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EQUITORSION HOLOMORPHICALLY PROJECTIVE MAPPINGS OF GENERALIZED KÄHLERIAN SPACE OF THE FIRST KIND

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Abstract. In this paper we define generalized Kählerian spaces of the first kind ($\underset{1}{GK}_N$) given by (2.1)–(2.3). For them we consider hollomorphically projective mappings with invariant complex structure. Also, we consider equitorsion geodesic mapping between these two spaces ($\underset{1}{GK}_N$ and $\underset{1}{G\overline{K}}_N$) and for them we find invariant geometric objects.

Keywords: generalized Riemannian space, Kählerian space, generalized Kählerian space of the first kind, equitorsion holomorphically projective mappings, holomorphically projective parameter.

MSC 2010: 53B05, 53B35

1. Motivation

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [6] is a differentiable N-dimensional manifold, equipped with a non-symmetric basic tensor g_{ij}. Connection coefficients of this space are generalized Cristoffel's symbols of the second kind. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

The use of non-symmetric basic tensor and non-symmetric connection became especially topical after the appearance of the papers of A. Einstein [2]–[5] related to the creation of the Unified Field Theory (UFT). We remark that in UFT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} is related to the gravitation, and the antisymmetric one $g_{\underset{\vee}{ij}}$ to the electromagnetism.

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In a generalized Riemannian space one can define four kinds of covariant derivatives [12], [13]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p, \tag{1.1}$$

$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p. \tag{1.2}$$

In the case of the space GR_N we have five independent curvature tensors [14]:

$$\underset{1}{R}{}^i_{jmn} = \Gamma^i_{j[m,n]} + \Gamma^p_{j[m}\Gamma^i_{pn]}, \quad \underset{2}{R}{}^i_{jmn} = \Gamma^i_{[mj,n]} + \Gamma^p_{[mj}\Gamma^i_{n]p}, \tag{1.3}$$

$$\underset{3}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}\Gamma^i_{[pj]}, \tag{1.4}$$

$$\underset{4}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{[pj]}, \tag{1.5}$$

$$\underset{5}{R}{}^i_{jmn} = \frac{1}{2}(\Gamma^i_{j[m,n]} + \Gamma^i_{[mj,n]} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}), \tag{1.6}$$

where $[i \ldots j]$ denotes antisymmetrization without division with respect to the indices i, j, and also $(i \ldots j)$ denotes symmetrization without division with respect to indices i, j.

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [23], [24], M. Prvanović [17], T. Otsuki [16], N. S. Sinyukov [20], J. Mikeš [11] and many others.

In [15], [21] we defined a generalized Kählerian space GK_N as a generalized N-dimensional Riemannian space with a (non-symmetric) metric tensor g_{ij} and an almost complex structure F^i_j such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{1.7}$$

$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}\,, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{1.8}$$

$$F^h_{i\underset{\theta}{|}j} = 0, \quad (\theta = 1, 2), \tag{1.9}$$

where $\underset{\theta}{|}$ denotes the covariant derivative of the kind θ with respect to the metric tensor g_{ij}.

2. Generalized Kählerian spaces of the first kind

A generalized N-dimensional Riemannian space with (non-symmetric) metric tensor g_{ij} is a *generalized Kählerian space of the first kind* $\underset{1}{GK}_N$ if there exists an almost complex structure $F^i_j(x)$ such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{2.1}$$

$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}\,, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{2.2}$$

$$F^h_{i\underset{1}{|}j} = 0, \tag{2.3}$$

where $\underset{1}{|}$ denotes the covariant derivative of the first kind with respect to the metric tensor g_{ij}. From (2.2), using (2.1), we get $F_{ij} = -F_{ji}$, $F^{ij} = -F^{ji}$, where we denote $F_{ji} = F^p_j g_{\underline{pi}}$, $F^{ji} = F^j_p g^{\underline{pi}}$.

From here we prove the following theorems.

Theorem 2.1. *For the almost complex structure F^i_j of a $\underset{1}{GK}_N$ the relations*

$$F^h_{i\underset{2}{|}j} = 2(F^p_i\Gamma^h_{\underset{\vee}{jp}} + F^h_p\Gamma^p_{\underset{\vee}{ij}}), \quad F^h_{i\underset{3}{|}j} = 2F^h_p\Gamma^p_{\underset{\vee}{ij}}, \quad F^h_{i\underset{4}{|}j} = 2F^p_i\Gamma^h_{\underset{\vee}{jp}} \tag{2.4}$$

are valid, where $\Gamma^h_{\underset{\vee}{ij}}$ is the torsion tensor.

P r o o f. We get the relations (2.4) by using the condition (2.3) and the covariant derivative (1.1), (1.2). □

Let us denote $\overline{F}^h_{ij} = F^p_i\Gamma^h_{\underset{\vee}{jp}}$ and $\overline{\overline{F}}^h_{ij} = F^h_p\Gamma^p_{\underset{\vee}{ij}}$. Then we have

Theorem 2.2. *For the curvature tensors $\underset{\theta}{R}$, $\theta = 1,\dots,5$, given by* (1.3)–(1.6) *of a $\underset{1}{GK}_N$ the relations*

$$\begin{aligned}
&F^p_i\underset{1}{R}{}^h_{pjk} = F^h_p\underset{1}{R}{}^p_{ijk},\\
&F^p_i\underset{2}{R}{}^h_{pjk} - F^h_p\underset{2}{R}{}^p_{ijk} + 4\Gamma^p_{\underset{\vee}{jk}}(F^q_i\Gamma^h_{\underset{\vee}{pq}} + F^h_q\Gamma^q_{\underset{\vee}{ip}}) = 2(\overline{F}^h_{i[j\underset{2}{|}k]} + \overline{\overline{F}}^h_{i[j\underset{2}{|}k]}),\\
&F^p_i\underset{3}{R}{}^h_{pjk} - F^h_p\underset{3}{R}{}^p_{ijk} = -2(\overline{F}^h_{ij\underset{1}{|}k} + \overline{\overline{F}}^h_{ij\underset{1}{|}k}),\\
&F^p_i\underset{4}{R}{}^h_{pjk} + F^h_p\underset{3}{R}{}^p_{ijk} = 2(\overline{\overline{F}}^h_{ik\underset{3}{|}j} - \overline{F}^h_{ij\underset{4}{|}k}),
\end{aligned} \tag{2.5}$$

are valid, where $\underset{\theta}{|}$ $(\theta = 1,2,3,4)$ denotes the covariant derivative of the kind θ.

P r o o f. The first equality follows directly from the Ricci identity for the tensor $\underset{1}{R}$ using (2.3).

From (2.4) by using the covariant derivative of the second kind we have

$$F^h_{i\underset{2}{|}j}k = 2\bar{F}^h_{ij\underset{2}{|}k} + 2\bar{\bar{F}}^h_{ij\underset{2}{|}k}. \tag{2.6}$$

Now, from (2.6) we obtain

$$F^h_{i\underset{2}{|}j}k - F^h_{i\underset{2}{|}k}j = 2(\bar{F}^h_{i[j\underset{2}{|}k]} + \bar{\bar{F}}^h_{i[j\underset{2}{|}k]}). \tag{2.7}$$

Using the Ricci identity [14], we get from (2.7)

$$F^p_i \underset{2}{R}^h_{pjk} - F^h_p \underset{2}{R}^p_{ijk} + 2\Gamma^p_{\underset{\vee}{jk}} F^h_{i\underset{2}{|}p} = 2(\bar{F}^h_{i[j\underset{2}{|}k]} + \bar{\bar{F}}^h_{i[j\underset{2}{|}k]}),$$

and from here the second equality (2.5) is valid.

From (2.4) and (2.3) we have

$$F^h_{i\underset{2}{|}j\underset{1}{|}k} = 2(\bar{F}^h_{ij\underset{1}{|}k} + \bar{\bar{F}}^h_{ij\underset{1}{|}k}), \quad F^h_{i\underset{1}{|}k\underset{2}{|}j} = 0.$$

Using the Ricci identity [14]

$$F^h_{i\underset{1}{|}j\underset{2}{|}k} - F^h_{i\underset{2}{|}k\underset{1}{|}j} = F^p_i \underset{3}{R}^h_{pjk} - F^h_p \underset{3}{R}^p_{ijk},$$

we find the third equality (2.5).

Finally, from (2.4) we have

$$F^h_{i\underset{3}{|}j\underset{4}{|}k} = 2\bar{F}^h_{ij\underset{4}{|}k}, \quad F^h_{i\underset{4}{|}k\underset{3}{|}j} = 2\bar{\bar{F}}^h_{ik\underset{3}{|}j}.$$

Using the Ricci identity [14]

$$F^h_{i\underset{3}{|}j\underset{4}{|}k} - F^h_{i\underset{4}{|}k\underset{3}{|}j} = \underset{4}{R}^h_{pjk} F^p_i + \underset{3}{R}^p_{ijk} F^h_p,$$

we have the fourth equality (2.5). □

Theorem 2.3. *For the Ricci tensor R_{ij} given by $g^{\underline{ij}}$ the relation*

$$R_{hk} = F^p_h F^q_k R_{pq} + g^{\underline{pq}} F^s_h \mathcal{D}_{(s.pqk)} \tag{2.8}$$

is valid, where

$$\mathcal{D}^h{}_{ijk} = F^p_{i;[k}\Gamma^h_{\underset{\vee}{j]}p} + F^p_i \Gamma^h_{\underset{\vee}{[jp;k]}} + F^h_{p;[k}\Gamma^p_{\underset{\vee}{ij]}} - F^h_p \Gamma^p_{\underset{\vee}{i[j;k]}} \tag{2.9}$$

and $\mathcal{D}_{h.ijk} = g_{ph}\mathcal{D}^p{}_{ijk}$.

P r o o f. From (2.3) and (2.4) we get

$$F^h_{i;j} = F^p_i \Gamma^h_{\underset{\vee}{jp}} + F^h_p \Gamma^p_{\underset{\vee}{ij}}. \tag{2.10}$$

The integrability conditions of the equality (2.10) are given by

$$F^h_{i;[jk]} = \mathcal{D}^h{}_{ijk}. \tag{2.11}$$

Using the Ricci identity in the symmetric case, from (2.11) we obtain

$$F^h_p R^p_{ijk} - F^p_i R^h_{pjk} = \mathcal{D}^h{}_{ijk}. \tag{2.12}$$

Here R^h_{ijk} is the curvature tensor with respect to the symmetric affine connection $\Gamma^h_{\underline{ij}}$. Composition with F^i_r in (2.12) gives

$$F^h_p F^q_i R^p_{qjk} + R^h_{ijk} = F^p_i \mathcal{D}^h{}_{pjk}. \tag{2.13}$$

Now, from (2.13) by composition with $g^{\underline{hr}}$ we get

$$F_{ph} F^q_i R^p_{qjk} + R_{hijk} = F^p_i \mathcal{D}_{h.pjk}. \tag{2.14}$$

From here we get

$$-F^h_p F^q_i R_{pqjk} + R_{hijk} = F^p_i \mathcal{D}_{h.pjk}. \tag{2.15}$$

From (2.15) by composition with F^i_r we have

$$F^p_{[h} R_{pi]jk} = -\mathcal{D}_{h.ijk}. \tag{2.16}$$

Using composition with $g^{\underline{ij}}$ in (2.16) we obtain

$$F^p_h R_{pk} - F^p_q R_{ph.k}{}^{q} = -g^{\underline{pq}} \mathcal{D}_{h.pqk}. \tag{2.17}$$

Symmetrization in (2.17) with respect to h, k gives the relation (2.8). □

Theorem 2.4. *The Ricci tensors* $\underset{\theta}{R}{}_{jm}$ $(\theta = 1, \dots, 5)$ *of the space* $G\underset{1}{K}{}_N$ *satisfy the relations*

$$\underset{\theta}{R}{}_{(pq)} F^p_j F^q_m = \underset{\theta}{R}{}_{(jm)} - 2\Gamma^p_{\underset{\vee}{rq}} \Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}} \Gamma^q_{\underset{\vee}{pm}} - 2g^{\underline{pq}} F^s_h \mathcal{D}_{(s.pqk)}, \quad \theta = 1, 2, 3, \tag{2.18}$$

$$\underset{4}{R}{}_{(pq)} F^p_j F^q_m = \underset{4}{R}{}_{(jm)} + 6\Gamma^p_{\underset{\vee}{rq}} \Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m - 6\Gamma^p_{\underset{\vee}{jq}} \Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}} F^s_h \mathcal{D}_{(s.pqk)}, \tag{2.19}$$

$$\underset{5}{R}{}_{(pq)} F^p_j F^q_m = \underset{5}{R}{}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rq}} \Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m - 2\Gamma^p_{\underset{\vee}{jq}} \Gamma^q_{\underset{\vee}{pm}} - 2g^{\underline{pq}} F^s_h \mathcal{D}_{(s.pqk)}, \tag{2.20}$$

where $(j \dots m)$ *denotes the symmetrization without division with respect to the indices* j, m.

P r o o f. We can express the tensor $\underset{1}{R}{}^{i}{}_{jmn}$ in the form [14]:

$$\underset{1}{R}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + \Gamma^{i}_{\underset{\vee}{j}[m;n]} + \Gamma^{p}_{\underset{\vee}{j}[m}\Gamma^{i}_{\underset{\vee}{pn}]}.$$

By contraction with respect to the indices i, n, and by symmetrization with respect to j, m, we get

$$\underset{1}{R}_{(jm)} = R_{(jm)} - 2\Gamma^{p}_{\underset{\vee}{jq}}\Gamma^{q}_{\underset{\vee}{pm}}. \tag{2.21}$$

From (2.8) and (2.21) we have (2.18) for curvature tensor $\underset{1}{R}$.

The tensor $\underset{2}{R}{}^{i}{}_{jmn}$ can be expressed in the form [14]:

$$\underset{2}{R}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + \Gamma^{i}_{\underset{\vee}{j}[n;m]} + \Gamma^{p}_{\underset{\vee}{j}[n}\Gamma^{i}_{\underset{\vee}{pm}]}.$$

By contraction with respect to i, n, and then by symmetrization with respect to j, m, we get

$$\underset{2}{R}_{(jm)} = R_{(jm)} - 2\Gamma^{p}_{\underset{\vee}{jq}}\Gamma^{q}_{\underset{\vee}{pm}},$$

from where, using (2.8), we get the relation (2.18) for the curvature tensor $\underset{2}{R}$.

For the tensor $\underset{3}{R}{}^{i}{}_{jmn}$ we have [14]:

$$\underset{3}{R}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + \Gamma^{i}_{\underset{\vee}{j}(m;n)} + \Gamma^{p}_{\underset{\vee}{j}[n}\Gamma^{i}_{\underset{\vee}{pm}]} - 2\Gamma^{p}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{pj}}.$$

Contracting with respect to i, n, and then symmetrizing in relation to j, m, we get

$$\underset{3}{R}_{(jm)} = R_{(jm)} - 2\Gamma^{p}_{\underset{\vee}{jq}}\Gamma^{q}_{\underset{\vee}{pm}},$$

from where, using (2.8), we can see that the relation (2.18) is valid for the curvature tensor $\underset{3}{R}$.

The tensor $\underset{4}{R}{}^{i}{}_{jmn}$ can be expressed in the form [14]:

$$\underset{4}{R}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + \Gamma^{i}_{\underset{\vee}{j}(m;n)} + \Gamma^{p}_{\underset{\vee}{j}[n}\Gamma^{i}_{\underset{\vee}{pm}]} + 2\Gamma^{p}_{\underset{\vee}{mn}}\Gamma^{i}_{\underset{\vee}{pj}}.$$

Contracting with respect to i, n, and symmetrizing with respect to j, m, we get

$$\underset{4}{R}_{(jm)} = R_{(jm)} + 6\Gamma^{p}_{\underset{\vee}{jq}}\Gamma^{q}_{\underset{\vee}{pm}}.$$

Using (2.8) we get the relation (2.19).

The tensor $\underset{5}{R}{}^i{}_{jmn}$ satisfies the relation [14]:

$$\underset{5}{R}{}^i{}_{jmn} = R^i{}_{jmn} + \Gamma^p_{\underset{\vee}{j(m}}\Gamma^i_{\underset{\vee}{pn)}}.$$

Contracting with respect to the of indices i, n, and then symmetrizing with respect to j, m, we get

$$\underset{5}{R}{}_{(jm)} = R_{(jm)} + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}},$$

from where, using (2.8), we get (2.20). □

3. Holomorphically projective mappings of generalized Kählerian space of the first kind which preserves complex structure

By generalizing the notion of analytic planar curve of Kählerian space [16], [20] we come to an analogous notion for generalized Kählerian spaces of the first kind.

Definition 3.1. A $G\underset{1}{K}_N$ space curve, which is, in parametric form, given by the equation

$$x^h = x^h(t) \quad (h = 1, 2, \dots, N) \tag{3.1}$$

will be called planar if:

$$\lambda^h{}_{\underset{\theta}{|}p}\lambda^p = a(t)\lambda^h + b(t)F^h_p\lambda^p \quad (\theta = 1, 2) \tag{3.2}$$

where $\lambda^h = \mathrm{d}x^h/\mathrm{d}t$, and $a(t)$ and $b(t)$ are functions of the parameter t.

Considering that

$$\lambda^h{}_{\underset{1}{|}p}\lambda^p = \frac{d\lambda^h}{dt} + \Gamma^h_{pq}\lambda^p\lambda^q = \lambda^h{}_{\underset{2}{|}p}\lambda^p,$$

we conclude that the expression on the left-hand side in (3.2) is the same with respect to both kinds of covariant derivatives, so we can define analytic planar curve in the space $G\underset{1}{K}_N$ by the following relation:

$$\frac{d\lambda^h}{dt} + \Gamma^h_{pq}\lambda^p\lambda^q = a(t)\lambda^h + b(t)F^h_p\lambda^p. \tag{3.3}$$

We can consider two N-dimensional generalized Kählerian spaces of the first kind $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ with complex structures F^h_i and $\overline{F}^h_i$, where:

$$F^h_i = \overline{F}^h_i \tag{3.4}$$

in the same local coordinate system, defined by the map $f\colon G\underset{1}{K}_N \to G\underset{1}{\overline{K}}_N$.

Definition 3.2. A diffeomorfism $f\colon G\underset{1}{K}_N \to G\underset{1}{\overline{K}}_N$ will be called holomorphically projective or analytic planar if it maps analytic planar curves of the space $G\underset{1}{K}_N$ into analytic planar curves of the space $G\underset{1}{\overline{K}}_N$.

We can denote

$$P^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij} \tag{3.5}$$

the deformation tensor of the connection under an analytic planar mapping. Here Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are the second kind Cristoffel's symbols of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$, respectively. Analytic planar curves of the space $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ are given by the following relations, respectively:

$$\begin{aligned}\frac{\mathrm{d}\lambda^h}{\mathrm{d}t} + \Gamma^h_{pq}\lambda^p\lambda^q &= a(t)\lambda^h + b(t)F^h_p\lambda^p,\\ \frac{\mathrm{d}\lambda^h}{\mathrm{d}t} + \overline{\Gamma}^h_{pq}\lambda^p\lambda^q &= \overline{a}(t)\lambda^h + \overline{b}(t)F^h_p\lambda^p.\end{aligned}$$

From the previous relations we have $(\overline{\Gamma}^h_{pq} - \Gamma^h_{pq})\lambda^p\lambda^q = \psi(t)\lambda^h + \sigma(t)F^h_p\lambda^p$, where we denote $\psi(t) = \overline{a}(t) - a(t)$, $\sigma(t) = \overline{b}(t) - b(t)$. We can now put: $\psi(t) = \psi_p\lambda^p$, $\sigma(t) = \sigma_q\lambda^q$. So we have

$$(\overline{\Gamma}^h_{pq} - \Gamma^h_{pq} - \psi_p\delta^h_q - \sigma_pF^h_q)\lambda^p\lambda^q = 0,$$

where from we can conclude that:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} + \xi^h_{ij}, \tag{3.6}$$

where ξ^h_{ij} is an arbitrary anti-symmetric tensor. In (3.6) we can select the vector σ_i so that $\sigma_i = -\psi_pF^p_i$. Because of that we have:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_pF^p_{(i}F^h_{j)} + \xi^h_{ij}. \tag{3.7}$$

Contracting over the indices h, i in (3.7) and using $F^p_p = 0$, $\xi^p_{pj} = 0$, we get:

$$\overline{\Gamma}^p_{pj} - \Gamma^p_{pj} = (N+2)\psi_j. \tag{3.8}$$

Thus from (3.8) we can see that ψ_j is obviously a gradient vector. If we substitute from (3.8) into (3.7) we have

$$\begin{aligned}\overline{\Gamma}^h_{ij} - \frac{1}{N+2}(\overline{\Gamma}^p_{p(i}\delta^h_{j)} - \overline{\Gamma}^q_{qp}\overline{F}^p_{(i}\overline{F}^h_{j)}) - \overline{\Gamma}^h_{\underset{\vee}{ij}}\\ = \Gamma^h_{ij} - \frac{1}{N+2}(\Gamma^p_{p(i}\delta^h_{j)} - \Gamma^q_{qp}F^p_{(i}F^h_{j)}) - \Gamma^h_{\underset{\vee}{ij}}.\end{aligned} \tag{3.9}$$

Denoting

$$HT^h_{ij} = \Gamma^h_{\underline{ij}} - \frac{1}{N+2}(\Gamma^p_{p(i}\delta^h_{j)} - \Gamma^q_{qp}F^p_{(i}F^h_{j)}), \tag{3.10}$$

we can present (3.9) in the form:

$$H\overline{T}^h_{ij} = HT^h_{ij}, \tag{3.11}$$

where by $H\overline{T}^h_{ij}$ we denoted the object of the form (3.10) for $G\underset{1}{\overline{K}}_N$. The quantity HT^h_{ij} is not a tensor. We will call it *holomorphically projective parameter of the type of Tomass projective parameter*. In this way, based on the above fact we have proved:

Theorem 3.1. *The quantities* (3.10) *represent invariants of holomorphically projective mapping of generalized Kählerian space of the first kind with equal complex structures.*

4. Holomorphically projective parameters of generalized Kählerian space of the first kind

If $f\colon G\underset{1}{K}_N \to G\underset{1}{\overline{K}}_N$ is a holomorphically projective mapping, and if the torsion tensors of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ satisfy

$$\overline{\Gamma}^h_{\underset{\vee}{ij}} = \Gamma^h_{\underset{\vee}{ij}}, \tag{4.1}$$

then we can tell that:

$$\xi^h_{ij} = 0. \tag{4.2}$$

4.1. Holomorphically projective parameter of the first kind

The relation between the curvature tensors $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ is given by

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + P^i_{j[m\underset{1}{|}n]} + P^p_{j[m}P^i_{pn]} + 2\Gamma^p_{\underset{\vee}{mn}}P^i_{jp}. \tag{4.3}$$

Substituting (3.5), (3.7) and (4.2) into (4.3) we get

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \delta^i_{[m}\,\underset{1}{\psi}_{jn]} + \delta^i_j\,\underset{1}{\psi}_{[mn]} + F^{(p}_j F^{i)}_{[n}\underset{1}{\psi}_{pm]} \\ + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p\delta^i_j - 2\Gamma^p_{\underset{\vee}{mn}}\psi_q F^q_{(j}F^i_{p)},\end{aligned} \tag{4.4}$$

where we denote

$$\underset{1}{\psi}_{ij} = \psi_{i\underset{1}{|}j} - \psi_i\psi_j + \psi_p F^p_i \psi_q F^q_j. \tag{4.5}$$

Contracting with respect to the indices i, n in (4.4) we get

$$\overline{\underset{1}{R}}_{jm} = \underset{1}{R}_{jm} + \underset{1}{\psi}_{[mj]} - N\underset{1}{\psi}_{jm} - F^p_j F^q_m \underset{1}{\psi}_{(pq)} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_{(j}F^r_{p)}. \tag{4.6}$$

Anti-symmetrization without division in (4.6) with respect to the indices j, m gives:

$$(N+2)\underset{1}{\psi}_{[jm]} = \underset{1}{R}_{[jm]} - \overline{\underset{1}{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^p_{\underset{\vee}{[mr}}\psi_q F^q_{(j]}F^r_{p)}. \tag{4.7}$$

By symmetrization without division in (4.6) with respect to the indices j, m we obtain:

$$\overline{\underset{1}{R}}_{(jm)} = \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{1}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{(j)}F^r_{p)}. \tag{4.8}$$

The analogous relation to the relation (2.18) for $\underset{1}{R}$ in the space $G\overline{\underset{1}{K}}_N$ is valid.

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and by use of the conditions (2.18) for $\underset{1}{R}$ in $G\underset{1}{K}_N$ and $G\overline{\underset{1}{K}}_N$ from (4.8) we get

$$\overline{\underset{1}{R}}_{(jm)} = \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(pq)}F^p_j F^q_m - 2\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)} + 2\Gamma^p_{\underset{\vee}{r(j}}\psi_q F^q_p F^r_{m)}. \tag{4.9}$$

From (4.8) and (4.9) we get:

$$(N-2)F^p_j F^q_m \underset{1}{\psi}_{(pq)} = (N-2)\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{j)}F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)}. \tag{4.10}$$

Replacing (4.10) in (4.9) we get:

$$\begin{aligned}(N+2)\underset{1}{\psi}_{(jm)} &= \underset{1}{R}_{(jm)} - \overline{\underset{1}{R}}_{(jm)} \\ &\quad - \frac{2}{N-2}\left(N\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{j)}F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)}\right) - 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_p F^r_{j)}.\end{aligned} \tag{4.11}$$

Using (4.7) and (4.11) we have:

$$\begin{aligned}(N+2)\underset{1}{\psi}_{jm} &= \underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_j F^r_p \\ &\quad - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_m F^r_p - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)} - 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_p F^r_j.\end{aligned} \tag{4.12}$$

Eliminating ψ_i and using the condition (3.8) the last equation becomes:

$$(N+2)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} + \underset{1}{\overline{P}}_{jm} - \underset{1}{P}_{jm}, \tag{4.13}$$

where we denoted

$$\underset{1}{P}_{jm} = \frac{2}{N+2}\Big(\Gamma^p_{\underset{\vee}{mj}}\Gamma^q_{qp} - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_jF^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_mF^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{s(j}F^r_pF^q_{m)} - \Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_pF^r_j\Big). \tag{4.14}$$

In the same way the object $\underset{1}{\overline{P}}_{jm}$ of the space $G\underset{1}{\overline{K}}_N$ is defined. Eliminating $\underset{1}{\psi}_{jm}$ from (4.4) we get

$$HP\underset{1}{\overline{W}}{}^i_{jmn} = HP\underset{1}{W}{}^i_{jmn}, \tag{4.15}$$

where following quantity

$$HP\underset{1}{W}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \frac{1}{N+2}\big[\delta^i_{[m}(\underset{1}{R}-\underset{1}{P})_{jn]} + \delta^i_j(\underset{1}{R}_{[mn]} - \underset{1}{P}_{[mn]}) + F^{(p}_jF^{i)}_{[n}(\underset{1}{R}-\underset{1}{P})_{pm]} - 2\Gamma^i_{\underset{\vee}{mn}}\Gamma^q_{qj} - 2\delta^i_j\Gamma^p_{\underset{\vee}{mn}}\Gamma^q_{qp} + 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_{(j}F^i_{p)}\big] \tag{4.16}$$

is an object of the space $G\underset{1}{K}_N$. We denoted in last equation $(\underset{1}{R}-\underset{1}{P})_{jm} = (\underset{1}{R}_{jm} - \underset{1}{P}_{jm})$. We see that the quantity $HP\underset{1}{\overline{W}}{}^i_{jmn}$ is expressed in the same way as the quantity $HP\underset{1}{W}{}^i_{jmn}$. Obviously, the quantity $HP\underset{1}{W}{}^i_{jmn}$ is not a tensor, so we shall call it an *equitorsion holomorphically projective parameter of the first kind* of the space $G\underset{1}{K}_N$. Because of all those facts the following theorem is proved:

Theorem 4.1. *The equitorsion holomorphically projective parameter of the first kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{1}{K}_N$ *and* $G\underset{1}{\overline{K}}_N$.

4.2. Holomorphically projective parameter of the second kind

The connection between the curvature tensors $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ is given by:

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + P^i_{[mj\underset{2}{|}n]} + P^p_{[mj}P^i_{n]p} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{pj}. \tag{4.17}$$

Replacing (3.5), (3.7) and (4.2) in (4.17) we have:

$$\tag{4.18} \begin{aligned} \underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i_{jmn} + \delta^i_{[m}\underset{1}{\psi}_{jn]} + \delta^i_j \underset{1}{\psi}_{[mn]} + F^{(p}_j F^{i)}_{[n}\underset{1}{\psi}_{pm]} + 2\Gamma^p_{\underset{\vee}{nm}}\psi_p\delta^i_j \\ &\quad + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j - 2\Gamma^{(p}_{\underset{\vee}{[nq}}\psi_p F^q_{(m]}F^{i)}_{j)} - 2\Gamma^p_{\underset{\vee}{mn}}\psi_q F^q_{(p}F^i_{j)} - 2\Gamma^p_{\underset{\vee}{j[n}}\psi_q F^i_{(p}F^q_{m])} \end{aligned}$$

where we denoted

$$\tag{4.19} \underset{2}{\psi}_{ij} = \psi_{i\underset{2}{|}j} - \psi_i\psi_j + \psi_p F^p_i \psi_q F^q_j .$$

Contracting with respect to the indices i, n in (4.18) we get

$$\tag{4.20} \underset{2}{\overline{R}}_{jm} = \underset{2}{R}_{jm} + \underset{2}{\psi}_{[mj]} - N\underset{2}{\psi}_{jm} - F^p_j F^q_m \underset{2}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^p_{\underset{\vee}{jr}}\psi_q F^r_{(p}F^q_{m)}.$$

Anti-symmetrization without division in (4.20) with respect to indices j, m gives:

$$\tag{4.21} (N+2)\underset{2}{\psi}_{[jm]} = \underset{2}{R}_{[jm]} - \underset{2}{\overline{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{jm}}\psi_p - 2\Gamma^p_{\underset{\vee}{[jr}}\psi_q F^q_{(m]}F^r_{p)}.$$

Symmetrization without division in (4.20) with respect to indices j, m gives:

$$\tag{4.22} \underset{2}{\overline{R}}_{(jm)} = \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{2}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{(j)}F^r_{p)}.$$

The relation analogous to the relation (2.18) for $\underset{2}{R}$ in the space $G\underset{1}{\overline{K}}_N$ is valid.

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and by use of the conditions (2.18) for $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ in $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$, respectively, from (4.22) we get

$$\tag{4.23} \underset{2}{\overline{R}}_{(jm)} = \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(pq)}F^p_j F^q_m - 2\underset{2}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)} + 2\Gamma^p_{\underset{\vee}{r(j}}\psi_q F^q_p F^r_{m)}.$$

From (4.22) and (4.23) we get:

$$\tag{4.24} (N-2)F^p_j F^q_m \underset{2}{\psi}_{(pq)} = (N-2)\underset{2}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{j)}F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)}.$$

Replacing (4.24) in (4.23) we get

$$\tag{4.25} \begin{aligned} (N+2)\underset{2}{\psi}_{(jm)} &= \underset{2}{R}_{(jm)} - \underset{2}{\overline{R}}_{(jm)} \\ &\quad - \frac{2}{N-2}\left(N\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_{j)}F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^r_p F^q_{m)}\right) - 2\Gamma^p_{\underset{\vee}{(mr}}\psi_q F^q_p F^r_{j)}. \end{aligned}$$

Using (4.21) and (4.25) we have:

$$(4.26)\quad (N+2)\underset{2}{\psi}_{jm} = \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{jm}}\psi_p - \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{jr}}\psi_q F^r_p F^q_m$$
$$- \frac{2}{N-2}\Gamma^p_{\underset{\vee}{mr}}\psi_q F^r_p F^q_j - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^q_{m)}F^r_p - 2\Gamma^p_{\underset{\vee}{jr}}\psi_q F^r_m F^q_p .$$

Eliminating ψ_i and using the condition (3.8) the last equation becomes:

$$(4.27)\qquad (N+2)\underset{2}{\psi}_{jm} = \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + \underset{2}{\overline{P}}_{jm} - \underset{2}{P}_{jm},$$

where we denoted

$$(4.28)\quad \underset{2}{P}_{jm} = \frac{2}{N+2}\Big(\Gamma^p_{\underset{\vee}{jm}}\Gamma^q_{qp} - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_m F^r_p$$
$$- \frac{1}{N-2}\Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_j F^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{s(j}F^r_p F^q_{m)} - \Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_p F^r_m\Big).$$

In the same way the object $\underset{2}{\overline{P}}_{jm}$ of the space $G\underset{1}{\overline{K}}_N$ is defined. Eliminating $\underset{2}{\psi}_{jm}$ from (4.18) we get

$$(4.29)\qquad HP\underset{2}{\overline{W}}{}^i_{jmn} = HP\underset{2}{W}{}^i_{jmn},$$

where we denoted

$$(4.30)\quad HP\underset{2}{W}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_{[m}(\underset{2}{R} - \underset{2}{P})_{jn]} + \delta^i_j(\underset{2}{R}_{[mn]} - \underset{2}{P}_{[mn]})$$
$$+ F^{(p}_j F^{i)}_{[n}(\underset{2}{R} - \underset{2}{P})_{pm]} + 2\Gamma^p_{\underset{\vee}{nm}}\Gamma^s_{sp}\delta^i_j + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j$$
$$- 2\Gamma^{(p}_{\underset{\vee}{[nq}}\Gamma^s_{sp}F^q_{(m]}F^{i)}_{j)} - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_{(p}F^i_{j)} - 2\Gamma^p_{\underset{\vee}{j[n}}\Gamma^s_{sq}F^i_{(p}F^q_{m])}.$$

It is easy to prove that the quantity $HP\underset{2}{W}{}^i_{jmn}$ is not a tensor, so we shall call it an *equitorsion holomorphically projective parameter of the second kind* of the space $G\underset{1}{K}_N$. And now we can formulate

Theorem 4.2. *The equitorsion holomorphically projective parameter of the second kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{1}{K}_N$ *and* $G\underset{1}{\overline{K}}_N$*.*

4.3. Holomorphically projective parameter of the third kind

The connection between the curvature tensors $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ is given by:

$$(4.31)\quad \underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} + 2P^p_{nm}\Gamma^i_{\underset{\vee}{pj}} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{\underset{\vee}{pj}}.$$

With the help of (4.1) and (4.2) we see that the tensor deformation (3.5) is symmetric, i.e. $P^i_{jk} = P^i_{kj}$. Now we can write

$$(4.32)\qquad \underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{j[m}P^i_{n]p} + 2P^p_{nm}\Gamma^i_{\underset{\vee}{pj}} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{\underset{\vee}{pj}}.$$

Replacing (3.5), (3.7) and (4.2) in (4.32) we have:

$$\begin{aligned}(4.33)\qquad \underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) - \delta^i_n \underset{1}{\psi}_{jm}\\ &+ F^p_j(F^i_n\underset{1}{\psi}_{pm} - F^i_m\underset{2}{\psi}_{pn}) + F^i_j(F^p_n\underset{1}{\psi}_{pm} - F^p_m\underset{2}{\psi}_{pn}) + 2\Gamma^i_{\underset{\vee}{(mj}}\psi_{n)}\\ &- 2\Gamma^i_{\underset{\vee}{[pj}}\psi_q F^q_{(n]}F^p_{m)} - 2\Gamma^{(p}_{\underset{\vee}{nq}}\psi_p F^q_{(m}F^{i)}_{j)} - 2\Gamma^p_{\underset{\vee}{mn}}\psi_q F^q_{(p}F^i_{j)},\end{aligned}$$

where we denoted

$$(4.34)\qquad \underset{\theta}{\psi}_{ij} = \psi_{i\underset{\theta}{|}j} - \psi_i\psi_j + \psi_p F^p_i\psi_q F^q_j \quad (\theta = 1, 2).$$

It is easy to prove that $\underset{2}{\psi}_{[mn]} = \underset{1}{\psi}_{[mn]} + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p$. Using the procedure given in the two previous cases we get

$$(4.35)\qquad (N+2)\underset{1}{\psi}_{jm} = \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} + \underset{3}{\overline{P}}_{jm} - \underset{3}{P}_{jm},$$

where

$$(4.36)\qquad \underset{3}{P}_{jm} = \underset{1}{P}_{jm}, \quad \underset{3}{\overline{P}}_{jm} = \underset{1}{\overline{P}}_{jm}.$$

The expressions for $\underset{1}{P}$ and $\underset{1}{\overline{P}}$ are given by (4.14). Eliminating $\underset{1}{\psi}_{jm}$ from (4.33) we get

$$(4.37)\qquad HP\underset{3}{\overline{W}}{}^i_{jmn} = HP\underset{3}{W}{}^i_{jmn},$$

where we denoted

$$
\begin{aligned}
HP\underset{3}{W}{}^{i}{}_{jmn} = \underset{3}{R}{}^{i}{}_{jmn} &+ \frac{1}{N+2}\big[\delta^{i}_{[m}(\underset{3}{R} - \underset{1}{P})_{jn]} + \delta^{i}_{j}(\underset{3}{R}_{[mn]} - \underset{1}{P}_{[mn]}) \\
&+ F^{(p}_{j}F^{i)}_{[n}(\underset{3}{R} - \underset{1}{P})_{pm]} + 2\Gamma^{p}_{\underset{\vee}{(jn}}\Gamma^{q}_{qp}\delta^{i}_{m)} + 2\Gamma^{i}_{\underset{\vee}{(mj}}\Gamma^{p}_{pn)} - 2\Gamma^{i}_{\underset{\vee}{pj}}\Gamma^{s}_{sq}F^{q}_{(n}F^{p}_{m)} \\
&- 2\Gamma^{p}_{\underset{\vee}{mn}}\Gamma^{s}_{sq}F^{q}_{(p}F^{i}_{j)} - 2\Gamma^{i}_{\underset{\vee}{nq}}\Gamma^{s}_{sp}F^{q}_{(j}F^{p}_{m)} - 2\Gamma^{p}_{\underset{\vee}{jn}}\Gamma^{s}_{sq}F^{i}_{(p}F^{q}_{m)}\big].
\end{aligned} \tag{4.38}
$$

Of course, $HP\underset{3}{\overline{W}}{}^{i}_{jmn}$ is expressed by geometric objects of the space $G\underset{1}{\overline{K}}_N$. It is not a tensor, so we shall call it an *equitorsion holomorphically projective parameter of the third kind* of the space $G\underset{1}{K}_N$. Finally, the next theorem is proved:

Theorem 4.3. *The equitorsion holomorphically projective parameter of the third kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{1}{K}_N$ *and* $G\underset{1}{\overline{K}}_N$.

4.4, 4.5. Holomorphically projective parameters of the fourth and fifth kind

The connections between the curvature tensors $\underset{4}{R}$ and $\underset{4}{\overline{R}}$, and the curvature tensors $\underset{5}{R}$ and $\underset{5}{\overline{R}}$ of the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ are given by:

$$
\underset{4}{\overline{R}}{}^{i}_{jmn} = \underset{4}{R}{}^{i}{}_{jmn} + P^{i}_{jm\underset{2}{|}n} - P^{i}_{nj\underset{1}{|}m} + P^{p}_{j[m}P^{i}_{n]p} + 2P^{p}_{mn}\Gamma^{i}_{\underset{\vee}{pj}} + 2\Gamma^{p}_{\underset{\vee}{mn}}P^{i}_{\underset{\vee}{pj}}, \tag{4.39}
$$

$$
\underset{5}{\overline{R}}{}^{i}_{jmn} = \underset{5}{R}{}^{i}{}_{jmn} + \frac{1}{2}(P^{i}_{[mj\underset{1}{|}n]} + P^{i}_{[mj\underset{2}{|}n]} + 2P^{p}_{j[m}P^{i}_{pn]} + 4\Gamma^{p}_{\underset{\vee}{j(n}}P^{i}_{\underset{\vee}{pm)}}). \tag{4.40}
$$

For the holomorphically projective parameters of the fourth and of the fifth kind we can do the same procedure that we used in the previous three cases, for the holomorphically projective parameters of the first, second and third kind. It is easy to prove that

$$
\underset{4}{P}_{jm} = \underset{1}{P}_{jm}, \quad \underset{4}{\overline{P}}_{jm} = \underset{1}{\overline{P}}_{jm}, \tag{4.41}
$$

where $\underset{1}{P}$ and $\underset{1}{\overline{P}}$ are given by (4.14). In the end we get for the fourth kind

$$
HP\underset{4}{\overline{W}}{}^{i}_{jmn} = HP\underset{4}{W}{}^{i}_{jmn}, \tag{4.42}
$$

where we introduced *equitorsion holomorphically projective parameter of the fourth kind*

$$
\begin{aligned}
HP\underset{4}{W}{}^{i}{}_{jmn} = \underset{4}{R}{}^{i}_{jmn} + \frac{1}{N+2}[&\delta^{i}_{[m}(\underset{4}{R} - \underset{1}{P})_{jn]} + \delta^{i}_{j}(\underset{4}{R}_{[mn]} - \underset{1}{P}_{[mn]}) \\
(4.43) \qquad &+ F^{(p}_{j}F^{i)}_{[n}(\underset{4}{R} - \underset{1}{P})_{pm]} + 2\Gamma^{p}_{\underset{\vee}{(jn}}\Gamma^{q}_{qp}\delta^{i}_{m)} + 2\Gamma^{i}_{\underset{\vee}{(mj}}\Gamma^{p}_{pn)} - 2\Gamma^{i}_{\underset{\vee}{pj}}\Gamma^{s}_{sq}F^{q}_{(n}F^{p}_{m)} \\
&- 2\Gamma^{p}_{\underset{\vee}{mn}}\Gamma^{s}_{sq}F^{q}_{(p}F^{i}_{j)} - 2\Gamma^{i}_{\underset{\vee}{nq}}\Gamma^{s}_{sp}F^{q}_{(j}F^{p}_{m)} - 2\Gamma^{p}_{\underset{\vee}{jn}}\Gamma^{s}_{sq}F^{i}_{(p}F^{q}_{m)}],
\end{aligned}
$$

and we have proved the following theorem:

Theorem 4.4. *The equitorsion holomorphically projective parameter of the fourth kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{1}{K}_N$ *and* $G\underset{1}{\overline{K}}_N$.

Replacing (3.5), (3.7) and (4.2) in (4.40) we have:

$$
\begin{aligned}
(4.44) \quad \underset{5}{\overline{R}}{}^{i}_{jmn} = \underset{5}{R}{}^{i}_{jmn} &+ \delta^{i}_{[m}\underset{12}{\psi}_{jn]} + \delta^{i}_{j}\underset{12}{\psi}_{[mn]} + F^{(p}_{j}F^{i)}_{[n}\underset{12}{\psi}_{pm]} \\
&- \Gamma^{(p}_{\underset{\vee}{nq}}\psi_{p}F^{q}_{(m}F^{i)}_{j)} - 2\Gamma^{p}_{\underset{\vee}{mn}}\psi_{q}F^{q}_{(p}F^{i}_{j)} - \Gamma^{p}_{\underset{\vee}{j[n}}\psi_{q}F^{i}_{(p}F^{q}_{m])} + \Gamma^{(p}_{\underset{\vee}{mq}}\psi_{p}F^{q}_{(j}F^{i)}_{n)},
\end{aligned}
$$

where we denote

$$
(4.45) \qquad \underset{12}{\psi}_{jm} = \frac{1}{2}(\psi_{j\underset{1}{|}m} + \psi_{j\underset{2}{|}m}) - \psi_{j}\psi_{m} + \psi_{p}F^{p}_{j}\psi_{q}F^{q}_{m}.
$$

Contracting with respect to the indices i, n in (4.44) we get

$$
(4.46) \qquad \underset{5}{\overline{R}}_{jm} = \underset{5}{R}_{jm} - \underset{12}{\psi}_{[jm]} - N\underset{12}{\psi}_{jm} - F^{p}_{j}F^{q}_{m}\underset{12}{\psi}_{(pq)} - \Gamma^{p}_{\underset{\vee}{(mr}}\psi_{q}F^{q}_{(p}F^{r}_{j))}.
$$

Anti-symmetrization without division in (4.46) with respect to the indices j, m gives:

$$
(4.47) \qquad (N+2)\underset{12}{\psi}_{[jm]} = \underset{5}{R}_{[jm]} - \underset{5}{\overline{R}}_{[jm]}.
$$

Symmetrization without division in (4.46) with respect to the indices j, m gives:

$$
(4.48) \qquad \underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} - N\underset{12}{\psi}_{(jm)} - 2F^{p}_{j}F^{q}_{m}\underset{12}{\psi}_{(pq)} - \Gamma^{p}_{\underset{\vee}{(mr}}\psi_{q}F^{q}_{(p}F^{r}_{j))}.
$$

The relation analogous to the relation (2.20) in the space $G\underset{1}{\overline{K}}_N$ is valid.

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and by use of the relation (2.20) in $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ from (4.48) we get

$$(4.49)\quad \underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} - N\underset{12}{\psi}_{(pq)}F^p_j F^q_m - 2\underset{12}{\psi}_{(jm)} - \Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^q_p F^r_{m)} - \Gamma^p_{\underset{\vee}{(jr}}\psi_q F^q_p F^r_{m)}.$$

From (4.48) and (4.49) we get:

$$(4.50)\qquad (N-2)F^p_j F^q_m \underset{12}{\psi}_{[pq]} = (N-2)\underset{12}{\psi}_{[jm]} + \Gamma^p_{\underset{\vee}{(mr}}\psi_q F^r_p F^q_{j)} - \Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^q_p F^r_{m)}.$$

Replacing (4.50) in (4.49) we get

$$(4.51)\qquad (N+2)\underset{12}{\psi}_{(jm)} = \underset{5}{R}_{(jm)} - \underset{5}{\overline{R}}_{(jm)} + \frac{2}{N-2}\left(\Gamma^p_{\underset{\vee}{qr}}\psi_{(j}F^q_p F^r_{m)} - (N-1)\Gamma^p_{\underset{\vee}{(jr}}\psi_q F^q_p F^r_{m)}\right).$$

Using (4.47) and (4.51) we have:

$$(4.52)\qquad (N+2)\underset{12}{\psi}_{jm} = \underset{5}{R}_{jm} - \underset{5}{\overline{R}}_{jm} + \underset{5}{\overline{P}}_{jm} - \underset{5}{P}_{jm},$$

where we denote

$$(4.53)\qquad \underset{5}{P}_{jm} = \frac{1}{N+2}\left(\frac{1}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{s(j}F^q_p F^r_{m)} - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{(jr}}\Gamma^s_{sq}F^q_p F^r_{m)}\right)$$

In the same way the object $\underset{5}{\overline{P}}_{jm}$ of the space $G\underset{1}{\overline{K}}_N$ is defined. Eliminating $\underset{12}{\psi}_{jm}$ from (4.44) we get

$$(4.54)\qquad HP\underset{5}{\overline{W}}{}^i_{jmn} = HP\underset{5}{W}{}^i_{jmn},$$

where

$$(4.55)\quad HP\underset{5}{W}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_{[m}(\underset{5}{R} - \underset{5}{P})_{jn]} + \delta^i_j(\underset{5}{R}_{[mn]} - \underset{5}{P}_{[mn]}) + F^{(p}_j F^{i)}_{[n}(\underset{5}{R} - \underset{5}{P})_{pm]} - \Gamma^{(p}_{\underset{\vee}{nq}}\Gamma^s_{sp}F^q_{(m}F^{i)}_{j)} - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_{(p}F^i_{j)} - \Gamma^p_{\underset{\vee}{j[n}}\Gamma^s_{sq}F^i_{(p}F^q_{m])} + \Gamma^{(p}_{\underset{\vee}{mq}}\Gamma^s_{sp}F^q_{(j}F^{i)}_{n)}.$$

This quantity $HP\underset{5}{W}{}^i_{jmn}$ is not a tensor, so we shall call it an *equitorsion holomorphically projective parameter of the fifth kind* of the space $G\underset{1}{K}_N$. And now we can formulate a theorem we have just proved:

Theorem 4.5. *The equitorsion holomorphically projective parameter of the fifth kind is an invariant of equitorsion holomorphically projective mapping which preserves the complex structure of the generalized Kählerian space* $G\underset{1}{K}_N$ *and* $G\underset{1}{\overline{K}}_N$.

5. Concluding remarks

1. For $g_{ij}(x) = g_{ji}(x)$ GR_N reduces to the Riemannian space R_N. The curvature tensors $\underset{\theta}{R}$, $\theta = 1, \dots, 5$ in generalized Riemannian space reduce to the single curvature tensor R in Riemannian space (in the symmetric case).
2. In the case of holomorphic mapping of the Kählerian spaces (in the symmetric case) $HP\underset{\theta}{W}{}^i{}_{jmn}$, $(\theta = 1, \dots, 5)$, given by the formulas (4.16), (4.30), (4.38), (4.43), (4.55) reduce to the holomorphically projective curvature tensor [20]

$$HPW^i{}_{jmn} = R^i_{jmn} + \frac{1}{N+2}(R_{j[n}\delta^i_{m]} + F^p_j R_{p[m}F^i_{n]} + 2F^i_j F^p_n R_{pm}).$$

3. In this paper by using the condition (2.3), non-symmetric metric tensor and equal torsion tensors in the spaces $G\underset{1}{K}_N$ and $G\underset{1}{\overline{K}}_N$ we get new quantities $HP\underset{\theta}{W}{}^i{}_{jmn}$, $(\theta = 1, \dots, 5)$ given by the formulas (4.16), (4.30), (4.38), (4.43), (4.55), and $\underset{1}{P}, \underset{2}{P}, \underset{5}{P}$ given by the formulas (4.14), (4.28), (4.53).

In the future work we can consider mappings between GK_N and $G\underset{1}{K}_N$, and probably get new quantities. All these quantities are interesting in constructions of new mathematical and physical structures.

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