Heiner Gonska; Radu Păltănea

Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 3, 783–799

Persistent URL: `http://dml.cz/dmlcz/140604`

Czechoslovak Mathematical Journal, 60 (135) (2010), 783–799

SIMULTANEOUS APPROXIMATION BY A CLASS OF BERNSTEIN-DURRMEYER OPERATORS PRESERVING LINEAR FUNCTIONS

Heiner Gonska, Duisburg, Radu Păltănea, Braşov

(Received March 2, 2009)

Abstract. We introduce and study a one-parameter class of positive linear operators constituting a link between the well-known operators of S. N. Bernstein and their genuine Bernstein-Durrmeyer variants. Several limiting cases are considered including one relating our operators to mappings investigated earlier by Mache and Zhou. A recursion formula for the moments is proved and estimates for simultaneous approximation of derivatives are given.

Keywords: positive linear operator, Bernstein-type operator, genuine Bernstein-Durrmeyer operator, simultaneous approximation, degree of approximation, moduli of continuity

MSC 2010: 41A10, 41A17, 41A25, 41A28, 41A36, 41A60

1. Introduction

Denote by $L_B[0,1]$ the space of bounded Lebesgue integrable functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n\in\mathbb{N}$. The operators $U_n\colon L_B[0,1]\to\Pi_n$, $n\geqslant 1$, given by

$$U_n(f,x)=(n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)\,\mathrm{d}t\right)\cdot p_{n,k}(x) + (1-x)^n f(0)+x^n f(1),\quad f\in L_B[0,1], \tag{1.1}$$

$$p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},\quad 0\leqslant k\leqslant n,\ \ k,n\in\mathbb{N},\ \ x\in[0,1], \tag{1.2}$$

were introduced simultaneously by Chen [2] and Goodman and Sharma [8] and studied, also in multidimensional settings, among others by Parvanov and Popov [13], Sauer [18], Gavrea [5], Păltănea [16], Waldron [19], Gonska, Kacsó and Raşa [7]. The operators U_n are limits of the Bernstein-Durrmeyer operators with Jacobi weights, $M_n^{a,b}$, $a, b > -1$, namely $U_n(f) = \lim\limits_{\substack{a\to-1\\b\to-1}} M_n^{a,b}(f)$, $f \in C[0,1]$, $n \in \mathbb{N}$, where

$$M_n^{a,b}(f,x) = \sum_{k=0}^{n} \frac{\int_0^1 f(t)t^a(1-t)^b p_{n,k}(t)\,\mathrm{d}t}{\int_0^1 t^a(1-t)^b p_{n,k}(t)\,\mathrm{d}t} \cdot p_{n,k}(x). \tag{1.3}$$

For $a = 0$ and $b = 0$, the operators $M_n^{a,b}$ reduce to the operators introduced independently by Durrmeyer [4] and Lupaş [10] and intensively studied also by Derrennic [3]. The operators $M_n^{a,b}$, studied by Păltănea [14], [15], Berens and Xu [1] and others have attractive properties of approximation, very similar to those of the Durrmeyer-Lupaş operators, including the characteristic property of representation as modified partial Fourier sums. However, for positive linear operators this property is incompatible with that of preserving linear functions. In contrast to the operators $M_n^{a,b}$, the limiting mappings U_n preserve linear functions. For this reason and for several other remarkable properties, some authors name the operators U_n the "genuine" Bernstein-Durrmeyer operators.

It is clear from the definition of U_n that they are closely related to the classical Bernstein operators given by

$$B_n(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) \cdot p_{n,k}(x), \quad f \in C[0,1]. \tag{1.4}$$

It is well known that these are also positive linear operators which reproduce linear functions, and there are thousands of articles and books dealing with their properties from various points of view.

To the authors' knowledge only trivial methods seem to be known bridging the gap between U_n and B_n, and such that important characteristics of the two like the reproduction of linear function, positivity, etc. are retained. One of these trivial methods is that of taking convex combinations $(1-\alpha)U_n + \alpha B_n$, $0 \leqslant \alpha \leqslant 1$.

In the present paper we introduce and investigate a one parameter class of Bernstein-Durrmeyer-type operators U_n^{ϱ}, $0 < \varrho < \infty$, which constitute a non-trivial link between U_n $(\varrho = 1)$ and B_n $(\varrho \to \infty)$. The U_n^{ϱ}, $0 < \varrho < \infty$, turn out to be the weak limits of certain operators studied earlier by Mache and Zhou [11]. As in their article, Euler's Beta function plays a fundamental role in our construction.

Our further investigation focusses on the moments and their recursion formula, the images of the monomial (Section 3) and in particular on the degree of simultaneous approximation (Section 5).

2. Definition and limiting cases

The following definition was first given in [17].

Definition 2.1. Let $\varrho > 0$ and $n \in \mathbb{N}$. Define an operator $U_n^\varrho\colon L_B[0,1] \to \Pi_n$ for $f \in L_B[0,1]$ and $x \in [0,1]$ by

$$\begin{aligned} U_n^\varrho(f,x) &:= \sum_{k=0}^{n} F_{n,k}^\varrho(f) \cdot p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 f(t)\mu_{n,k}^\varrho(t)\,\mathrm{d}t \right) \cdot p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n. \end{aligned} \tag{2.1}$$

Here, for $1 \leqslant k \leqslant n-1$,

$$\mu_{n,k}^\varrho(t) := \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)}, \tag{2.2}$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \quad x,y > 0, \tag{2.3}$$

is Euler's Beta function.

In what follows we write $e_j(t) = t^j$, $t \in [0,1]$, for $j \geqslant 0$.

Remark 2.2. Basic properties of the positive functionals $F_{n,k}^\varrho\colon L_B[0,1] \to \mathbb{R}$, $\varrho > 0$, are the following:

$$F_{n,k}^\varrho(e_0) = 1, \quad F_{n,k}^\varrho(e_1) = \frac{k}{n}, \quad 0 \leqslant k \leqslant n. \tag{2.4}$$

This implies

$$U_n^\varrho(e_0) = e_0, \quad U_n^\varrho(e_1) = e_1, \tag{2.5}$$

i.e., the operators U_n^ϱ preserve linear functions.

It is easily seen that, for $\varrho = 1$, we obtain $U_n^1 = U_n$. On the other hand, we have the following interesting limiting behavior.

Theorem 2.3. *For any $f \in C[0,1]$ we have*

$$\lim_{\varrho\to\infty} U_n^{\varrho}(f) = B_n(f), \quad \textit{uniformly}. \tag{2.6}$$

P r o o f. Let $f \in C[0,1]$ and let $n \in \mathbb{N}$ be fixed. It is sufficient to show that for fixed k and n with $1 \leqslant k \leqslant n-1$ one has

$$\lim_{\varrho\to\infty} F_{n,k}^{\varrho}(f) = f\Big(\frac{k}{n}\Big).$$

But this is a consequence of Korovkin's famous theorem, applied to the situation

$$F_{n,k}^{\varrho}\colon\ C[0,1] \to C\Big[\frac{k}{n}, \frac{k}{n}\Big].$$

Indeed, first we have relations (2.4). Moreover, using the representation of the Beta function in terms of the Gamma function, it is easy to see that for $\varrho > 0$ one has

$$F_{n,k}^{\varrho}(e_2) = \frac{k}{n} \cdot \frac{k+1/\varrho}{n+1/\varrho} \to \frac{k^2}{n^2} \quad \text{for } \varrho \to \infty. \tag{2.7}$$

Hence for all $f \in C[0,1]$

$$F_{n,k}^{\varrho}(f) \to f\Big(\frac{k}{n}\Big) \quad \text{for } \varrho \to \infty. \tag{2.8}$$

□

Next we show that the operators U_n^{ϱ} are the limit case of the operators studied by Mache and Zhou [11], which we denote by $P_n^{\varrho,a,b}$. These operators are given, for $n \in \mathbb{N}$, $\varrho > 0$, $a > -1$, $b > -1$, by

$$\begin{aligned} P_n^{\varrho,a,b}(f,x) &= \sum_{k=0}^{n} T_{n,k}^{\varrho,a,b}(f) \cdot p_{n,k}(x), \\ T_{n,k}^{\varrho,a,b}(f) &= \frac{\int_0^1 t^{k\varrho+a}(1-t)^{(n-k)\varrho+b} f(t)\,\mathrm{d}t}{B(k\varrho+a+1, (n-k)\varrho+b+1)}, \end{aligned}$$

where $f \in C[0,1]$, $x \in [0,1]$.

Theorem 2.4. *For any $f \in C[0,1]$ and $\varrho > 0$ we have*

$$\lim_{\substack{a\to -1\\ b\to -1}} P_n^{\varrho,a,b}(f) = U_n^{\varrho}(f), \quad \textit{uniformly.} \tag{2.9}$$

P r o o f. Fix f, ϱ and n. We have to prove

$$\lim_{\substack{a\to -1\\ b\to -1}} T_{n,k}^{\varrho,a,b}(f) = F_{n,k}^{\varrho}(f) \quad \text{for } 0 \leqslant k \leqslant n. \tag{2.10}$$

For $1 \leqslant k \leqslant n-1$ this limit is immediate. For $k = 0$ let us denote $R = T_{n,0}^{\varrho,a,b}(f) - f(0)$. We have

$$R = \int_0^1 \frac{(f(t) - f(0))t^a(1-t)^{n\varrho+b}}{B(a+1, n\varrho+b+1)}\,\mathrm{d}t.$$

Let $\varepsilon > 0$ be arbitrarily given. There exists $0 < \delta < 1$ such that $|f(t) - f(0)| < \frac{1}{2}\varepsilon$ for any $t \in [0,\delta]$. Then

$$\left|\int_0^\delta \frac{(f(t) - f(0))t^a(1-t)^{n\varrho+b}}{B(a+1, n\varrho+b+1)}\,\mathrm{d}t\right| < \frac{1}{2}\varepsilon, \quad a, b > -1.$$

Also

$$\left|\int_\delta^1 \frac{(f(t) - f(0))t^a(1-t)^{n\varrho+b}}{B(a+1, n\varrho+b+1)}\,\mathrm{d}t\right| \leqslant 2\|f\| \int_\delta^1 \frac{t^a(1-t)^{n\varrho+b}}{B(a+1, n\varrho+b+1)}\,\mathrm{d}t.$$

Let $a \in (-1, 0)$, $b \in (-1, 0)$. First, we have

$$\int_\delta^1 t^a(1-t)^{n\varrho+b}\,\mathrm{d}t \leqslant \delta^a \int_\delta^1 (1-t)^{n\varrho+b}\,\mathrm{d}t \leqslant \delta^{-1}B(1, n\varrho).$$

On the other hand, we have

$$\begin{aligned} B(a+1, n\varrho+b+1) &\geqslant \int_0^1 t^a(1-t)^{n\varrho}\,\mathrm{d}t \geqslant \int_0^{1/2} t^a(1-t)^{n\varrho}\,\mathrm{d}t \\ \geqslant \Big(\frac{1}{2}\Big)^{n\varrho} \int_0^{1/2} t^a\,\mathrm{d}t &= (a+1)^{-1}\Big(\frac{1}{2}\Big)^{n\varrho+a+1} \geqslant (a+1)^{-1}\Big(\frac{1}{2}\Big)^{n\varrho+1}. \end{aligned}$$

It follows that if $|a+1|$ is sufficiently small, then

$$2\|f\| \int_\delta^1 \frac{t^a(1-t)^{n\varrho+b}}{B(a+1, n\varrho+b+1)}\,\mathrm{d}t < \frac{1}{2}\varepsilon$$

and hence $|R| < \varepsilon$. For $k = n$ the proof is similar, using a symmetric argument. $\square$

3. Moments and their recursion

Below we will repeatedly use the function $\Psi(t)=t(1-t)$, $t\in[0,1]$.

Theorem 3.1. *For $x,y\in[0,1]$ we have*

$$U_n^\varrho(e_0,x)=1,\quad U_n^\varrho(e_1-ye_0,x)=x-y, \tag{3.1}$$

and for $r\geqslant 1$

$$\begin{aligned}
&U_n^\varrho((e_1-ye_0)^{r+1},x)\\
&\quad=\frac{\varrho\Psi(x)}{n\varrho+r}\cdot(U_n^\varrho((e_1-ye_0)^r,x))'_x+\frac{(1-2y)r+n\varrho(x-y)}{n\varrho+r}\cdot U_n^\varrho((e_1-ye_0)^r,x)\\
&\qquad+\frac{r\Psi(y)}{n\varrho+r}\cdot U_n^\varrho((e_1-ye_0)^{r-1},x).
\end{aligned}$$

Proof. The first two relations are immediate, see Remark 2.2. Let now $n\geqslant 2$ and fix $1\leqslant k\leqslant n-1$ and $r\geqslant 1$. Let $x\in[0,1]$. We have

$$\begin{aligned}
&(k-nx)F_{n,k}^\varrho((e_1-y)^r)\\
&\quad=\int_0^1(k-nt)(t-y)^r\mu_{n,k}^\varrho(t)\,\mathrm{d}t+nF_{n,k}^\varrho((e_1-y)^{r+1})+n(y-x)F_{n,k}^\varrho((e_1-y)^r).
\end{aligned}$$

The following relation is immediate:

$$(\mu_{n,k}^\varrho(t)\Psi(t))'_t=\varrho(k-nt)\mu_{n,k}^\varrho(t),\quad 1\leqslant k\leqslant n-1.$$

Consequentely, integrating by parts we obtain

$$\begin{aligned}
\int_0^1(k-nt)(t-y)^r\mu_{n,k}^\varrho(t)\,\mathrm{d}t&=\frac{1}{\varrho}\int_0^1(\mu_{n,k}^\varrho(t)\Psi(t))'_t(t-y)^r\,\mathrm{d}t\\
&=-\frac{r}{\varrho}\int_0^1\mu_{n,k}^\varrho(t)\Psi(t)(t-y)^{r-1}\,\mathrm{d}t.
\end{aligned}$$

Using the identity $t(1-t)=-(t-y)^2+(1-2y)(t-y)+y(1-y)$, we arrive at

$$\begin{aligned}
&(k-nx)F_{n,k}^\varrho((e_1-y)^r)\\
&\quad=\frac{n\varrho+r}{\varrho}\cdot F_{n,k}^\varrho((e_1-y)^{r+1})+\frac{r(2y-1)+(y-x)n\varrho}{\varrho}\cdot F_{n,k}^\varrho((e_1-y)^r)\\
&\qquad-\frac{r\Psi(y)}{\varrho}\cdot F_{n,k}^\varrho((e_1-y)^{r-1}).
\end{aligned}$$

It is immediately seen that this relation holds also for $k=0$ and $k=n$. We multiply each of these relations by $p_{n,k}(x)$, add them and by taking into account the relation

$$\Psi(x)(U_n^\varrho(g,x))'_x = \sum_{k=0}^{n} p_{n,k}(x)(k-nx)F_{n,k}^\varrho(g), \quad g\in C[0,1],$$

we arrive at the claim of the theorem. □

For brevity we will write in the sequel $M_{n,r}(x) := M_{n,r}^\varrho(x) := U_n^\varrho((e_1 - xe_0)^r, x)$, $n \geqslant 1$, $r \geqslant 0$, $x \in [0,1]$. It is immediate that

$$(M_{n,r}(x))' = (U_n^\varrho((e_1-ye_0)^r,x))'_x|_{y=x} - rM_{r-1}(x). \tag{3.2}$$

Using (3.2) and substituting $y=x$ in Theorem 3.1 we arrive at

Corollary 3.2.

$$M_{n,0}(x)=1, \quad M_{n,1}(x)=0,$$

and, for $r\geqslant 1$,

$$\begin{aligned} M_{n,r+1}(x) = {} & \frac{r(\varrho+1)\Psi(x)}{n\varrho+r}\cdot M_{n,r-1}(x) \\ & + \frac{r(1-2x)}{n\varrho+r}\cdot M_{n,r}(x) + \frac{\varrho\Psi(x)}{n\varrho+r}\cdot (M_{n,r}(x))'. \end{aligned} \tag{3.3}$$

Here are some important particular cases:

Corollary 3.3.

$$\begin{aligned} M_2(x) &= \frac{(\varrho+1)x(1-x)}{n\varrho+1}, \\ M_3(x) &= \frac{(\varrho+1)(\varrho+2)x(1-x)(1-2x)}{(n\varrho+1)(n\varrho+2)}, \\ M_4(x) &= \frac{3\varrho(\varrho+1)^2\Psi^2(x)n}{(n\varrho+1)(n\varrho+2)(n\varrho+3)} \\ &\quad + \frac{-6(\varrho+1)(\varrho^2+3\varrho+3)\Psi^2(x)+(\varrho+1)(\varrho+2)(\varrho+3)\Psi(x)}{(n\varrho+1)(n\varrho+2)(n\varrho+3)}. \end{aligned}$$

Next we give representations of the images of the monomials under the U_n^ϱ. Writing for shortness also $T_{n,r}(x) := T_{n,r}^\varrho(x) := U_n^\varrho(e_r,x)$, $n\geqslant 1$, $r\geqslant 0$, $x\in[0,1]$, we have

Corollary 3.4.

$$T_{n,0}(x) = 1, \quad T_{n,1}(x) = x,$$

and for $r \geqslant 1$

$$T_{n,r+1}(x) = \frac{n\varrho x + r}{n\varrho + r} \cdot T_{n,r}(x) + \frac{\varrho \Psi(x)}{n\varrho + r} \cdot (T_{n,r}(x))'. \tag{3.4}$$

In the next lemma we give a more explicit representation of the polynomials $T_{n,r}(x)$. We use the following notation: if a is a real number and $r \geqslant 0$ is an integer, then put $(a)_r := a(a-1)\dots(a-r+1)$ and $[a]_r := a(a+1)\dots(a+r-1)$. Hence $(a)_0 = 1 = [a]_0$. Also, for $a \geqslant 0$ and an integer $r > 0$, define $(a)_{-r} = ((a+1)\dots(a+r))^{-1}$. Moreover, for $m < j$ we put $\binom{m}{j} = 0$.

Lemma 3.5. *For* $n \geqslant 1$, $r \geqslant 0$, $x \in [0,1]$, *we have:*

$$T_{n,r}(x) = A_{n,r}x^r + B_{n,r}x^{r-1} + C_{n,r}x^{r-2} + R_{n,r}(x), \quad \text{where} \tag{3.5}$$

$$A_{n,r} = \varrho^r \cdot \frac{(n)_r}{[n\varrho]_r}, \tag{3.6}$$

$$B_{n,r} = \varrho^{r-1} \cdot \frac{(n)_{r-1}}{[n\varrho]_r} \cdot (\varrho+1)\binom{r}{2}, \tag{3.7}$$

$$C_{n,r} = \varrho^{r-2} \cdot \frac{(n)_{r-2}}{[n\varrho]_r} \cdot (\varrho+1)\left[3(\varrho+1)\binom{r}{4} + (\varrho+2)\binom{r}{3}\right] \tag{3.8}$$

and $R_{n,r}$ *is a polynomial of degree* $r-3$.

P r o o f. From (3.4) it follows immediately by induction that $T_{n,r}(x)$ are polynomials of degree r in x. It remains to prove the representations claimed. For $r = 0$ the correctness of the coefficients is immediately verified. Using formula (3.4) we arrive, for $r \geqslant 0$, at the following equations by equating coefficients:

$$\begin{aligned}
A_{n,r+1} &= \frac{\varrho(n-r)}{n\varrho + r} \cdot A_{n,r}, \\
B_{n,r+1} &= \frac{r(\varrho+1)}{n\varrho + r} \cdot A_{n,r} + \frac{\varrho(n-r+1)}{n\varrho + r} \cdot B_{n,r}, \\
C_{n,r+1} &= \frac{(\varrho r - \varrho + r)}{n\varrho + r} \cdot B_{n,r} + \frac{\varrho(n-r+2)}{n\varrho + r} \cdot C_{n,r}.
\end{aligned}$$

By the induction hypothesis we infer:

$$\begin{aligned}
A_{n,r+1} &= \varrho^r \cdot \frac{\varrho(n-r)}{n\varrho + r} \cdot \frac{(n)_r}{[n\varrho]_r} = \varrho^{r+1} \cdot \frac{(n)_{r+1}}{[n\varrho]_{r+1}}, \\
B_{n,r+1} &= \varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+1}}\left[r + \binom{r}{2}\right] = \varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+1}} \cdot \binom{r+1}{2}, \\
C_{n,r+1} &= \varrho^{r-1}(\varrho+1) \cdot \frac{(n)_{r-1}}{[n\varrho]_{r+1}} \cdot \left[(\varrho r - \varrho + r)\binom{r}{2} + 3(\varrho+1)\binom{r}{4} + (\varrho+2)\binom{r}{3}\right],
\end{aligned}$$

and

$$
\begin{aligned}
(\varrho r-\varrho+r)\binom{r}{2}&+3(\varrho+1)\binom{r}{4}+(\varrho+2)\binom{r}{3}\\
&=[(\varrho+1)(r-2)+\varrho+2]\binom{r}{2}+3(\varrho+1)\binom{r}{4}+(\varrho+2)\binom{r}{3}\\
&=(\varrho+1)\left[(r-2)\binom{r}{2}+3\binom{r}{4}\right]+(\varrho+2)\left[\binom{r}{2}+\binom{r}{3}\right]\\
&=3(\varrho+1)\binom{r+1}{4}+(\varrho+2)\binom{r+1}{3}.
\end{aligned}
$$

This completes the proof. □

4. Preservation of convexity

An important shape preservation feature of both the U_n and the B_n is that they preserve convexity of all orders. In fact, this very property of the B_n is the main reason for using Bézier curves as main (conceptual) tools in Computer Aided Geometric Design. In this section we first prove convexity preservation of all orders $k \geqslant -1$ by the operators U_n^ϱ.

We briefly recall the definition of the above notion (also in the hope to remove some confusion). The term *convex of order* $k \geqslant -1$ was used by Tiberiu Popoviciu in Romanian literature of the 1930's, and convexity of order $k=-1$, i.e., $k+1=0$, meant positivity to him. The reason for this is that positivity of a function can be expressed in terms of divided differences of order $k+1=-1+1$, i.e., in terms of the inequality

$$[f;x_0,\dots x_{k+1}] \geqslant 0 \quad \text{with } \ k=-1.$$

We also recall this concept. Let f be defined on a compact interval $[a,b]$, and let $k \geqslant -1$. Consider $k+1$ distinct points $x_i \in [a,b]$, $0 \leqslant i \leqslant k$, and denote by $[f;x_0,\dots x_k]$ the *divided difference of order* $k+1$ of f, relative to these $k+1$ points. A function f is named convex of order k if all its divided differences of order $k+1$ are not negative. Hence in what follows convexity of order -1 will mean positivity in the usual sense, convexity of order 0 will stand for monotonic increasement, and convexity of order $k=1$ will stand for convexity as used in the Western hemisphere. The reader should be aware of the fact that the terms *convexity/concavity* have always carried different mathematical connotations in the East and the West and normally require further explanation.

An operator acting on a space of functions defined over an interval $[a,b]$ is named convex of order $k \geqslant -1$ if it transforms any k-convex function from its domain of definition into a k-convex function in its range.

The next assertation shows that the bridging operators U_n^ϱ, $\varrho > 0$, $n \geqslant 1$, have this property.

Theorem 4.1. *The operators U_n^ϱ, $\varrho > 0$, $n \geqslant 1$, are convex of order $r-1$ for all $0 \leqslant r \leqslant n$.*

Proof. In order to show that the operators U_n^ϱ are convex of order $r-1$, $0 \leqslant r \leqslant n$, it suffices to show that we have $(U_n^\varrho(f))^{(r)} \geqslant 0$ for any function $f \in C^r[0,1]$ such that $f^{(r)} \geqslant 0$. For $r=0$ this fact is obvious, since U_n^ϱ is positive. So let $r \geqslant 1$ and fix a function $f \in C^r[0,1]$ such that $f^{(r)} \geqslant 0$.

We make the convention that if the integers m, q do not satisfy the condition $0 \leqslant q \leqslant m$, then $p_{m,q}(x) = 0$ for $x \in [0,1]$. The following formula can be proved by induction with regard to r:

$$(p_{n,k}(x))^{(r)} = \frac{n!}{(n-r)!} \sum_{j=0}^{r} \binom{r}{j} (-1)^j p_{n-r,k-r+j}(x), \quad 0 \leqslant k \leqslant n, \ \ x \in [0,1].$$

We thus obtain

$$\begin{aligned}(U_n^\varrho(f,x))^{(r)} &= \frac{n!}{(n-r)!} \sum_{k=0}^{n} \sum_{j=0}^{r} \binom{r}{j} (-1)^j p_{n-r,k-r+j}(x) F_{n,k}^\varrho(f) \\ &= \frac{n!}{(n-r)!} \sum_{l=0}^{n-r} p_{n-r,l}(x) \Phi_{n,l}(f),\end{aligned}$$

where the functionals $\Phi_{n,l}$, $0 \leqslant l \leqslant n-r$ are defined by

$$\Phi_{n,l}(g) := \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} F_{n,l+i}^\varrho(g), \quad g \in C[0,1].$$

In order to prove that $(U_n^\varrho(f,x))^{(r)} \geqslant 0$, $x \in [0,1]$, it suffices to prove $\Phi_{n,l}(f) \geqslant 0$ for $0 \leqslant l \leqslant n-r$.

We have

$$\Phi_{n,l}(e_j) = 0, \quad 0 \leqslant j < r, \ \ 0 \leqslant l \leqslant n-r. \tag{4.1}$$

Indeed, for $j = 0$ we use the fact that $F^{\varrho}_{n,l+i}(e_0) = 1$, $0 \leqslant i \leqslant r$, $0 \leqslant l \leqslant n - r$. If $j \geqslant 1$ we have

$$
\begin{aligned}
\Phi_{n,l}(e_j) &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} F^{\varrho}_{n,l+i}(e_j) \\
&= \frac{\Gamma(n\varrho)}{\Gamma(n\varrho + j)} \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} (t^{(l+i)\varrho+j-1})^{(j)}|_{t=1} \\
&= (-1)^r \frac{\Gamma(n\varrho)}{\Gamma(n\varrho + j)} (t^{l\varrho+j-1}(1 - t^{\varrho})^r)^{(j)}|_{t=1} \\
&= (-1)^r \frac{\Gamma(n\varrho)}{\Gamma(n\varrho + j)} \sum_{s=0}^{j} \binom{j}{s} (t^{l\varrho+j-1})^{(j-s)}((1 - t^{\varrho})^r)^{(s)}|_{t=1}.
\end{aligned}
$$

Note that this calculus is valid also for $l = 0$ or $l = n - r$, where the functionals $F^{\varrho}_{n,0}$ and $F^{\varrho}_{n,n}$ appear, since we have $F^{\varrho}_{n,0}(e_j) = 0 = \Gamma(n\varrho)/\Gamma(n\varrho + j) \cdot (t^{j-1})^{(j)}|_{t=1}$ and $F^{\varrho}_{n,n}(e_j) = 1 = \Gamma(n\varrho)/\Gamma(n\varrho + j) \cdot (t^{\varrho n+j-1})^{(j)}|_{t=1}$.

Next, note the following formula, which can be proved by induction:

$$
((1 - t^{\varrho})^r)^{(s)} = (1 - t^{\varrho})^{r-s} Q_s(t), \quad 0 \leqslant s < r,
$$

where Q_s is an infinitely times differentiable function. This yields relation (4.1).

From the Taylor formula we obtain, for $t \in [0, 1]$:

$$
f(t) = \sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!} \cdot t^j + R_r(t), \quad \text{where } R_r(t) = \int_0^t \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)\,\mathrm{d}u.
$$

Using relation (4.1) we obtain, for $0 < l < n - r$:

$$
\begin{aligned}
\text{(4.2)} \quad \Phi_{n,l}(f) &= \Phi_{n,l}\bigg(\sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!} \cdot e_j + R_r\bigg) = \Phi_{n,l}(R_r) \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} \int_0^1 \mu^{\varrho}_{n,l+i}(t) \bigg(\int_0^t \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)\,\mathrm{d}u\bigg)\,\mathrm{d}t \\
&= \int_0^1 f^{(r)}(u) \bigg(\sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} \int_u^1 \mu^{\varrho}_{n,l+i}(t) \frac{(t-u)^{r-1}}{(r-1)!}\,\mathrm{d}t\bigg)\,\mathrm{d}u.
\end{aligned}
$$

In the case $l = 0$, for $i = 0$, since $F^{\varrho}_{n,0}(R_r) = 0$, we obtain a formula similar to (4.2) but in which the term $\int_u^1 \mu^{\varrho}_{n,0}(t)(t-u)^{r-1}/(r-1)!\,\mathrm{d}t$ must be replaced by 0. In the case $l = n-r$, for $i = r$, since $F^{\varrho}_{n,n}(R_r) = \int_0^1 (1-u)^{r-1}/(r-1)! \cdot f^{(r)}(u)\,\mathrm{d}u$, we obtain a formula similar to (4.2), but in which the term $\int_u^1 \mu_{n,n}^{\varrho}(t)(t-u)^{r-1}/(r-1)!\,\mathrm{d}t$ must be replaced by $(1-u)^{r-1}/(r-1)!$.

From relation (4.2) it follows that in order to prove that $\Phi_{n,l}(f) \geqslant 0$ for $0 \leqslant l \leqslant n-r$ it is sufficient to prove that $G_l(u) \geqslant 0$, $u \in [0,1]$, $0 \leqslant l \leqslant n-r$, where

$$G_l(u) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} G_{n,l+i}(u)$$

and

$$G_{n,l+i}(u) = \begin{cases} \displaystyle\int_u^1 \mu_{n,l+i}^{\varrho}(t) \frac{(t-u)^{r-1}}{(r-1)!}\,\mathrm{d}t, & 1 \leqslant l+i \leqslant n-1, \\ 0, & l+i=0, \\ \dfrac{(1-u)^{r-1}}{(r-1)!}, & l+i=n. \end{cases}$$

For $0 \leqslant k \leqslant r-1$ we obtain

$$(G_{n,l+i})^{(k)}(u) = \begin{cases} (-1)^k \displaystyle\int_u^1 \mu_{n,l+i}^{\varrho}(t) \frac{(t-u)^{r-1-k}}{(r-1-k)!}\,\mathrm{d}t, & 1 \leqslant l+i \leqslant n-1, \\ 0, & l+i=0, \\ (-1)^k \dfrac{(1-u)^{r-1-k}}{(r-1-k)!}, & l+i=n, \end{cases}$$

while

$$(G_{n,l+i})^{(r)}(u)) = \begin{cases} (-1)^r \mu_{n,l+i}^{\varrho}(u), & 1 \leqslant l+i \leqslant n-1, \\ 0, & l+i=0, \\ 0, & l+i=n. \end{cases}$$

For $0 \leqslant k \leqslant r-1$ and $0 \leqslant l \leqslant n-r$ we get

$$G_l^{(k)}(0) = 0, \tag{4.3}$$

$$G_l^{(k)}(1) = 0. \tag{4.4}$$

Indeed, relation (4.3) for $0 \leqslant k \leqslant r-1$ follows from formula (4.1) since $G_l^{(k)}(0) = (-1)^k/(r-1-k)! \cdot \Phi_{n,l}(e_{r-1-k})$. Relation (4.4) is immediate.

On the other hand, we can write for $0 \leqslant l \leqslant n-r$ and $u \in (0,1)$:

$$G_l^{(r)}(u) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(\frac{u}{1-u}\right)^{i\varrho} \cdot \frac{\Gamma(n\varrho) u^{l\varrho-1}(1-u)^{(n-l)\varrho-1}}{\Gamma((l+i)\varrho)\Gamma((n-l-i)\varrho)}, \tag{4.5}$$

with the observation that in the case $l=0$ the term for $i=0$ drops and in the case $l=n-r$ the term for $i=r$ drops. From relation (4.5) we obtain that the function $G_l^{(r)}$ has at most r changes of sign in the interval $(0,1)$. Now using relations (4.3), (4.4) and Rolle's theorem we infer, inductively, that the function $G_l^{(k)}$ has at most k changes of sign. Indeed, suppose the assertion to be true for k, $1 \leqslant k \leqslant r$ and let $x_1 < \ldots < x_m$ be the interior points where $G_l^{(k)}$ changes the sign, where, by the induction hypothesis, $m \leqslant k$. It follows that the function $G_l^{(k-1)}$ can have at most one point at which it changes the sign in each of the intervals (x_i, x_{i+1}), $1 \leqslant i \leqslant m-1$, but it cannot have a point of change of sign in the interval $(0, x_1)$ or in the interval $(x_m, 1)$, since $G_l^{(k-1)}(0) = 0$ and $G_l^{(k-1)}(1) = 0$. So, $G_l^{(k-1)}$ can have at most $m-1 \leqslant k-1$ changes of sign.

Finally, we obtain that G_l has no changes of sign, i.e., G_l has constant sign on the interval $[0,1]$. From the relation

$$G_l(u) = \int_u^1 \sum_{i=0}^{r} (-1)^{i+r} \binom{r}{i} \Big(\frac{t}{1-t}\Big)^{i\varrho} \cdot \frac{t^{l\varrho-1}(1-t)^{(n-l)\varrho-1}}{B((l+i)\varrho, (n-l-i)\varrho)} \frac{(t-u)^{r-1}}{(r-1)!} \,\mathrm{d}t$$

and the limit

$$\lim_{t\to 1} \sum_{i=0}^{r} (-1)^{i+r} \binom{r}{i} \Big(\frac{t}{1-t}\Big)^{i\varrho} \frac{1}{B((l+i)\varrho, (n-l-i)\varrho)} = \infty$$

we obtain that G_l is non negative on the interval $[0,1]$. The proof is complete. $\square$

The corollary below is a useful consequence of our previous observations. Its proof shows that operators that are convex of order k automatically map Π_k into Π_k.

Corollary 4.2. *We have*

$$U_n^{\varrho}(\Pi_k) \subset \Pi_k \quad \textit{for } 0 \leqslant k \leqslant n,\ \varrho > 0. \tag{4.6}$$

P r o o f. If $f \in \Pi_k$, $k < n$, then $f^{(k+1)} = 0$. Since U_n^{ϱ} is convex of order k, it follows that $(U_n^{\varrho}(f))^{(k+1)} \geqslant 0$ and $(U_n^{\varrho}(f))^{(k+1)} \leqslant 0$, i.e., $(U_n^{\varrho}(f))^{(k+1)} = 0$. This means that $U_n^{\varrho}(f) \in \Pi_k$. The case $k = n$ is obvious. $\square$

5. Degree of simultaneous approximation

We now give estimates for simultaneous approximation for the operators U_n^{ϱ} in terms of the first and second order moduli. There exist estimates for simultaneous approximation for general convex operators, using the classical second modulus, first given by Gonska [6], see also Kascó [9]. Here we use a similar method in order to obtain estimates using also other types of moduli. Consider operators $Q_{n,r}$ defined by $Q_{n,r} := D^r \circ U_n^\varrho \circ I_r$, where D^r denotes the differential operator of the rth order and I_r produces the corresponding antiderivatives, and is given by

$$I_r(f,x) := \int_0^x \frac{(x-t)^{r-1}}{(r-1)!} \cdot f(t)\,\mathrm{d}t, \quad f \in C[0,1], \ x \in [0,1].$$

The convexity of order $r-1$ of the operators U_n^ϱ ensures that the operators $Q_{n,r}$ are positive operators. Also, we have the important equality

$$Q_{n,r}(f^{(r)},x) = (D^r U_n^\varrho)(f,x) \quad \text{for } f \in C^r[0,1]. \tag{5.1}$$

We next find estimates for the pointwise difference $D^r U_n^\varrho(f,x) - f^{(r)}(x) = Q_{n,r}(f^{(r)},x) - f^{(r)}(x)$ using the first three moments of the operator $Q_{n,r}$. We use the abbreviations $\alpha_{n,r}^\varrho(x) = Q_{n,r}(e_0,x)$, $\beta_{n,r}^\varrho(x) = Q_{n,r}(e_1 - xe_0, x)$, and $\gamma_{n,r}^\varrho(x) = Q_{n,r}((e_1 - xe_0)^2, x)$.

Lemma 5.1. *For $n \geqslant 1$, $x \in [0,1]$ and $\varrho > 0$ we have*

$$\begin{aligned}
\alpha_{n,r}^\varrho(x) &= \varrho^r \cdot \frac{(n)_r}{[n\varrho]_r},\\
\beta_{n,r}^\varrho(x) &= \varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+1}} \cdot \frac{r}{2} \cdot (1-2x),\\
\gamma_{n,r}^\varrho(x) &= \varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+2}}\\
&\quad \times \Big((n\varrho - (\varrho+1)r(r+1))x(1-x) + \frac{r}{12} \cdot ((3r+1)(\varrho+1)+4)\Big).
\end{aligned}$$

Proof. We apply formula (5.1) and Lemma 3.5. We have

$$\begin{aligned}
\alpha_{n,r}^\varrho &= (D^r U_n^\varrho)\Big(\frac{1}{r!} \cdot e_r, x\Big) = A_{n,r} = \varrho^r \cdot \frac{(n)_r}{[n\varrho]_r},\\
\beta_{n,r}^\varrho &= (D^r U_n^\varrho)\Big(\frac{1}{(r+1)!} \cdot e_{r+1} - \frac{x}{r!} \cdot e_r, x\Big) = (A_{n,r+1} - A_{n,r})x + \frac{1}{r+1} \cdot B_{n,r+1}\\
&= \varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+1}} \cdot \frac{r}{2} \cdot (1-2x),\\
\gamma_{n,r}^\varrho &= (D^r U_n^\varrho)\Big(\frac{2}{(r+2)!} \cdot e_{r+2} - \frac{2x}{(r+1)!} \cdot e_{r+1} + \frac{x^2}{r!} \cdot e_r, x\Big)\\
&= (A_{n,r+2} - 2A_{n,r+1} + A_{n,r})x^2\\
&\quad + \Big(\frac{2}{r+2} \cdot B_{n,r+2} - \frac{2}{r+1} \cdot B_{n,r+1}\Big)x + \frac{2}{(r+1)(r+2)} \cdot C_{n,r+2}.
\end{aligned}$$

After simple computations we get for $\gamma_{n,r}^\varrho$ the form given in the claim. $\square$

We denote by ω_1, ω_2 the usual first order and second order moduli of continuity and by ω_1^φ, ω_2^φ the first and second order Ditzian-Totik moduli, where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Applying to the operator $Q_{n,r}$ and to the function $f^{(r)}$ the general estimates for positive linear operators, namely, the estimate with modulus ω_1 of Mond [12] and the estimates with the combination of the first and second order moduli given in Păltănea [16], we obtain directly the following results.

Theorem 5.2. *For $f \in C^r[0,1]$, $r \geqslant 0$, $x \in [0,1]$ and $0 < h \leqslant 1$, we have*

$$\tag{5.2} \begin{aligned} &|(U_n^\varrho(f,x) - f(x))^{(r)}| \\ &\quad \leqslant (1-\alpha_{n,r}^\varrho(x)) \cdot |f^{(r)}(x)| + \Big(\alpha_{n,r}^\varrho(x) + \frac{1}{h^2} \cdot \gamma_{n,r}^\varrho(x)\Big)\omega_1(f^{(r)}, h); \end{aligned}$$

for $f \in C^r[0,1]$, $r \geqslant 0$, $x \in [0,1]$ and $0 < h \leqslant \frac{1}{2}$, we obtain

$$\tag{5.3} \begin{aligned} &|(U_n^\varrho(f,x) - f(x))^{(r)}| \\ &\quad \leqslant (1-\alpha_{n,r}^\varrho(x)) \cdot |f^{(r)}(x)| + |\beta_{n,r}^\varrho(x)| \frac{1}{h}\omega_1(f^{(r)}, h) \\ &\quad\quad + \Big(\alpha_{n,r}^\varrho(x) + \frac{1}{2h^2} \cdot \gamma_{n,r}^\varrho(x)\Big)\omega_2(f^{(r)}, h); \end{aligned}$$

and for $f \in C^r[0,1]$, $r \geqslant 0$, $x \in (0,1)$ and $0 < h \leqslant \frac{1}{2}$, we have

$$\tag{5.4} \begin{aligned} &|(U_n^\varrho(f,x) - f(x))^{(r)}| \\ &\quad \leqslant (1-\alpha_{n,r}^\varrho(x)) \cdot |f^{(r)}(x)| + \frac{|\beta_{n,r}^\varrho(x)|}{2h\varphi(x)} \cdot \omega_1^\varphi(f^{(r)}, 2h) \\ &\quad\quad + \Big(\alpha_{n,r}^\varrho(x) + \frac{3}{2} \cdot \frac{\gamma_{n,r}^\varrho(x)}{(h\varphi(x))^2}\Big)\omega_2^\varphi(f^{(r)}, h), \end{aligned}$$

where the coefficients $\alpha_{n,r}^\varrho(x)$, $\beta_{n,r}^\varrho(x)$ and $\gamma_{n,r}^\varrho(x)$ are given in Lemma 5.1.

Remark 5.3. For $n\varrho \geqslant (\varrho+1)r(r+1)$ and $\varrho \geqslant 1$ the following upper bounds are valid:

$$1 - \alpha_{n,r}^\varrho(x) = 1 - \prod_{i=0}^{r-1} \frac{(n-r+1+i)\varrho}{n\varrho + i} \leqslant 1 - \Big(1 - \frac{r-1}{n}\Big)^r \leqslant \frac{r(r-1)}{n},$$

$$|\beta_{n,r}^\varrho(x)| = |\varrho^r(\varrho+1) \cdot \frac{(n)_r}{[n\varrho]_{r+1}} \cdot \frac{r}{2} \cdot (1-2x)| \leqslant \frac{\varrho+1}{\varrho} \cdot \frac{r}{2n},$$

$$\gamma_{n,r}^\varrho(x) \leqslant \frac{\varrho+1}{(n\varrho)^2}\Big(\frac{n\varrho}{4} + \frac{r(1-\varrho)}{6}\Big) \leqslant \frac{\varrho+1}{\varrho} \cdot \frac{1}{4n}.$$

Remark 5.4. For $r = 0$ we have $1 - \alpha_{n,r}^\varrho(x) = 0$ and $\beta_{n,r}^\varrho(x) = 0$, so that the estimates given in Theorems 5.2 and 5.5 reduce to estimates with a single term only.

Letting $\varrho \to \infty$, we obtain

Theorem 5.5. *The Bernstein operators satisfy estimates similar to those in Theorem 5.2, if we replace the quantities* $1-\alpha_{n,r}^{\varrho}(x)$, $|\beta_{n,r}^{\varrho}(x)|$ *and* $\gamma_{n,r}^{\varrho}(x)$ *by the quantities* $1-\alpha_{n,r}(x)$, $|\beta_{n,r}(x)|$ *and* $\gamma_{n,r}(x)$, *respectively, where*

$$\begin{aligned} 1-\alpha_{n,r}(x) &= 1-\frac{(n)_r}{n^r} \leqslant \frac{r(r-1)}{2n}, \\ |\beta_{n,r}(x)| &= \frac{(n)_r}{n^r} \cdot \frac{r}{2n} \cdot |1-2x| \leqslant \frac{r}{2n}, \\ \gamma_{n,r}(x) &= \frac{(n)_r}{n^r} \Big(\frac{n-r(r+1)}{n^2} \cdot x(1-x) + \frac{r(3r+1)}{12n^2} \Big) \leqslant \frac{1}{4n} \end{aligned}$$

and where the last inequality is true for $n \geqslant r(r+1)$.

The estimates given in Theorems 5.2 and 5.5 are quantitative results for pointwise and also global simultaneous approximation by the operators U_n^{ϱ} and B_n. Indeed, if we put for instance $h := 1/\sqrt{n}$, we obtain, with the notation $U_n^{\infty} = B_n$,

Theorem 5.6.

$$\lim_{n\to\infty} \|(U_n^{\varrho}(f))^{(r)} - f^{(r)}\|_{\infty} = 0 \tag{5.5}$$

for $\varrho \in (0,\infty]$ *and* $f \in C^r[0,1]$.

References

[1] *H. Berens, Y. Xu*: On Bernstein-Durrmeyer polynomials with Jacobi weights. In: Approximation Theory and Functional Analysis. Academic Press, Boston, 1991, pp. 25–46.

[2] *W. Chen*: On the modified Bernstein-Durrmeyer operator. Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China. 1987.

[3] *M. M. Derriennic*: Sur l'approximation de fonctions intégrables sur $[0,1]$ par des polynômes de Bernstein modifiés. J. Approx. Theory *31* (1981), 325–343. (In French.)

[4] *J. L. Durrmeyer*: Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments. Thèse de 3ème cycle. Faculté des Sciences Univ., Paris, 1967.

[5] *I. Gavrea*: The approximation of the continuous functions by means of some positive operators. Result. Math. *30* (1996), 55–66.

[6] *H. Gonska*: Quantitative Korovkin-type theorems on simultaneous approximation. Math. Z. *186* (1984), 419–433.

[7] *H. H. Gonska, D. Kacsó, I. Raşa*: On genuine Bernstein-Durrmeyer operators. Result. Math. *50* (2007), 213–225.

[8] *T. N. T. Goodman, A. Sharma*: A modified Bernstein-Schoenberg operator. Proc. Conf. Constructive Theory of Functions, Varna 1987 (Bl. Sendov et al., eds.). Publ. House Bulg. Acad. Sci., Sofia, 1988, pp. 166–173.

[9] *D. Kacsó*: Certain Bernstein-Durrmeyer type operators preserving linear functions. Habilitationschrift. Universität Duisburg-Essen, 2006.

[10] *A. Lupaş*: Die Folge der Betaoperatoren. Dissertation. Universität Stuttgart, 1972.

[11] *D. H. Mache, D. X. Zhou*: Characterization theorems for the approximation by a family of operators. J. Approx. Theory *84* (1996), 145–161.
[12] *B. Mond*: On the degree of approximation by linear positive operators. J. Approx. Theory *18* (1976), 304–306.
[13] *P. E. Parvanov, B. D. Popov*: The limit case of Bernstein's operators with Jacobi weights. Math. Balk. (N.S.) *8* (1994), 165–177.
[14] *R. Păltănea*: Sur un operateur polynômial défini sur l'ensemble des fonctions intégrables. Babeş Bolyai Univ., Fac. Math., Res. Semin. *2* (1983), 101–106. (In French.)
[15] *R. Păltănea*: Une propriété d'extrémalité des valeurs propres des opérateurs polynômiaux de Durrmeyer généralisés. Math., Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation *15* (1986), 57–64. (In French.)
[16] *R. Păltănea*: Approximation Theory Using Positive Linear Operators. Birkhäuser, Boston, 2004.
[17] *R. Păltănea*: A class of Durrmeyer type operators preserving linear functions. Annals of the Tiberiu Popoviciu Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca), vol. 5. 2007, pp. 109–118.
[18] *T. Sauer*: The genuine Bernstein-Durrmeyer operator on a simplex. Result. Math. *26* (1994), 99–130.
[19] *S. Waldron*: A generalised beta integral and the limit of the Bernstein-Durrmeyer operator with Jacobi weights. J. Approx. Theory *122* (2003), 141–150.

Authors' addresses: Heiner Gonska (corresponding author), University of Duisburg-Essen, D-47048 Duisburg, Germany, e-mail: `heiner.gonska@uni-due.de`; Radu Păltănea, Transilvania University, RO-500036 Braşov, Romania, e-mail: `radupaltanea@yahoo.com`.