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# ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $n$-TH ORDER EMDEN-FOWLER DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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*Abstract.* We study oscillatory properties of solutions of the Emden-Fowler type differential equation

$$u^{(n)}(t)+p(t)\big|u(\sigma(t))\big|^{\lambda}\operatorname{sign}u(\sigma(t))=0,$$

where $0<\lambda<1$, $p\in L_{\rm loc}(\mathbb{R}_+;\mathbb{R})$, $\sigma\in C(\mathbb{R}_+;\mathbb{R}_+)$ and $\sigma(t)\geqslant t$ for $t\in\mathbb{R}_+$.

Sufficient (necessary and sufficient) conditions of new type for oscillation of solutions of the above equation are established.

Some results given in this paper generalize the results obtained in the paper by Kiguradze and Stavroulakis (1998).

*Keywords*: proper solution, property **A**, property **B**

*MSC 2010*: 34K15, 34C10

## 1. Introduction

This work concerns the study of oscillatory properties of the differential equation

$$u^{(n)}(t)+p(t)\big|u(\sigma(t))\big|^{\lambda}\operatorname{sign}u(\sigma(t))=0, \tag{1.1}$$

where $p\in L_{\rm loc}(\mathbb{R}_+;\mathbb{R})$, $\sigma\in C(\mathbb{R}_+;\mathbb{R}_+)$ and

$$0<\lambda<1,\quad \sigma(t)\geqslant t\quad \text{for } t\in\mathbb{R}_+. \tag{1.2}$$

It will always be assumed that either the condition

$$p(t)\geqslant 0\quad \text{for } t\in\mathbb{R}_+ \tag{1.3}$$

or the condition

$$p(t) \leqslant 0 \quad \text{for } t \in \mathbb{R}_+ \tag{1.4}$$

is fulfilled.

Let $t_0 \in \mathbb{R}_+$. A function $u\colon [t_0, +\infty) \to \mathbb{R}$ is said to be a proper solution of Eq. (1.1) if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, satisfies (1.1) almost everywhere on $[t_0, +\infty)$ and $\sup\{|u(s)|\colon s \in [t, +\infty)\} > 0$ for any $t \in [t_0, +\infty)$.

A proper solution $u\colon [t_0, +\infty) \to \mathbb{R}$ of Eq. (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution $u$ is said to be nonoscillatory.

**Definition 1.1** ([2]). We say that Eq. (1.1) has Property **A** if any of its proper solutions is oscillatory when $n$ is even and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as } t \uparrow +\infty \quad (i = 0, \ldots, n-1) \tag{1.5}$$

when $n$ is odd.

**Definition 1.2** ([1]). We say that Eq. (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies (1.5) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as } t \uparrow +\infty \quad (i = 0, \ldots, n-1) \tag{1.6}$$

when $n$ is even, and either is oscillatory or satisfies (1.6) when $n$ is odd.

A number of survey papers and monographs have been devoted to various aspects of oscillation of nonlinear differential equations (see, for example, [3]–[10]).

Some results analogous to those of this paper are given without proofs in [11]–[13].

## 2. Some auxiliary lemmas

In the sequel, $\widetilde{C}_{\text{loc}}([t_0, +\infty))$ will denote the set of all functions $u\colon [t_0, +\infty) \to \mathbb{R}$ absolutely continuous on any finite subinterval of $[t_0, +\infty)$ along with their derivatives of order up to and including $n-1$.

**Lemma 2.1** ([1]). *Let $u \in \widetilde{C}^{n-1}_{\mathrm{loc}}([t_0,+\infty))$, $u(t) > 0$ for $t \geqslant t_0$, $u^{(n)}(t) \not\equiv 0$ in any neighborhood of $+\infty$ and $u^{(n)}(t) \leqslant 0$ ($u^{(n)}(t) \geqslant 0$) for $t \geqslant t_0$. Then there exist $t_1 \geqslant t_0$ and $l \in \{0,\dots,n\}$ such that $l+n$ is odd ($l+n$ is even) and*

$$\tag{$2.1_l$} \begin{aligned} u^{(i)}(t) > 0 \quad (i = 0,\dots,l-1) \quad \text{for } t \geqslant t_1,\\ (-1)^{i+l}\, u^{(i)}(t) > 0 \quad (i = l,\dots,n-1) \quad \text{for } t \geqslant t_1. \end{aligned}$$

In the case $l = 0$ we mean that only the second inequality in $(2.1_l)$ holds, while if $l = n$ only the first holds and $u^{(n)}(t) \geqslant 0$.

**Lemma 2.2** ([14]). *Let $u \in \widetilde{C}_{\mathrm{loc}}([t_0,+\infty))$, $u^{(n)}(t) \leqslant 0$ ($u^{(n)}(t) \geqslant 0$) and $(2.1_l)$ be satisfied for some $l \in \{1,\dots,n-1\}$, where $l+n$ is odd ($l+n$ is even). Then*

$$\tag{2.2} \int_{t_0}^{+\infty} t^{n-l-1}\big|u^{(n)}(t)\big|\,\mathrm{d}t < +\infty.$$

*Moreover, if*

$$\tag{2.3} \int_{t_0}^{+\infty} t^{n-l}\big|u^{(n)}(t)\big|\,\mathrm{d}t = +\infty,$$

*then there exists $t_* \geqslant t_0$ such that*

$$\tag{2.4} \frac{u^{(i)}(t)}{t^{l-i}} \downarrow 0, \quad \frac{u^{(i)}(t)}{t^{l-i-1}} \uparrow +\infty \quad (i = 0,\dots,l-1),$$

$$\tag{2.5} u(t) \geqslant \frac{t^{l-1}}{l!}\, u^{(l-1)}(t) \quad \text{for } t \geqslant t_*$$

*and*

$$\tag{2.6} \begin{aligned} u^{(l-1)}(t) \geqslant \frac{t}{(n-l)!} \int_t^{+\infty} s^{n-l-1}\big|u^{(n)}(s)\big|\,\mathrm{d}s \\ + \frac{1}{(n-l)!} \int_{t_*}^{t} s^{n-l}\big|u^{(n)}(s)\big|\,\mathrm{d}s \quad \text{for } t \geqslant t_*. \end{aligned}$$

## 3. Necessary conditions for existence of solutions of type $(2.1_l)$

The results of this section play an important role in establishing sufficient conditions for Eq. (1.1) to have Properties **A** and **B**.

Let $t_0 \in \mathbb{R}_+$. By $\mathbf{U}_{l,t_0}$ we denote the set of all solutions of Eq. (1.1) satisfying the condition $(2.1_l)$.

**Theorem 3.1.** *Let the conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd* ($l+n$ *even*) *and*

$$\int_0^{+\infty} t^{n-l}\big(\sigma(t)\big)^{\lambda(l-1)}|p(t)|\,\mathrm{d}t = +\infty. \tag{3.1$_l$}$$

*If, moreover,* $\mathbf{U}_{l,t_0} \neq \emptyset$ *for some* $t_0 \in \mathbb{R}_+$, *then for any* $\delta \in [0,\lambda]$ *and* $k \in \mathbb{N}$ *we have*

$$\int_0^{+\infty} t^{n-l-1+\lambda-\delta}\big(\sigma(t)\big)^{\lambda(l-1)}\big(\varrho_{l,k}(\sigma(t))\big)^{\delta}|p(t)|\,\mathrm{d}t < +\infty, \tag{3.2$_l$}$$

*where*

$$\varrho_{l,1}(t) = \left(\frac{1-\lambda}{l!\,(n-l)!}\int_0^t\int_s^{+\infty}\xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s\right)^{\frac{1}{1-\lambda}}, \tag{3.3$_l$}$$

$$\begin{aligned}\varrho_{l,k}(t) = \frac{1}{l!\,(n-l)!}\int_0^t\int_s^{+\infty}&\xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}\\ &\times\big(\varrho_{l,k-1}(\sigma(\xi))\big)^{\lambda}|p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s, \quad k=2,3,\dots\end{aligned} \tag{3.4$_l$}$$

P r o o f. Let $t_0 \in \mathbb{R}_+$ and $\mathbf{U}_{l,t_0} \neq \emptyset$. By definition of the set $\mathbf{U}_{l,t_0}$, Eq. (1.1) has a proper solution $u \in \mathbf{U}_{l,t_0}$ satisfying the condition (2.1$_l$). By (2.1$_l$) and (3.1$_l$) it is clear that the condition (2.3) holds. Thus, by Lemma 2.2, (2.4)–(2.6) are fulfilled and

$$\begin{aligned}u^{(l-1)}(t) \geqslant{}& \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|p(s)|\,u^{\lambda}(\sigma(s))\,\mathrm{d}s\\ &+\frac{1}{(n-l)!}\int_{t_*}^{t} s^{n-l}|p(s)|\,u^{\lambda}(\sigma(s))\,\mathrm{d}s \quad \text{for } t \geqslant t_*,\end{aligned} \tag{3.5}$$

where $t_*$ is a sufficiently large number.

According to (2.5), from (3.5) we get

$$\begin{aligned}u^{(l-1)}(t) \geqslant{}& \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|p(s)|\,u^{\lambda}(\sigma(s))\,\mathrm{d}s\\ &-\frac{1}{(n-l)!}\int_{t_*}^{t} s\,\mathrm{d}\int_s^{+\infty}\xi^{n-l-1}|p(\xi)|\,u^{\lambda}(\sigma(\xi))\,\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&= \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|p(s)|\,u^{\lambda}(\sigma(s))\,\mathrm{d}s\\
&\quad - \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|p(\xi)|\,u^{\lambda}(\sigma(\xi))\,\mathrm{d}\xi\\
&\quad + \frac{t_*}{(n-l)!}\int_{t_*}^{+\infty} \xi^{n-l-1}|p(\xi)|\,u^{\lambda}(\sigma(\xi))\,\mathrm{d}\xi\\
&\quad + \frac{1}{(n-l)!}\int_{t_*}^{+\infty}\int_s^{+\infty} \xi^{n-l-1}|p(\xi)|\,u^{\lambda}(\sigma(\xi))\,\mathrm{d}\xi\,\mathrm{d}s\\
&\geqslant \frac{1}{(n-l)!}\int_{t_*}^{t}\int_s^{+\infty} \xi^{n-l-1}|p(\xi)|\,u^{\lambda}(\sigma(\xi))\,\mathrm{d}\xi\,\mathrm{d}s\\
&\geqslant \frac{1}{l!\,(n-l)!}\int_{t_*}^{t}\int_s^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\\
&\qquad \times \big(u^{(l-1)}(\sigma(\xi))\big)^{\lambda}\,\mathrm{d}\xi\,\mathrm{d}s \quad \text{for } t \geqslant t_*.
\end{aligned}
$$

Therefore, by (1.2) and the second condition of (2.4) we have

$$
x'(t) \geqslant \frac{(u^{(l-1)}(t))^{\lambda}}{l!\,(n-l)!}\int_t^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\,\mathrm{d}\xi \quad \text{for } t \geqslant t_*, \tag{3.7}
$$

where

$$
x(t) = \frac{1}{l!\,(n-l)!}\int_{t_*}^{t}\int_s^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}\big(u^{(l-1)}(\sigma(\xi))\big)^{\lambda}|p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s. \tag{3.8}
$$

Thus, according to (3.6) and (3.8), from (3.7) we get

$$
x'(t) \geqslant \frac{x^{\lambda}(t)}{l!\,(n-l)!}\int_t^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\,\mathrm{d}\xi \quad \text{for } t \geqslant t_*.
$$

Therefore,

$$
x(t) \geqslant \left(\frac{1-\lambda}{l!\,(n-l)!}\int_{t_*}^{t}\int_s^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s\right)^{\frac{1}{1-\lambda}} \quad \text{for } t \geqslant t_*.
$$

Hence, according to (3.6) and (3.8), we have

$$
u^{(l-1)}(t) \geqslant \varrho_{t_*,l,1}(t) \quad \text{for } t \geqslant t_*, \tag{3.9}
$$

where

$$
\varrho_{t_*,l,1} = \left(\frac{1-\lambda}{l!\,(n-l)!}\int_{t_*}^{t}\int_s^{+\infty} \xi^{n-l-1}\big(\sigma(\xi)\big)^{\lambda(l-1)}|p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s\right)^{\frac{1}{1-\lambda}}. \tag{3.10}
$$

Thus, by (3.6), (3.9) and (3.10), we get

$$u^{(l-1)}(t) \geqslant \varrho_{t_*,l,k}(t) \quad \text{for } t \geqslant t_*,\ k = 2, 3, \ldots, \tag{3.11}$$

where

$$\begin{aligned} \varrho_{t_*,l,k}(t) = \frac{1}{l!\,(n-l)!} \int_{t_*}^{t} \int_{s}^{+\infty} & \xi^{n-l-1} \big(\sigma(\xi)\big)^{\lambda(l-1)} |p(\xi)| \\ & \times \big(\varrho_{t_*,l,k-1}(\sigma(\xi))\big)^{\lambda} \,\mathrm{d}\xi \,\mathrm{d}s, \quad k = 2, 3, \ldots. \end{aligned} \tag{3.12}$$

On the other hand, by (1.2), ($2.1_l$), (2.5) and (3.11), from (3.5) for any $\delta \in [0, \lambda]$ we have

$$\begin{aligned} u^{(l-1)}(t) \geqslant \frac{t}{l!\,(n-l)!} \int_{t}^{+\infty} & s^{n-l-1} \big(\sigma(s)\big)^{\lambda(l-1)} \big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta} |p(s)| \\ & \times \big(u^{(l-1)}(s)\big)^{\lambda-\delta} \,\mathrm{d}s \text{ for } t \geqslant t_*,\ k = 1, 2, \ldots. \end{aligned} \tag{3.13}$$

If $\delta = \lambda$, then from (3.13) we get

$$\begin{aligned} \int_{t_*}^{+\infty} s^{n-l-1} \big(\sigma(s)\big)^{\lambda(l-1)} \big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\lambda} |p(s)| \,\mathrm{d}s \\ \leqslant l!\,(n-l)!\, \frac{u^{(l-1)}(t_*)}{t_*}, \quad k = 1, 2, \ldots. \end{aligned} \tag{3.14}$$

Let $\delta \in [0, \lambda)$. Then (3.13) implies

$$\begin{aligned} \frac{\varphi(t)}{\big(\int_t^{+\infty} \varphi(s)\,\mathrm{d}s\big)^{\lambda-\delta}} \geqslant \frac{1}{(l!\,(n-l)!)^{\lambda-\delta}} & \, t^{n-l-1+\lambda-\delta} \big(\sigma(t)\big)^{\lambda(l-1)} \\ & \times \big(\varrho_{t_*,l,k}(\sigma(t))\big)^{\delta} |p(t)| \quad \text{for } t \geqslant t_*, \end{aligned}$$

where

$$\varphi(t) = t^{n-l-1} \big(\sigma(t)\big)^{\lambda(l-1)} |p(t)| \big(\varrho_{t_*,l,k}(\sigma(t))\big)^{\delta} \big(u^{(l-1)}(t)\big)^{\lambda-\delta}. \tag{3.15}$$

Thus, from the last inequality we get

$$\begin{aligned} -\int_{y(t_*)}^{y(t)} \frac{\mathrm{d}s}{s^{\lambda-\delta}} \geqslant \frac{1}{(l!\,(n-l)!)^{\lambda-\delta}} \int_{t_*}^{t} & s^{n-l-1+\lambda-\delta} \big(\sigma(s)\big)^{\lambda(l-1)} \\ & \times |p(s)| \big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta} \,\mathrm{d}s, \end{aligned} \tag{3.16}$$

where

$$y(t) = \int_t^{+\infty} \varphi(s)\,\mathrm{d}s. \tag{3.17}$$

Therefore,

$$\begin{aligned}\int_{t_*}^{t} s^{n-l-1+\lambda-\delta}\big(\sigma(s)\big)^{\lambda(l-1)}|p(s)|\big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta}\,\mathrm{d}s \\ \leqslant \big(l!(n-l)!\big)^{\lambda-\delta}\int_0^{y(t_*)}\frac{\mathrm{d}s}{s^{\lambda-\delta}}.\end{aligned} \tag{3.18}$$

By (3.17), without loss of generality we can assume that $y(t_*) \leqslant 1$. Thus from (3.18) we have

$$\begin{aligned}&\int_{t_*}^{t} s^{n-l-1+\lambda-\delta}\big(\sigma(s)\big)^{\lambda(l-1)}|p(s)|\big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta}\,\mathrm{d}s \\ &\qquad \leqslant (l!(n-l)!)^{\lambda-\delta}\int_0^1\frac{\mathrm{d}s}{s^{\lambda-\delta}} = \frac{(l!(n-l)!)^{\lambda-\delta}}{1-\lambda+\delta} \quad \text{for } t \geqslant t_*.\end{aligned}$$

Passing to the limit in the latter inequality, we obtain

$$\int_{t_*}^{+\infty} s^{n-l-1+\lambda-\delta}\big(\sigma(s)\big)^{\lambda(l-1)}\big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta}|p(s)|\,\mathrm{d}s \leqslant \frac{(l!(n-l)!)^{\lambda-\delta}}{1-\lambda+\delta}. \tag{3.19}$$

According to (3.14) and (3.19), for any $\delta \in [0,\lambda]$ and $k \in \mathbb{N}$ we have

$$\int_{t_*}^{+\infty} s^{n-l-1+\lambda-\delta}\big(\sigma(s)\big)^{\lambda(l-1)}\big(\varrho_{t_*,l,k}(\sigma(s))\big)^{\delta}|p(s)|\,\mathrm{d}s < +\infty. \tag{3.20}$$

On the other hand, since

$$\lim_{t\to+\infty}\frac{\varrho_{l,k}(t)}{\varrho_{t_*,l,k}} = 1,$$

by (3.20) it is obvious that $(3.2_l)$ holds, which proves the validity of the theorem. $\square$

## 4. Sufficient conditions of nonexistence of solutions of type $(2.1_l)$

**Theorem 4.1.** *Let the conditions* (1.2), (1.3) ((1.4)), $(3.1_l)$ *be fulfilled,* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd (*$l+n$ *even), and assume that for some* $\delta \in [0,\lambda]$ *and* $k \in \mathbb{N}$

$$(4.1_l) \qquad \int_0^{+\infty} t^{n-l-1+\lambda-\delta}\big(\sigma(t)\big)^{\lambda(l-1)}\big(\varrho_{l,k}(\sigma(t))\big)^{\delta}|p(t)|\,\mathrm{d}t = +\infty,$$

*where* $\varrho_{l,k}$ *is defined by* $(3.3_l)$ *and* $(3.4_l)$*. Then* $\mathbf{U}_{l,t_0} = \emptyset$ *for any* $t_0 \in \mathbb{R}_+$.

Proof. Assume the contrary. Let there exist $t_0 \in \mathbb{R}_+$ such that $\mathbf{U}_{l,t_0} \neq \emptyset$. Thus Eq. (1.1) has a proper solution $u\colon [t_0,+\infty) \to (0,+\infty)$ satisfying the conditions $(2.1_l)$. Since the conditions of Theorem 3.1 are fulfilled, $(3.2_l)$ holds for any $\delta \in [0,\lambda]$ and $k \in \mathbb{N}$, which contradicts $(4.1_l)$. The obtained contradiction proves the validity of the theorem. $\square$

**Theorem 4.2.** *Let the conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd (*$l+n$ *even), and for some* $\gamma \in (0,1)$ *let*

$$(4.2_l) \qquad \liminf_{t\to+\infty} t^{\gamma}\int_t^{+\infty} s^{n-l-1}\big(\sigma(s)\big)^{\lambda(l-1)}|p(s)|\,\mathrm{d}s > 0.$$

*If, moreover, for some* $\delta \in [0,\lambda]$

$$(4.3_l) \qquad \int_0^{+\infty} t^{n-l-1+\lambda-\delta}\big(\sigma(t)\big)^{\lambda(l-1)}\Big(\big(\sigma(t)\big)^{\frac{\delta(1-\gamma)}{1-\lambda}}\Big)|p(t)|\,\mathrm{d}t = +\infty,$$

*then* $\mathbf{U}_{l,t_0} = \emptyset$ *for any* $t_0 \in \mathbb{R}_+$.

Proof. Clearly the condition $(3.1_l)$ is fulfilled by virtue of $(4.2_l)$. On the other hand, according to $(3.3_l)$ and $(4.2_l)$ there exist $c > 0$ and $t_1 \in \mathbb{R}_+$ such that

$$\varrho_{l,1}(t) \geqslant c\,t^{\frac{1-\gamma}{1-\lambda}} \quad \text{for } t \geqslant t_1.$$

Therefore, for $k = 1$ $(4.1_l)$ follows from $(4.3_l)$, and all conditions of Theorem 4.1 hold, which proves the validity of the theorem. $\square$

In a similar manner one can prove the following theorem.

**Theorem 4.3.** *Let the conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $l \in \{1,\ldots,n-1\}$ *with* $l+n$ *odd* ($l+n$ *even*), *and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-l-1}\big(\sigma(s)\big)^{\lambda(l-1)}|p(s)|\,\mathrm{d}s > 0. \tag{$4.4_l$}$$

*If, moreover,*

$$\int_0^{+\infty} t^{n-l-1+\lambda}\big(\sigma(s)\big)^{\lambda(l-1)}\big(\ln(\sigma(s))\big)^{\frac{\lambda}{1-\lambda}}|p(s)|\,\mathrm{d}s = +\infty, \tag{$4.5_l$}$$

*then* $\mathbf{U}_{l,t_0} = \emptyset$ *for any* $t_0 \in \mathbb{R}_+$.

**Theorem 4.4.** *Let the conditions* (1.2), (1.3), ((1.4)) *and* ($4.2_l$) *hold,* $l \in \{1,\ldots,n-1\}$ *with* $l+n$ *odd* ($l+n$ *even*) *and for some* $\alpha \in (1,+\infty)$ *let*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\alpha}} > 0. \tag{4.6}$$

*If, moreover, at least one of the conditions:*

$$\alpha\,\lambda \geqslant 1 \tag{4.7}$$

*or* $\alpha\lambda < 1$ *and for some* $\varepsilon > 0$

$$\int_0^{+\infty} s^{n-l-1+\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon}\big(\sigma(t)\big)^{\lambda(l-1)}|p(t)|\,\mathrm{d}t = +\infty \tag{$4.8_l$}$$

*holds, then* $\mathbf{U}_{l,t_0} = \emptyset$ *for any* $t_0 \in \mathbb{R}_+$.

Proof. It suffices to show that the condition ($4.1_l$) is satisfied for some $k \in \mathbb{N}$ and $\sigma = \lambda$. Indeed, according to ($4.2_l$) and (4.6) there exist $\alpha > 1$, $\gamma \in (0,1)$, $c > 0$ and $t_1 \in \mathbb{R}_+$ such that

$$t^{\gamma}\int_t^{+\infty} s^{n-l-1}\big(\sigma(s)\big)^{\lambda(l-1)}|p(s)|\,\mathrm{d}s \geqslant c \quad \text{for } t \geqslant t_1 \tag{4.9}$$

and

$$\sigma(t) \geqslant c\,t^{\alpha} \quad \text{for } t \geqslant t_1. \tag{4.10}$$

By ($3.3_l$) and ($4.2_l$), it is obvious that $\lim\limits_{t\to+\infty} \varrho_{l,1}(t) = +\infty$. Therefore, without loss of generality we can assume that $\varrho_{l,1}(t) \geqslant 1$ for $t \geqslant t_1$. Thus, by (4.9) from ($3.4_l$) we get

$$\varrho_{l,2}(t) \geqslant \frac{c}{l!\,(n-l)!}\int_{t_1}^{t} s^{-\gamma}\,\mathrm{d}s = \frac{c}{l!\,(n-l)!\,(1-\gamma)}\big(t^{1-\gamma} - t_1^{1-\gamma}\big).$$

We can choose $t_2 > t_1$ such that

$$\varrho_{l,2}(t) \geqslant \frac{c\,t^{1-\gamma}}{2l!\,(n-l)!\,(1-\gamma)} \quad \text{for } t \geqslant t_2.$$

Thus, by (4.9) and (4.10), from ($3.4_l$) for $k = 3$ we have

$$\varrho_{l,3}(t) \geqslant \left(\frac{c}{2l!\,(n-l)!\,(1-\gamma)}\right)^{1+\lambda} t^{(1-\gamma)(1+\alpha\lambda)} \quad \text{for } t \geqslant t_3,$$

where $t_3 > t_2$ is a sufficiently large number. Therefore, for any $k_0 \in \mathbb{N}$ there exists $t_{k_0} \in \mathbb{R}_+$ such that for $t \geqslant t_{k_0}$,

$$\varrho_{l,k_0}(t) \geqslant \left(\frac{c}{2l!\,(n-l)!\,(1-\gamma)}\right)^{1+\lambda+\ldots+\lambda^{k_0-2}} t^{(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{k_0-2})}. \tag{4.11}$$

Assume that (4.7) is fulfilled. Choose $k_0 \in \mathbb{N}$ such that $(1-\gamma)(k_0-1) \geqslant 1/\lambda$. Then according to (4.10) and (4.11) we have

$$\varrho^{\lambda}_{l,k_0}(t) \geqslant c_0\,t \quad \text{for } t \geqslant t_{k_0},$$

where $c_0 > 0$. Therefore, by (4.9) and (4.10) it is obvious that ($4.1_l$) holds for $\delta = \lambda$ and $k = k_0$. In the case when (4.7) holds, the validity of the theorem has been already proved.

Assume now that ($4.8_l$) is fulfilled. Let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that

$$1 + \alpha\lambda + \ldots + (\alpha\lambda)^{k_0-2} > \frac{1}{1-\alpha\lambda} - \frac{\varepsilon}{(1-\gamma)\alpha}.$$

Then from (4.11) we have

$$\varrho_{l,k_0}(t) \geqslant c_0\,t^{\frac{1-\gamma}{1-\alpha\lambda}-\frac{\varepsilon}{\alpha}} \quad \text{for } t \geqslant t_{k_0},$$

where $c_0 > 0$. Therefore, by (4.10)

$$\varrho^{\lambda}_{l,k_0}\big(\sigma(t)\big) \geqslant c_1\,t^{\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon} \quad \text{for } t \geqslant t_{k_0},$$

where $c_1 > 0$. Consequently, according to ($4.8_l$) it is obvious that ($4.1_l$) holds for $k = k_0$ and $\delta = \lambda$. The proof of the theorem is complete. □

In a similar manner one can prove the following theorem.

**Theorem 4.5.** *Let the conditions* (1.2), (1.3), ((1.4)) *and* (4.4$_l$) *be fulfilled,* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd* ($l+n$ *even*), *and let there exist* $\alpha > 0$ *such that*

$$\liminf_{t\to+\infty} t^{-\alpha} \ln \sigma(t) > 0. \tag{4.12}$$

*Then* $\mathbf{U}_{l,t_0} = \emptyset$ *for any* $t_0 \in \mathbb{R}_+$.

## 5. Differential equations with property A and B

**Theorem 5.1.** *Let the conditions* (1.2), (1.3) ((1.4)) *be fulfilled and for any* $l \in \{1,\dots,n-1\}$ *with* $l+n$ *odd* ($l+n$ *even*) *let* (3.1$_l$) *as well as* (4.1$_l$) *hold for some* $\delta \in [0,\lambda]$ *and* $k \in \mathbb{N}$. *Let, moreover,*

$$\int_0^{+\infty} t^{n-1}|p(t)|\,\mathrm{d}t = +\infty \tag{5.1}$$

*when* $n$ *is odd* ($n$ *is even*) *and*

$$\int_0^{+\infty} \big(\sigma(t)\big)^{\lambda(n-1)}|p(t)|\,\mathrm{d}t = +\infty \tag{5.2}$$

*in the case when* (1.4) *holds. Then Eq.* (1.1) *has Property* $\mathbf{A}(\mathbf{B})$.

P r o o f. Let Eq. (1.1) have a proper nonoscillatory solution $u\colon [t_0,+\infty) \to (0,+\infty)$ (the case $u(t) < 0$ is similar). Then by (1.1), (1.3) ((1.4)) and Lemma 2.1, there exists $l \in \{0,\dots,n\}$ such that $l+n$ is odd ($l+n$ is even) and the condition (2.1$_l$) holds. Since the conditions of Theorem 4.1 are fulfilled for any $l \in \{1,\dots,n-1\}$ with $l+n$ odd ($l+n$ even), we have $l \notin \{1,\dots,n-1\}$. Therefore, $n$ is odd ($n$ is even) and $l = 0$, or (1.4) holds and $l = n$. Let $l = 0$. Then we will show that the conditions (1.5) hold. If that is not the case, there exists $c > 0$ such that $u(t) \geqslant c$ for sufficiently large $t$. According to (2.1$_0$), from (1.1) we have

$$\sum_{i=0}^{n-1}(n-i-1)!\,t_1^i\big|u^{(i)}(t_1)\big| \geqslant c^\lambda \int_{t_1}^t s^{n-1}|p(s)|\,\mathrm{d}s \quad \text{for } t \geqslant t_1,$$

where $t_1$ is a sufficiently large number. The latter inequality contradicts the condition (5.1). Thus, (1.5) is fulfilled.

Now assume that (1.4) holds and $l = n$. To complete the proof, it suffices to show that (1.6) is valid when $l = n$.

From $(2.1_n)$ we have $u(\sigma(t)) \geqslant c(\sigma(t))^{n-1}$ for $t \geqslant t_1$, where $c > 0$ and $t_1$ is a sufficiently large number. Therefore, by (1.4), (5.2) and $(2.1_n)$, from (1.1) we get

$$u^{(n-1)}(t) \geqslant u^{(n-1)}(t_1) + c^{\lambda} \int_{t_1}^{t} (\sigma(s))^{\lambda(n-1)} |p(s)| \, \mathrm{d}s \to +\infty \quad \text{for } t \to +\infty,$$

i.e., (1.6) holds. Therefore Eq. (1.1) has Property **A** (**B**). □

**Remark 5.1.** Theorem 5.1 is a generalization of Theorem 1.1 [1].

**Theorem 5.2.** *Let the conditions* (1.2), (1.3) *as well as* (5.1) *be fulfilled for odd* $n$, *and let*

$$\liminf_{t \to +\infty} \frac{\sigma^{\lambda}(t)}{t} > 0. \tag{5.3}$$

*If, moreover, for some* $\delta \in [0, \lambda]$ *and* $k \in \mathbb{N}$, $(4.1_1)$ *holds when* $n$ *is even and* $(4.1_2)$ *holds when* $n$ *is odd, then Eq.* (1.1) *has Property* **A**.

Proof. It is obvious that, according to $(4.1_1)$, $(4.1_2)$ and (5.3), for any $l \in \{1, \ldots, n-1\}$, where $l+n$ is odd, the conditions $(4.1_l)$ hold. Therefore, all conditions of Theorem 5.1 for the case of Property **A** hold, which proves the validity of the theorem. □

**Corollary 5.1.** *Let the conditions* (1.2), (1.3) *and* (5.3) *be fulfilled and for even* $n$ *let*

$$\int_0^{+\infty} t^{n-2+\lambda} p(t) \, \mathrm{d}t = +\infty \tag{5.4}$$

*hold. If, moreover,* (5.1) *and*

$$\int_0^{+\infty} t^{n-3+\lambda} \big(\sigma(t)\big)^{\lambda} p(t) \, \mathrm{d}t = +\infty \tag{5.5}$$

*hold for odd* $n$, *then Eq.* (1.1) *has Property* **A**.

Proof. It is obvious by (5.4) and (5.5) that for $\delta = 0$, $(4.1_1)$ and $(4.1_2)$ are fulfilled. Thus, all conditions of Theorem 5.2 are fulfilled, which proves the validity of the theorem. □

Analogously of Theorem 5.2 we can prove

**Theorem 5.3.** *Let the conditions* (1.2), (1.4), (5.2) *and* (5.3) *as well as* (5.1) *be fulfilled for even* $n$. *If, moreover, for some* $\delta \in [0, \lambda]$ *and* $k \in \mathbb{N}$, $(4.1_1)$ *holds when* $n$ *is odd and* $(4.1_2)$ *holds when* $n$ *is even. Then Eq.* (1.1) *has Property* **B**.

**Corollary 5.2.** *Let the conditions* (1.2), (1.4) *and* (5.3) *be fulfilled and for odd* $n$ *let*

$$\int_0^{+\infty} t^{n-2+\lambda} |p(t)| \, \mathrm{d}t = +\infty \tag{5.6}$$

*hold. If, moreover,* (5.1) *and*

$$\int_0^{+\infty} t^{n-3+\lambda} \big(\sigma(t)\big)^{\lambda} |p(t)| \, \mathrm{d}t = +\infty \tag{5.7}$$

*hold for even* $n$, *then Eq.* (1.1) *has Property* **B**.

**Theorem 5.4.** *Let the conditions* (1.2) *and* (1.3) *as well as* (5.1) *be fulfilled for odd* $n$, *and let*

$$\limsup_{t \to +\infty} \frac{\sigma^{\lambda}(t)}{t} < +\infty. \tag{5.8}$$

*Then for Eq.* (1.1) *to have Property* **A** *it is sufficient that for some* $k \in \mathbb{N}$ *and* $\delta \in [0, \lambda]$, $(4.1_{n-1})$ *hold, where* $\varrho_{n-1,k}$ *is defined by* $(3.4_{n-1})$.

P r o o f. By (5.8), there exist $c > 0$ and $t_1 \in \mathbb{R}_+$ such that $t \geqslant c\sigma^{\lambda}(t)$ for $t \geqslant t_1$. Therefore, according to $(4.1_{n-1})$, for any $l \in \{1, \ldots, n-1\}$ with $l + n$ odd $(4.1_l)$ holds, which proves the validity of the theorem. □

**Corollary 5.3.** *Let the conditions* (1.2), (1.3), (5.8) *be fulfilled and let*

$$\int_0^{+\infty} t^{\lambda} \big(\sigma(t)\big)^{\lambda(n-2)} p(t) \, \mathrm{d}t = +\infty. \tag{5.9}$$

*Then Eq.* (1.1) *has Property* **A**.

Analogously to Theorem 5.4 we can prove

**Theorem 5.5.** *Let the conditions* (1.2), (1.4), (5.2), (5.8) *be fulfilled. Then for Eq.* (1.1) *to have Property* **B** *it is sufficient that for some* $k \in \mathbb{N}$ *and* $\delta \in [0, \lambda]$ $(4.1_{n-2})$ *hold, where* $\varrho_{n-2,k}$ *is defined by* $(3.4_{n-2})$.

**Corollary 5.4.** *Let the conditions* (1.2), (1.4), (5.2), (5.8) *be fulfilled and let*

$$\int_0^{+\infty} t^{1+\lambda}\big(\sigma(t)\big)^{\lambda(n-2)}|p(t)|\,\mathrm{d}t = +\infty. \tag{5.10}$$

*Then Eq.* (1.1) *has Property* **B**.

**Theorem 5.6.** *Let the conditions* (1.2), (1.3) *and* (5.3) *be fulfilled and for even* $n$ *let the conditions* ($4.2_1$) *and* ($4.3_1$) *hold. If, moreover,* (5.1), ($4.2_2$) *and* ($4.3_2$) *hold for odd* $n$, *then Eq.* (1.1) *has Property* **A**.

**Theorem 5.7.** *Let the conditions* (1.2), (1.4) *and* (5.3) *be fulfilled and for odd* $n$ *let the conditions* ($4.2_1$) *and* ($4.3_1$) *hold. If, moreover,* (5.1), ($4.2_2$) *and* ($4.3_2$) *hold for even* $n$, *then Eq.* (1.1) *has Property* **B**.

For proving Theorems 5.6 and 5.7, it suffices to note that by (5.3), ($4.2_1$) and ($4.3_1$) (by (5.3), ($4.2_2$) and ($4.3_2$)), for any $l \in \{1,\dots,n-1\}$ (for any $l \in \{2,\dots,n-1\}$), ($4.2_l$) and ($4.3_l$) hold.

Analogously to Theorems 5.5 and 5.6, using Theorems 4.3 and 4.5, we can prove Theorems 5.8–5.13.

**Theorem 5.8.** *Let the conditions* (1.2), (1.3) *and* (5.3) *be fulfilled and for even* $n$ *let the conditions* ($4.4_1$) *and* ($4.5_1$) *hold. If, moreover,* (5.1), ($4.4_2$) *and* ($4.5_2$) *hold for odd* $n$, *then Eq.* (1.1) *has Property* **A**.

**Theorem 5.9.** *Let the conditions* (1.2), (1.4) *and* (5.3) *be fulfilled and for odd* $n$ *let the conditions* ($4.4_1$) *and* ($4.5_1$) *hold. If, moreover,* (5.1), ($4.4_2$) *and* ($4.5_2$) *hold for even* $n$, *then Eq.* (1.1) *has Property* **B**.

**Theorem 5.10.** *Let the conditions* (1.2), (1.3), (4.6) *and* (4.7) *be fulfilled and for even* $n$ (*for odd* $n$) *let the conditions* ($4.1_1$) (($4.1_2$) *and* (5.1)) *hold. Then Eq.* (1.1) *has Property* **A**.

**Theorem 5.11.** *Let the conditions* (1.2), (1.4), (4.6) *and* (4.7) *be fulfilled and for odd* $n$ (*for even* $n$) *let the conditions* ($4.1_1$) (($4.1_2$) *and* (5.1)) *hold. Then Eq.* (1.1) *has Property* **B**.

**Theorem 5.12.** *Let the conditions* (1.2), (1.3), (4.6) *and* (5.3) *be fulfilled, where* $\alpha\lambda < 1$. *If, moreover, for even* $n$ (*odd* $n$) *the conditions* ($4.2_1$) *and* ($4.8_1$) ((5.1), ($4.2_2$) *and* ($4.8_2$)) *hold, then Eq.* (1.1) *has Property* **A**.

**Theorem 5.13.** *Let the conditions* (1.2), (1.4), (4.6) *and* (5.3) *be fulfilled, where* $\alpha\lambda < 1$. *If, moreover, for odd* $n$ *(for even* $n$*) the conditions* $(4.2_1)$ *and* $(4.8_1)$ ((5.1), $(4.2_2)$ *and* $(4.8_2)$)) *hold, then Eq.* (1.1) *has Property* **B**.

**Remark 5.2.** Using Theorems 4.1–4.5 it is possible to get effective conditions for the validity of properties **A** and **B** different from the conditions given above.

## 6. Necessary and sufficient conditions

**Theorem 6.1.** *Let* $n$ *be odd (*$n$ *be even), the conditions* (1.2), (1.3) ((1.4)) *be fulfilled and let*

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}}} > 0. \tag{6.1}$$

*Then the condition* (5.1) *is necessary and sufficient for Eq.* (1.1) *to have Property* **A**(**B**).

P r o o f. Sufficiency. By virtue of (5.1) and (6.1), when $n$ is odd (when $n$ is even) the conditions of Corollary 5.1 (of Corollary 5.2) are satisfied. Therefore, according to the same corollaries, Eq. (1.1) has Property **A**(**B**).

Necessity. Assume that Eq. (1.1) has Property **A**(**B**) and

$$\int_0^{+\infty} t^{n-1}|p(t)|\,\mathrm{d}t < +\infty.$$

Then by [7, Lemma 4.1] there exists $c \neq 0$ such that (1.1) has a proper solution $u\colon [t_0,+\infty) \to \mathbb{R}$ satisfying the condition $\lim_{t\to+\infty} u(t) = c$. But this contradicts the fact that Eq. (1.1) has Property **A**(**B**). □

**Remark 6.1.** Theorem 6.1 is a generalization of Theorem 1.2 [1].

**Remark 6.2.** The condition (6.1) defines the set of the functions $\sigma$ for which the condition (5.1) is necessary and sufficient. It turns out that the number $(2-\lambda)/\lambda$ is optimal. Indeed, let $\varepsilon \in (0,(1+\lambda)/\lambda)$, $\lambda \in [1/(1+\varepsilon),1)$ and $\gamma \in (1,2)$. Consider the differential Eq. (1.1), where

$$p(t) = -\gamma(\gamma-1)\dots(\gamma-n+1)\,t^{-n-\gamma(1-\lambda)(1+\varepsilon))},\ \sigma(t) = t^{\frac{2-\lambda}{\lambda}-\varepsilon},\ t \geqslant 1.$$

It is obvious that the condition (5.1) is fulfilled and

$$\liminf_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}}} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{2-\lambda}{\lambda}-\varepsilon}} > 0.$$

On the other hand, for odd $n$ (for even $n$), $u(t) = t^{\gamma}$ is a solution of Eq. (1.1). Therefore, for odd $n$ (for even $n$) Eq. (1.1) does not have Property **A**(**B**).

**Theorem 6.2.** *Let $n \geqslant 3$, let the conditions* (1.2), (1.4) *be fulfilled and*

$$\limsup_{t\to+\infty} \frac{\sigma(t)}{t^{\frac{1+\lambda}{2\lambda}}} < +\infty. \tag{6.2}$$

*Then the condition* (5.2) *is necessary and sufficient for Eq.* (1.1) *to have Property* **B**.

P r o o f. Necessity follows from [7, Lemma 4.1]. On the other hand, by (5.2) and (6.2) the condition (5.10) is fulfilled. Therefore, sufficiency follows from Corollary 5.4. □

**Remark 6.3.** The condition (6.2) defines the set of the functions $\sigma$ for which the condition (5.2) is necessary and sufficient. It turns out that the number $\frac{1}{2}(1+\lambda)/\lambda$ is optimal. Indeed, let $\varepsilon > 0$, $\lambda \in [1/(1+2\varepsilon), 1)$ and $\gamma \in (n-3, n-2)$. Consider Eq. (1.1), where

$$p(t) = -\gamma(\gamma-1)\dots(\gamma-n+1)\, t^{-n+\gamma\frac{1-\lambda-2\varepsilon\lambda}{2}}, \quad \sigma(t) = t^{\frac{1+\lambda}{2\lambda}+\varepsilon},\ t \geqslant 1.$$

Since $\lambda(1+2\varepsilon) \geqslant 1$ and $\gamma \in (n-3, n-2)$, we have

$$\int_1^{+\infty} (\sigma(t))^{\lambda(n-1)} |p(t)|\,\mathrm{d}t = -\gamma(\gamma-1)\dots(\gamma-n+1) \times \int_1^{+\infty} t^{-1}\cdot t^{((n-1)-\gamma)(\frac{\lambda+2\varepsilon\lambda-1}{2})}\,\mathrm{d}t = +\infty.$$

Thus, all conditions of Theorem 6.2 hold, except the condition (6.2). On the other hand, $u(t) = t^\gamma$ is a solution of Eq. (1.1). Therefore, Eq. (1.1) does not have Property **B**.

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