the results in [10] we define the following function spaces:

$$Z:=\widetilde{L}^q(\Omega)^n,\ W:=\widetilde{W}_\sigma^{-1,\frac{q}{2}}(\Omega),\ \text{ and } X:=\widetilde{X}^q_{p,\sigma}(\Omega),$$

where α and p are as in the theorem, i.e. $\alpha\geqslant 0$, $p\in(2,\infty]$ with $\alpha+1/p=\frac{1}{2}-\frac{1}{2}n/q$. The next proposition establishes the properties we have to check for the nonlinearity.

Proposition 3.1. (a) *The map $Z\times Z\to\widetilde{W}^{-1,\frac{q}{2}}(\Omega)^n,(u,v)\mapsto\nabla\cdot(u\otimes v)$ is well-defined, bilinear and continuous.*

(b) *The Helmholtz projection $\widetilde{\mathbb{P}}_{q/2}$ has a continuous linear extension $\widetilde{P}_{q/2}\colon\widetilde{W}^{-1,\frac{q}{2}}(\Omega)^n\to W$ given by restriction*

$$\widetilde{P}_{q/2}\varphi=\varphi|_{\widetilde{W}_{0,\sigma}^{1,(q/2)'}(\Omega)},\quad\varphi\in(\widetilde{W}^{-1,\frac{q}{2}}(\Omega)^n)'. \tag{3.1}$$

(c) *The map $Z\times Z\to W,(u,v)\mapsto-\widetilde{P}_{q/2}\nabla\cdot(u\otimes v)$ is well-defined, bilinear and continuous.*

Proof. (a): As mentioned above we distinguish two cases.

Case $q>n$ **and** $q\geqslant 4$. For $u,v\in Z$ we then have $u,v\in L^q\cap L^2$ which yields $u\otimes v\in L^{q/2}\cap L^1$ by Hölder's inequality. By $q\geqslant 4$ we have $L^{q/2}\cap L^1\subset L^2$, and thus we obtain that

$$\begin{aligned}&(u,v)\mapsto u\otimes v \text{ is bilinear and continuous}\\ &Z\times Z\to\widetilde{L}^{q/2}(\Omega)^{n\times n}=L^{\frac{q}{2}}(\Omega)^{n\times n}\cap L^2(\Omega)^{n\times n}.\end{aligned}$$

By $q \geqslant 4$ we have $(\frac{1}{2}q)' \leqslant 2$ and $\widetilde{W}_0^{1,(q/2)'}(\Omega) = W_0^{1,(q/2)'}(\Omega) + W_0^{1,2}(\Omega)$ which leads to $\widetilde{W}^{-1,q/2}(\Omega)^n = W^{-1,q/2}(\Omega)^n \cap W^{-1,2}(\Omega)^n$. We conclude that

$$Z \times Z \to \widetilde{W}^{-1,q/2}(\Omega)^n, \quad (u,v) \mapsto \nabla \cdot (u \otimes v)$$

is bilinear and continuous.

Case $n \in \{2,3\}$ **and** $q \in (n,4)$. For $u, v \in Z = L^q \cap L^2$ we have $u \otimes v \in L^{q/2}$ by Hölder. Since $\frac{1}{2}q < 2$, we conclude that

$$(u,v) \mapsto u \otimes v \text{ is continuous } Z \times Z \to \widetilde{L}^{q/2}(\Omega)^{n\times n} = L^{q/2}(\Omega)^{n\times n} + L^2(\Omega)^{n\times n}.$$

Hence $\nabla \cdot (u \otimes v)$ is an element of

$$\begin{aligned} W^{-1,q/2}(\Omega)^n + W^{-1,2}(\Omega)^n &= (W_0^{1,(q/2)'}(\Omega)^n \cap W_0^{1,2}(\Omega)^n)' \\ &= (\widetilde{W}_0^{1,(q/2)'}(\Omega)^n)' = \widetilde{W}^{-1,q/2}(\Omega)^n, \end{aligned}$$

and $(u,v) \mapsto \nabla \cdot (u \otimes v)$ is bilinear and continuous $Z \times Z \to \widetilde{W}^{-1,q/2}(\Omega)^n$.

(b): Since (3.1) defines a continuous linear map, we only have to check for consistency with $\widetilde{\mathbb{P}}_{q/2}$. For a $\varphi \in \widetilde{W}^{-1,q/2}(\Omega)^n$ which coincides with $\langle f, \cdot \rangle$, where $f \in \widetilde{L}^{q/2}(\Omega)^n$, and for any $v \in \widetilde{W}_{0,\sigma}^{1,(q/2)'}(\Omega)$ we have

$$\varphi(v) = \langle f, v \rangle = \langle f, \widetilde{\mathbb{P}}_{(q/2)'} v \rangle = \langle \widetilde{\mathbb{P}}_{q/2} f, v \rangle,$$

and hence $\widetilde{P}_{q/2}\varphi = \langle \widetilde{\mathbb{P}}_{q/2} f, \cdot \rangle$ as desired.

(c) follows from (a) and (b). □

The next proposition contains the properties we have to check for the Stokes semigroup.

Proposition 3.2. (a) *The Stokes semigroup* $\widetilde{T}_{q/2}^{(-1)}(t)$ *acts continuously* $W \to Z$ *with*

$$\|\widetilde{T}_{q/2}^{(-1)}(t)\|_{W\to Z} \leqslant C_\tau t^{-(1/2+n/2q)}, \quad t \in [0,\tau],$$

for any $\tau \in (0,\infty)$.

(b) *The Stokes semigroup acts continuously* $X \to Z$ *and* $W \to X$ *with*

$$\|\widetilde{T}_{q,p}(t)\|_{X\to Z} \leqslant C_\tau t^{-1/2+n/2q}, \quad \|\widetilde{T}_{q/2}^{(-1)}(t)\|_{W\to X} \leqslant C_\tau t^{-n/q}, \quad t \in [0,\tau],$$

for any $\tau \in (0,\infty)$.

P r o o f. (a): More generally, we shall show that, for $1 < r < s < \infty$ and $\tau \in (0,\infty)$,

$$\|\widetilde{T}_r(t)\|_{\widetilde{L}^r_\sigma(\Omega)\to\widetilde{L}^s_\sigma(\Omega)} \leqslant C_{r,s,\tau} t^{-\frac{1}{2}n(1/r-1/s)}, \quad t \in [0,\tau].$$

By duality, interpolation (cf. [14, Lem. 4.1], where [20, 1.2.4] is used), and the semigroup property it suffices to show this for $r = 2$ and some $s > 2$.

For $n \geqslant 3$ the usual Sobolev embedding $W^{1,2}_{0,\sigma}(\Omega) \subseteq W^{1,2}_0(\Omega)^n \hookrightarrow L^{2n/(n-2)}(\Omega)^n$ leads to $\widetilde{W}^{1,2}_{0,\sigma}(\Omega) \hookrightarrow \widetilde{L}^{2n/(n-2)}_\sigma(\Omega)$, and this in turn yields

$$\|\widetilde{T}_2(t)\|_{\widetilde{L}^2_\sigma(\Omega)\to\widetilde{L}^{2n/(n-2)}_\sigma(\Omega)} \leqslant C_\tau t^{-1/2}, \quad t \in [0,\tau];$$

notice that $\frac{1}{2}n\big(\frac{1}{2} - \frac{1}{2}(n-2)/n\big) = \frac{1}{2}$.

For $n = 2$ we use the inequality $\|u\|_{L^4} \leqslant C\|u\|^{1/2}_{W^{1,2}}\|u\|^{1/2}_{L^2}$ (cf. [16, p. 70]) that leads to

$$\|\widetilde{T}_2(t)\|_{\widetilde{L}^2_\sigma(\Omega)\to\widetilde{L}^4_\sigma(\Omega)} \leqslant C_\tau t^{-1/4}, \quad t \in [0,\tau];$$

observe here that $\frac{1}{2}n(\frac{1}{2} - \frac{1}{4}) = \frac{1}{4}$ since $n = 2$.

(b) The first assertion follows by interpolation from the observation that the Stokes semigroup acting $\widetilde{W}^{-1,q}_\sigma(\Omega) \to \widetilde{L}^q_\sigma(\Omega)$ has norm $\leqslant C_\tau t^{-1/2}$, $t \in [0,\tau]$, for any finite $\tau > 0$ (here we use Corollary 2.6).

Part (a) and Corollary 2.6 imply that the Stokes semigroup acts continuously $W \to \widetilde{W}^{-1,q}_\sigma(\Omega)$ with norm $\leqslant C_\tau t^{-n/2q}$, $t \in [0,\tau]$, for any finite $\tau > 0$. Now interpolation yields the second assertion. □

Let $\gamma = \frac{1}{2} + n/2q$. Then $\gamma \in (\frac{1}{2},1)$ and $\alpha + 1/p = 1 - \gamma$. By reiteration it is clear that the space X satisfies $(X_{-1}, X_1)_{1/2,p} = X$ where X_1 denotes the domain of the Stokes operator in X, equipped with the graph norm, and X_{-1} denotes the first extrapolation space of X with respect to the Stokes operator, i.e.

$$X_{-1} = (X, \|(1+A)^{-1}\cdot\|_X)^\sim,$$

we refer to [10] for more details. Now the results of [10, Sect. 3] prove the assertion of Theorem 1.1.

4. Maximal L^∞-regularity and proof of theorem 1.5

In this section we prove Theorem 1.5. First of all we extend the definition (1.5) to all exponents $q \in (1, \infty)$ by letting

$$\widetilde{L}^q_\infty(\Omega) := (\widetilde{L}^{q_0}(\Omega), \widetilde{L}^{q_1}(\Omega))_{\theta,\infty} \tag{4.1}$$

where $1 < q_0 < q < q_1 < \infty$, $\theta \in (0,1)$ with $1/q = (1-\theta)/q_0 + \theta/q_1$ and $q_0, q_1 \in (2, \infty)$ if $q > 2$, $q_0, q_1 \in (1, 2)$ if $q < 2$ and $q_0 < 2 < q_1$ if $q = 2$. By reiteration this definition does not depend on the particular choice of q_0 and q_1 since $(\widetilde{L}^q(\Omega))_{q \geqslant 2}$ and $(\widetilde{L}^q(\Omega))_{q\in(1,2]}$ are complex interpolation scales (cf. [14, Lem. 4.1]). By continuity of the Helmholtz projection in the scale $(\widetilde{L}^q(\Omega))$ we obtain boundedness of the Helmholtz projection $\widetilde{\mathbb{P}}_{q,\infty}$ in $\widetilde{L}^q_\infty(\Omega)^n$. By [20, 1.2.4] we see that $\widetilde{L}^q_{\infty,\sigma}(\Omega) := \widetilde{\mathbb{P}}_{q,\infty}\widetilde{L}^n_\infty(\Omega)^n$ satisfies

$$\widetilde{L}^q_{\infty,\sigma}(\Omega) = (\widetilde{L}^{q_0}_\sigma(\Omega), \widetilde{L}^{q_1}_\sigma(\Omega))_{\theta,\infty}$$

where q_0, q_1, θ are as in (4.1). In particular, we have consistency with the definition in (1.5). We also use real interpolation $(\cdot,\cdot)_{\theta,\infty}$ and the same q_0, q_1 to define

$$\widetilde{W}^{1,q}_{0,\infty}(\Omega) := (\widetilde{W}^{1,q_0}_0(\Omega), \widetilde{W}^{1,q_1}_0(\Omega))_{\theta,\infty},$$

and similarly $\widetilde{L}^q_{\infty,\sigma}(\Omega)$, $\widetilde{W}^{1,q}_{0,\infty,\sigma}(\Omega)$, $\widetilde{W}^{-1,q}_\infty(\Omega)$, $\widetilde{W}^{-1,q}_{\infty,\sigma}(\Omega)$. In the following we shall abbreviate $X := \widetilde{L}^n_{\infty,\sigma}(\Omega)$, $X^\flat := \widetilde{L}^{n,\flat}_{\infty,\sigma}(\Omega)$, $Z := \widetilde{L}^n_\infty(\Omega)^n$, and $W := \widetilde{W}^{-1,n/2}_{\infty,\sigma}(\Omega)$.

Lemma 4.1. (a) *For $q \in (1,\infty)$, the Helmholtz projection $\widetilde{\mathbb{P}}_{q,\infty}$ has a continuous extension $\widetilde{P}_{q,\infty}\colon \widetilde{W}^{-1,q}_\infty(\Omega)^n \to \widetilde{W}^{-1,q}_{\infty,\sigma}(\Omega)$ which acts by restriction.*

(b) *The map $(u,v) \mapsto \widetilde{P}_{n/2,\infty}\nabla\cdot(u\otimes v)$ is bilinear and continuous $Z \times Z \to W$.*

P r o o f. (a) follows from Proposition 3.1 (b) by real interpolation. For the proof of (b) we use real interpolation for the assertion of Proposition 3.1 (a) and combine with the assertion on the Helmholtz projection in (a). □

In the following lemma we understand in the assertions (a) and (b) that $q \in (1,\infty)$ and that q_0, q_1, θ are as in (4.1).

Lemma 4.2. (a) *The Stokes semigroup acts as an analytic semigroup* $\widetilde{T}_{(q,\infty)}(\cdot)$ *in* $\widetilde{L}^q_{\infty,\sigma}(\Omega)$. *Denoting by* $\widetilde{A}_{(q,\infty)}$ *its negative generator,* $\varepsilon + \widetilde{A}_{(q,\infty)}$ *has a bounded* H^∞*-calculus for each* $\varepsilon > 0$.

(b) *For any* $s \in (0,1]$ *and* $\varepsilon > 0$ *we have*

$$D((\varepsilon + \widetilde{A}_{(q,\infty)})^s) = (D((\varepsilon + \widetilde{A}_{q_0})^s), D((\varepsilon + \widetilde{A}_{q_1})^s))_{\theta,\infty}.$$

Moreover,

$$\widetilde{W}^{-1,q}_{\infty,\sigma}(\Omega) = (\widetilde{L}^q_{\infty,\sigma}(\Omega), \|(\varepsilon + \widetilde{A}_{(q,\infty)})^{-1/2}\cdot\|)^{\sim}.$$

(c) *For any* $\tau > 0$ *the convolution operator*

$$\widetilde{T}_{(n/2,\infty)}(\cdot)*\colon\ L^\infty(0,\tau;W) \to L^\infty(0,\tau;X)$$

is bounded.

P r o o f. (a) is obtained by real interpolation from the corresponding properties in the $\widetilde{L}^q_\sigma$-scale.

(b) follows by real interpolation, since $(\varepsilon + \widetilde{A})^{-s}$ acts as an isomorphism $\widetilde{L}^{q_j}_\sigma \to D((\varepsilon + \widetilde{A}_{q_j})^s)$, $j = 0, 1$. The same argument applies to the negative Sobolev type space.

(c): By [10, Lem. 3.11] it is sufficient to check the inclusion $(W, W_2)_{1/2,\infty} \hookrightarrow X$ where W_2 denotes $D((\widetilde{A}^{(-1)}_{(n/2,\infty)})^2)$ and $\widetilde{A}^{(-1)}_{(n/2,\infty)}$ is the extrapolated version of the Stokes operator $\widetilde{A}_{(n/2,\infty)}$ on W (recall (b) for $q = n/2$). By reiteration, it is sufficient to check

$$(D((\varepsilon + \widetilde{A}_{(n/2,\infty)})^{(1-\delta)/2}), D((\varepsilon + \widetilde{A}_{(n/2,\infty)})^{(1+\delta)/2}))_{1/2,\infty} \hookrightarrow \widetilde{L}^n_{\infty,\sigma}(\Omega) \tag{4.2}$$

for some small $\delta \in (0,1)$. We use (b) for $s = s_\pm = (1 \pm \delta)/2$ and fixed q_0, q_1 where we arrange for $\theta = 1/2$. By Corollary 2.7 we have

$$D((\varepsilon + \widetilde{A}_{q_j})^s) \hookrightarrow \widetilde{L}^{r_j}_\sigma(\Omega),\ j = 0, 1, \quad \text{where } r_j^{-1} = q_j^{-1} - 2s/n.$$

By real interpolation and (b) we obtain that

$$D((\varepsilon + \widetilde{A}_{(n/2,\infty)})^{s_\pm}) \hookrightarrow \widetilde{L}^{r_\pm}_{\infty,\sigma}(\Omega) \quad \text{where } r_\pm^{-1} = (2 - 2s_\pm)/n.$$

By reiteration we then conclude that (4.2) holds. □

P r o o f of Theorem 1.5. Lemma 4.2 yields that

$$L^\infty(0,\tau;X) \times L^\infty(0,\tau;X) \to L^\infty(0,\tau;X),$$
$$(u,v) \mapsto \widetilde{T}_{(n/2,\infty)} * \widetilde{P}_{(n/2,\infty)} \nabla\cdot(u \otimes v)$$

is continuous. It is well known that the proof can then be finished by a fixed point argument (cf., e.g. [2, Lem. 1.2.6]). □

5. The pressure term and proof of theorem 1.6

Our starting point is (1.3). For a given initial value $u_0 \in \widetilde{X}^{q,\flat}_{p,\sigma}(\Omega)$ we decompose $u(t) = u_1(t) + u_2(t)$, where

$$u_1(t) = \widetilde{T}_{q,p}(t)u_0, \quad u_2(t) = \int_0^t \widetilde{T}^{(-1)}_{q/2}(t-s)\widetilde{P}_{q/2}(f - \nabla\cdot(u(s)\otimes u(s)))\,\mathrm{d}s,$$

and we look on u_1, u_2 separately.

We start with u_1. Since the Stokes semigroup is analytic in $\widetilde{X}^q_{p,\sigma}(\Omega)$ and in $\widetilde{L}^q_\sigma(\Omega)$, we obtain

$$t \mapsto t\partial_t u_1(t) = tAu_1(t) \in C([0,\tau), \widetilde{X}^q_{p,\sigma}(\Omega)) \cap L^p_\alpha(0,\tau;\widetilde{L}^q_\sigma(\Omega)).$$

In particular, we have $u_1 \in L^p_{\alpha+1}(0,\tau;D(\widetilde{A}_q))$ and, recalling the definition of $\widetilde{A}_q$,

$$\partial_t u_1, \widetilde{\Delta}_q u_1 \in L^p_{\alpha+1}(0,\tau;\widetilde{L}^q(\Omega)^n), \quad \widetilde{A}_q u_1 = -\widetilde{\mathbb{P}}_q\widetilde{\Delta}_q u_1.$$

Hence $\partial_t u_1 - \widetilde{\Delta}_q u_1 \in L^p_{\alpha+1}(0,\tau;\widetilde{L}^q(\Omega)^n)$ and

$$\widetilde{\mathbb{P}}_q(\partial_t u_1 - \widetilde{\Delta}_q u_1) = \partial_t u_1 + \widetilde{A}_q u_1 = 0. \tag{5.1}$$

By the Helmholtz decomposition in $\widetilde{L}^q(\Omega)^n$ we thus obtain a gradient term $\nabla p_1 \in L^p_{\alpha+1}(0,\tau;\widetilde{L}^q(\Omega)^n)$ such that

$$\partial_t u_1 - \widetilde{\Delta}_q u_1 + \nabla p_1 = 0. \tag{5.2}$$

We turn to u_2. Since $u \in L^p_\alpha(0,\tau;\widetilde{L}^q(\Omega)^n)$ we have, by the assumption on f and the arguments in the proof of Theorem 1.1, that

$$w := \widetilde{P}_{q/2}(f - \nabla\cdot(u\otimes u)) \in L^{p/2}_{2\alpha}(0,\tau;\widetilde{W}^{-1,q/2}_\sigma(\Omega)).$$

By Corollary 2.6 maximal L^s-regularity of $\widetilde{A}_{q/2}$ carries over to the operator $\widetilde{A}^{(-1)}_{q/2}$ in $\widetilde{W}^{-1,q/2}_\sigma(\Omega)$. By [18, Thm. 2.4] (cf. also [9, Thm. 1.13]) the operator $\widetilde{A}^{(-1)}_{q/2}$ has maximal $L^{p/2}_{2\alpha}$-regularity in $\widetilde{W}^{-1,q/2}_\sigma(\Omega)$. This means that

$$u_2 = \widetilde{T}^{(-1)}_{q/2}(\cdot) * w \in L^{p/2}_{2\alpha}(0,\tau;\widetilde{W}^{1,q}_{0,\sigma}(\Omega)) \quad \text{and} \quad \partial_t u_2 \in L^{p/2}_{2\alpha}(0,\tau;\widetilde{W}^{-1,q}_\sigma(\Omega)).$$

Similarly as for u_1, we want to apply the Dirichlet Laplacian to u_2. We need a lemma on the relation of extrapolated versions of the Stokes operator to extrapolated versions of the Dirichlet Laplacian.

Lemma 5.1. *We have* $\widetilde{A}_{q/2}^{(-1)} = -\widetilde{P}_{q/2}\widetilde{\Delta}_{q/2}^{(-1)}$ *on* $\widetilde{W}_{0,\sigma}^{1,q/2}(\Omega)$, *where* $\widetilde{\Delta}_{q/2}^{(-1)}$ *denotes the extrapolated version of the Dirichlet Laplacian to the space*

$$\widetilde{W}^{-1,q/2}(\Omega)^n = (\widetilde{L}^{q/2}(\Omega)^n, \|(1-\widetilde{\Delta}_{q/2})^{-1/2}\cdot\|_{\widetilde{L}^{q/2}})^{\sim}.$$

Observe that the last equality holds dy dualization and the identity

$$D((1-\widetilde{\Delta}_{(q/2)'})^{1/2}) = \widetilde{W}_0^{1,(q/2)'}(\Omega)^n.$$

P r o o f of Lemma 5.1. By definition of $\widetilde{A}_{q/2}$, equality holds on the dense subset $D(\widetilde{A}_{q/2})$. Hence the proof can be finished by approximation. □

By the lemma we have $\widetilde{\Delta}_{q/2}^{(-1)} u_2 \in L_{2\alpha}^{p/2}(0,\tau;\widetilde{W}^{-1,q/2}(\Omega)^n)$, in particular, this function has values in the distributions on Ω. We would like to have the same for $\partial_t u_2$, but this function has values in $\widetilde{W}_\sigma^{-1,q/2}(\Omega)$, which is not a space of distributions. We proceed as in [19, p. 247] and integrate $\int_0^t \cdot\,\mathrm{d}s$ with respect to time: let $v(t) := \int_0^t u(s)\,\mathrm{d}s$ and define v_1, v_2 similarly by integrating u_1, u_2, respectively. Moreover, let $g(t) := \int_0^t w(s)\,\mathrm{d}s$ and

$$h(t) := \int_0^t f(s) - \nabla\cdot(u(s)\otimes u(s))\,\mathrm{d}s.$$

Then $v_2 = \widetilde{T}(\cdot)*g$, $u_2 + \widetilde{A}v_2 = g$, and we see that $\partial_t v_2 = u_2 \in L_{2\alpha}^{p/2}(0,\tau;\widetilde{W}_{0,\sigma}^{1,q/2}(\Omega))$. By the lemma we obtain

$$\partial_t v_2 - \widetilde{\Delta}_{q/2}^{(-1/2)} v_2 - h \in L_{2\alpha}^{p/2}(0,\tau;\widetilde{W}^{-1,q/2}(\Omega)^n),$$
$$\widetilde{P}_{q/2}(\partial_t v_2 - \widetilde{\Delta}_{q/2}^{(-1)} v_2 - h) = u_2 + \widetilde{A}_{q/2}^{(-1)} v_2 - g = 0.$$

Hence there is a gradient term $\nabla\hat{p_2} \in L_{2\alpha}^{p/2}(0,\tau;\widetilde{W}^{-1,q/2}(\Omega)^n)$ such that

$$\partial_t v_2 - \widetilde{\Delta}_{q/2}^{(-1/2)} v_2 - h - \nabla\hat{p_2} = 0. \tag{5.3}$$

We let $\nabla p := \nabla p_1 + \partial_t\nabla\hat{p_2}$ where ∂_t is taken in distributional sense. Putting together (5.2) and (5.3) we see that construction of the pressure term is achieved.

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Author's address: Peer Christian Kunstmann, Institut für Analysis, Universität Karlsruhe, Kaiserstr. 89, D-76128 Karlsruhe, Germany, e-mail: peer.kunstmann@math.uni-karlsruhe.de.