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GROTHENDIECK RING OF QUANTUM DOUBLE OF FINITE GROUPS

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Abstract. Let kG be a group algebra, and $D(kG)$ its quantum double. We first prove that the structure of the Grothendieck ring of $D(kG)$ can be induced from the Grothendieck ring of centralizers of representatives of conjugate classes of G. As a special case, we then give an application to the group algebra kD_n, where k is a field of characteristic 2 and D_n is a dihedral group of order $2n$.

Keywords: Grothendieck ring, quantum double, Yetter-Drinfeld module, dihedral group

MSC 2010: 13D15

1. Introduction

The quantum double of a Hopf algebra, also called the Drinfeld double, was defined by Drinfeld in the study of quantum Yang-Baxter equations. It is defined in terms of what Drinfeld calls the "quasitriangular Hopf algebra", and its construction is based on a general procedure, also due to Drinfeld, assigning to a Hopf algebra H a quasitriangular Hopf algebra $D(H)$ (see Section 10 in [2]). The Hopf algebra $D(H)$ is called the quantum double of H. It has brought remarkable applications to new aspects of representation theory, theoretical physics, non-commutative geometry, low-dimensional topology and so on.

For a Hopf algebra H with a bijective antipode S, a Yetter-Drinfeld H-module M is both a left H-module and a right H-comodule satisfying the two equivalent

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compatibility conditions

$$\sum h_1 \cdot m_0 \otimes h_2 m_1 = \sum (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1 h_1,$$
$$\sum (h \cdot m)_0 \otimes (h \cdot m)_1 = \sum h_2 \cdot m_0 \otimes h_3 m_1 S^{-1}(h_1)$$

for all $h \in H$ and $m \in M$. The category of Yetter-Drinfeld H-modules is denoted by ${}_H\mathcal{YD}^H$. Majid first proved that the Yetter-Drinfeld category ${}_H\mathcal{YD}^H$ can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the quantum double $D(H)$ (see Proposition 2.1 of [4]).

Now we return to the case that $H = kG$ is a finite dimensional group algebra. Let $\mathcal{K}(G)$ be the set of conjugate classes in G. For any $g \in G$, let $C_G(g) = \{x \in G : xg = gx\}$ be the centralizer of g in G. For any $C \in \mathcal{K}(G)$, fix a $g_C \in C$. Then $\{g_C : C \in \mathcal{K}(G)\}$ is a set of representatives of conjugate classes in G. Now suppose that N is a left $kC_G(g_C)$-module. Then $N{\uparrow}^G = kG \otimes_{kC_G(g_C)} N$ is a left kG-module. Define a k-linear map $\varphi\colon\ N{\uparrow}^G \to N{\uparrow}^G \otimes kG$ by $\varphi(g \otimes n) = (g \otimes n) \otimes g g_C g^{-1}$ for all $g \in G$, $n \in \mathbb{N}$. A straightforward verification shows that $(N{\uparrow}^G, \varphi)$ is a Yetter-Drinfeld kG-module, denoted by $D(N)$.

Let M be a Yetter-Drinfeld kG-module with coaction $\varphi\colon\ M \to M \otimes kG$. For $C \in \mathcal{K}(C)$, let

$$M_C = \bigoplus_{g \in C} M_g$$

where $M_g = \{m \in M : \varphi(m) = m \otimes g\}$. An easy computation shows that M_g is a $kC_G(g)$-submodule of $M \downarrow_{C_G(g)}$ and M_C is a Yetter-Drinfeld kG-submodule of M.

By Corollary 2.3 of [6], we have the following characterization of Yetter-Drinfeld kG-modules. Let $C \in \mathcal{K}(G)$ and let N be a $kC_G(g_C)$-module. Then $D(N)$ is an indecomposable (or simple) Yetter-Drinfeld kG-module if and only if N is an indecomposable (or simple) $kC_G(g_C)$-module. Let M be an indecomposable (simple) Yetter-Drinfeld kG-module. Then there exists a conjugate class $C \in \mathcal{K}(G)$ such that $M = M_C \cong D(M_{g_C})$. Let N_1 and N_2 be an indecomposable (simple) $kC_G(g_C)$-module. Then $D(N_1) \cong D(N_2)$ if and only if $N_1 \cong N_2$.

Thus up to isomorphism, there is a 1-1 correspondence between the indecomposable (simple) $kC_G(g_C)$-modules and indecomposable (simple) Yetter-Drinfeld kG-modules.

Following the characterization of Yetter-Drinfeld kG-modules and Majid's result, we prove that the structure of the Grothendieck ring of $D(kG)$ can be induced from the Grothendieck ring of centralizers of representatives of conjugate classes of G. We then give an application to the group algebra kD_n, where k is a field of characteristic 2, and

$$D_n = \langle a, b : a^n = 1, b^2 = 1, (ab)^2 = 1 \rangle = \{1, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$$

is a dihedral group of order $2n$.

Throughout this paper we will work over a field k. All modules are left modules, all comodules are right comodules, and moreover they are finite dimensional over k, $\otimes$ means $\otimes_k$, and $\mathbb{C}$ denotes the complex number field. We refer the reader to [3], [5] for standard definitions and results concerning Hopf algebras and quantum doubles.

2. Grothendieck ring of finite groups

Let A be a finite-dimensional algebra over k. We fix a full set of non-isomorphic simple left A-modules $V_1, V_2, \ldots, V_m$. Let $G_0(A)$ denote the Grothendieck group of the category of finite dimensional left A-modules. This is the abelian group generated by the isomorphism classes $[V]$ of left A-modules V modulo the relation $[V] = [U] + [W]$ for each short exact sequence of A-modules $0 \to U \to V \to W \to 0$. It is well known (see Theorem 1.7 of [1]) that $G_0(A)$ is a free abelian group with the basis $\{[V_i] : i = 1, \ldots, m\}$.

Let H be a finite-dimensional Hopf algebra over k, and let U and V be left H-modules. Then $U \otimes V$ is also a left H-module with the H-action given by $h(u \otimes v) = \sum h_1 u \otimes h_2 v$, where $h \in H, u \in U$ and $v \in V$. Define $[U][V] = [U \otimes V]$, then $G_0(H)$ is a ring. If H is a quasitriangular Hopf algebra, then $U \otimes V \cong V \otimes U$ as left H-modules, and so $G_0(H)$ is a commutative ring (see Proposition VIII.3.1 of [3]). In particular, the Grothendieck ring of the quantum double of a finite group is commutative.

Lemma 2.1. *Let G be a finite group and φ_L the Brauer character afforded by a finite dimensional kG-module L. Then we have the following well-known results.*

(1) $\varphi_L(1) = \dim_k L$;

(2) *let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of finite dimensional kG-modules, then

$$\varphi_M = \varphi_L + \varphi_N;$$

(3) $\varphi_{M \otimes N} = \varphi_M \varphi_N$.

Let $B(kG)$ denote the ring generated by $\{\varphi_L : L$ is a finite dimensional kG-module$\}$, which is called the character ring of kG. Then the following proposition implies that the structure of $G_0(kG)$ is totally determined by $B(kG)$.

Proposition 2.2. *Let G be a finite group. Then there is an isomorphism between the rings $G_0(kG)$ and $B(kG)$.*

Proof. Define maps $\varphi\colon G_0(kG) \to B(kG)$ by $\varphi([M]) = \varphi_M$ and $\varphi^{-1}\colon B(kG) \to G_0(kG)$ by $\varphi^{-1}(\varphi_M) = [M]$. It follows from Lemma 2.1 that φ is well defined. Then $\varphi([M]+[N]) = \varphi([M \oplus N]) = \varphi_{M\oplus N} = \varphi_M + \varphi_N = \varphi([M]) + \varphi([N])$ and $\varphi([M][N]) = \varphi([M \otimes N]) = \varphi_{M\otimes N} = \varphi_M\varphi_N = \varphi([M])\varphi([N])$. Thus φ is an isomorphism of rings. $\square$

Before discussing the structure of the Grothendieck ring of kD_n, we fix some notation. Let x be any integer, and let y, z non-negative integers. We define

$$[x,y,z] = \begin{cases} x, & x \leqslant y; \\ z - x, & x \geqslant y+1, \end{cases} \qquad \{x\}_y = \begin{cases} x, & x \geqslant 0; \\ y + x, & x < 0, \end{cases} \qquad \text{and} \quad [x]_y = x \bmod y.$$

Let

$$D_n = \langle a, b : a^n = 1, b^2 = 1, (ab)^2 = 1\rangle = \{1, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$$

be a dihedral group of order $2n$, k a field of characteristic 2. Suppose that $n = 2s+1$, $s \geqslant 1$. Then there are $s+2$ conjugate classes in D_n:

$$\{1\},\ \{a^i, a^{n-i}\}\ (1 \leqslant i \leqslant s),\ \{a^j b : 0 \leqslant j \leqslant 2s\},$$

with representatives $1, a, a^2, \ldots, a^s, b$ respectively, where $1, a, a^2, \ldots, a^s$ are 2-regular. Hence, there are only $s+1$ distinct simple kD_n-modules up to isomorphism.

Lemma 2.3. *Let $M_2(k)$ be the algebra of 2×2 matrices over k, and ξ the n-th primitive root of unity in k. For any $1 \leqslant i \leqslant s$, define A_i, B in $M_2(k)$ by*

$$A_i = \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then there is a unique algebra morphism Ω_i from kD_n to $M_2(k)$ such that $\Omega_i(a) = A_i, \Omega_i(b) = B$ for every i.

Proof. One can easily check that $A_i^n = B^2 = (A_iB)^2 = E$, where E is the identity matrix in $M_2(k)$. Then it follows by the definition of D_n that there is a unique algebra morphism Ω_i from kD_n to $M_2(k)$. $\square$

We also define $\Omega_0\colon kD_n \to k$, $\Omega_0(a) = \Omega_0(b) = 1$. Then we can show that $\{\Omega_i : 0 \leqslant i \leqslant s\}$ is a complete set of irreducible representations of kD_n up to isomorphism. We denote their corresponding modules by W_i and their Brauer characters by χ_i. Then the Brauer character table of kD_n is

	1	a	a^2	$\dots$	a^s
χ_0	1	1	1	$\dots$	1
χ_i	2	$\omega^i + \omega^{-i}$	$\omega^{2i} + \omega^{-2i}$	$\dots$	$\omega^{si} + \omega^{-si}$

where $\omega \in \mathbb{C}$ is the n-th primitive root of unity, $1 \leqslant i \leqslant s$.

Theorem 2.4. *Let $1 \leqslant i, j \leqslant s$. Then the structure of the Grothendieck ring of kD_n can be described as follows:*

$$[W_0]^2 = [W_0], \quad [W_0][W_i] = [W_i], \quad [W_i]^2 = 2[W_0] + [W_{[2i,s,n]}],$$
$$[W_i][W_j] = [W_{[i+j,s,n]}] + [W_{i-j}], \quad j < i.$$

Proof. From the above discussion, we can compute the Brauer character of kD_n:

$$[\chi_0]^2 = [\chi_0], \quad [\chi_0][\chi_i] = [\chi_i], \quad [\chi_i]^2 = 2[\chi_0] + [\chi_{[2i,s,n]}],$$
$$[\chi_i][\chi_j] = [\chi_{[i+j,s,n]}] + [\chi_{i-j}], \quad j < i.$$

Hence, the result follows from Proposition 2.2. $\square$

3. Grothendieck ring of quantum double of finite groups

We first give a general method used later for determining the structure of the Grothendieck ring of the quantum double of finite groups.

Theorem 3.1. *Let G be a finite group, k an arbitrary field, and M a Yetter-Drinfeld kG-module. Let $D(N) = kG \otimes_{kC_G(g_C)} N$ be a simple Yetter-Drinfeld kG-module, where g_C is a representative of some conjugate class C and N is a simple $kC_G(g_C)$-module. Then the multiplicity of $D(N)$ in M is equal to the multiplicity of $N = D(N)_{g_C}$ in M_{g_C} as a $kC_G(g_C)$-module.*

Proof. Let

$$M_0 = M_{g_C} \supset M_1 \supset \ldots \supset M_{n-1} \supset M_n = 0$$

be a composition series of M_{g_C}, and N a composition factor of M_{g_C}. Then there exist M_i, M_{i+1} such that

$$M_i/M_{i+1} \cong N$$

as $kC_G(g_C)$-modules. This implies that

$$D(N) \cong D(M_i/M_{i+1}) \cong D(M_i)/D(M_{i+1})$$

as Yetter-Drinfeld kG-modules. Hence, $D(N)$ is a composition factor of M.

Conversely, let

$$M^0 = M \supset M^1 \supset \ldots \supset M^{m-1} \supset M^m = 0$$

be a composition series of M, and let $D(N)$ be a composition factor of M. Then there exist M^i, M^{i+1} such that

$$M^i/M^{i+1} \cong D(N)$$

as Yetter-Drinfeld kG-modules. Consequently,

$$N = D(N)_{g_C} \cong (M_i/M_{i+1})_{g_C} \cong M^i_{g_C}/M^{i+1}_{g_C}$$

as $kC_G(g_C)$-modules. Hence, N is a composition factor of M_{g_C}. This completes the proof. $\square$

Let C_1, C_2 be conjugate classes in G. Take $x \in C_1$, $y \in C_2$, and let M, N be $kC_G(x)$- and $kC_G(y)$-modules, respectively. Let $\{g_1, \ldots, g_m\}$ and $\{h_1, \ldots, h_l\}$ be the sets of left coset representatives of $C_G(x)$ and $C_G(y)$ in G, respectively. For any $z \in G$ and a simple $kC_G(z)$-module P we can determine the multiplicity of $D(P)$ in $D(M) \otimes D(N)$ by Theorem 3.1. Consider the tensor product $D(M) \otimes D(N)$ as kG-modules. It contains the subspace

$$S(z) = \bigoplus_{i,j} (g_i \otimes_{kC_G(x)} M) \otimes (h_j \otimes_{kC_G(y)} N)$$

where the sum is taken over all indices i, j such that $(h_j y h_j^{-1})(g_i x g_i^{-1}) = z$. Then $S(z) = (D(M) \otimes D(N))_z$ is a $kC_G(z)$-module, and the multiplicity of $D(P)$ in $D(M) \otimes D(N)$ is equal to the multiplicity of P in $S(z)$ by Theorem 3.1.

In the rest of this section, we assume that k is a field of characteristic 2, $n = 2s+1$, $s \geqslant 1$. From the discussion in the preceding section we know that the representatives of conjugate classes of D_n are 1, a^i $(1 \leqslant i \leqslant s)$, b. Their centralizers are $C_{D_n}(1) = D_n$, $C_{D_n}(a^i) = C_n$, $C_{D_n}(b) = \{1, b\}$, respectively. Thus we can construct all simple Yetter-Drinfeld kD_n-modules by the characterization in Introduction.

Let W_i $(0 \leqslant i \leqslant s)$ be the simple kD_n-modules defined in the preceding section. Since the centralizer $C_{D_n}(1) = D_n$, $D(W_i) = W_i$ $(0 \leqslant i \leqslant s)$ are non-isomorphic simple Yetter-Drinfeld kD_n- modules, with the comodule structures given by

$$\varphi\colon\ W_i \to W_i \otimes kD_n, \varphi(w_i) = w_i \otimes 1,$$

where $w_i \in W_i$.

Now $C_{D_n}(a^i) = C_n$ $(1 \leqslant i \leqslant s)$ is a cyclic group of order n. The representatives set of left cosets of $C_{D_n}(a^i)$ in D_n is $\{1, b\}$. Let V_j be the simple 1-dimensional kC_n-module given by

$$a \cdot v = \xi^j v, \quad v \in V_j, \quad 0 \leqslant j \leqslant n-1,$$

where ξ is the n-th primitive root of unity in k. For any i $(1 \leqslant i \leqslant s)$, let $V_j^i = V_j$ as kC_n-modules, where the upper index indicates that it is a $kC_{D_n}(a^i)$-module. Then the induced kD_n-modules $D(V_j^i) = kD_n \otimes_{kC_n} V_j^i$ are non-isomorphic simple Yetter-Drinfeld kD_n-modules of dimension 2, with the comodule structures given by

$$\varphi\colon\ D(V_j^i) \to D(V_j^i) \otimes kD_n,$$
$$\varphi(b \otimes_{kC_n} v_j^i) = (b \otimes_{kC_n} v_j^i) \otimes a^{n-i}, \varphi(1 \otimes_{kC_n} v_j^i) = (1 \otimes_{kC_n} v_j^i) \otimes a^i,$$

where $v_j^i \in V_j^i$.

Further, $C_{D_n}(b) = \langle b \rangle$ is a cyclic group of order 2, the representatives set of left cosets of $C_{D_n}(b)$ in D_n is C_n. Let U be the only simple 1-dimensional $kC_{D_n}(b)$-module. Then the induced kD_n-module $D(U) = kD_n \otimes_{k\langle b \rangle} U$ is a simple Yetter-Drinfeld kD_n-module of dimension n, with the comodule structure given by

$$\varphi\colon\ D(U) \to D(U) \otimes kD_n, \varphi(a^i \otimes_{k\langle b \rangle} u) = (a^i \otimes_{k\langle b \rangle} u) \otimes a^{2i} b,$$

where $1 \leqslant i \leqslant n$, $u \in U$.

Up to now, we know that $D(W_i)$ $(0 \leqslant i \leqslant s)$, $D(V_j^i)$ $(0 \leqslant j \leqslant n-1, 1 \leqslant i \leqslant s)$ and $D(U)$ are all simple Yetter-Drinfeld kD_n-modules up to isomorphism.

Theorem 3.2. *The ring structure of $G_0(D(kD_n))$ can be described as follows.*

(1) *Let $1 \leqslant i, j \leqslant s$, then*

$$[D(W_0)][D(W_i)] = [D(W_i)], \quad [D(W_i)]^2 = 2[D(W_0)] + [D(W_{[2i,s,n]})],$$
$$[D(W_0)]^2 = [D(W_0)], \quad [D(W_i)][D(W_j)] = [D(W_{[i+j,s,n]})] + [D(W_{i-j})], \quad j < i.$$

(2) *Let $1 \leqslant i \leqslant s$, then*

$$[D(U)][D(W_0)] = [D(U)], \quad [D(U)][D(W_i)] = 2[D(U)].$$

(3) *Let* $1 \leqslant i, k \leqslant s$, $0 \leqslant j \leqslant n-1$, *then*

$$[D(W_0)][D(V_j^i)] = [D(V_j^i)], \quad [D(V_j^i)][D(W_k)] = [D(V^i_{[j+k]_n})] + [D(V^i_{\{j-k\}_n})].$$

(4) *We have*

$$[D(U)]^2 = \sum_{i=0}^{s}[D(W_i)] + \sum_{j=0}^{n-1}\sum_{k=1}^{s}[D(V_j^k)].$$

(5) *Let* $1 \leqslant i_1, i_2 \leqslant s$, $0 \leqslant j_1, j_2 \leqslant n-1$, *then*

$$\begin{aligned}[D(V_{j_1}^{i_1})][D(V_{j_2}^{i_2})] &= [D(V^{i_2-i_1}_{\{j_2-j_1\}_n})] + [D(V^{[i_1+i_2,s,n]}_{[j_1+j_2]_n})], \quad i_1 < i_2;\\ [D(V_{j_1}^{i})][D(V_{j_2}^{i})] &= [D(W_{[j_2-j_1,s,n]})] + [D(V^{[2i,s,n]}_{[j_1+j_2]_n})], \quad j_1 < j_2;\\ [D(V_j^i)]^2 &= 2[D(W_0)] + [D(V^{[2i,s,n]}_{[2j]_n})].\end{aligned}$$

(6) *Let* $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant n$, *then*

$$[D(V_j^i)][D(U)] = 2[D(U)].$$

P r o o f. (1) The result follows from Theorem 3.1 and the structure of $D(W_i)$ $(0 \leqslant i \leqslant s)$.

(2) Take $w \in D_n$. Consider the subspace

$$S(w) = \oplus_i (a^i \otimes_{k\langle b\rangle} U) \otimes (1 \otimes_{kD_n} W_j)$$

of $D(U) \otimes D(W_j)$ $(0 \leqslant j \leqslant s)$, where the sum is taken over all indices i such that $a^i b a^{-i} = a^{2i} b = w$. If $w = b$, then

$$S(b) = (1 \otimes_{k\langle b\rangle} U) \otimes (1 \otimes_{kD_n} W_j) \cong U \otimes W_j$$

is a $kC_{D_n}(b) = k\langle b\rangle$-module.

When $j = 0$ one can easily check that $S(b) \cong U$. When $j \neq 0$, the Brauer character $\chi_{S(b)} = 2\chi$, where χ is the Brauer character afforded by U. By Proposition 2.2, the multiplicity of U in $S(b)$ is 2. Consequently, from the discussion following Theorem 3.1, the multiplicity of $D(U)$ in $D(U) \otimes D(W_j)$ is 2. Hence, we have $[D(U)][D(W_j)] = 2[D(U)]$ by comparing the dimensions of both sides. We will use this method frequently in the rest of this section. However, we will not explicitly mention it in the proof, for the sake of simplicity.

(3) Take $w \in D_n$. Consider the subspace

$$S(w) = \oplus_m (b^m \otimes_{kC_n} V_j^i) \otimes (1 \otimes_{kD_n} W_l)$$

of $D(V_j^i) \otimes D(W_l)$ $(0 \leqslant l \leqslant s)$, where the sum is taken over all indices m such that $b^m a^i b^{-m} = w$. If $w = a^i$, then

$$S(a^i) = (1 \otimes_{kC_n} V_j^i) \otimes (1 \otimes_{kD_n} W_l) \cong V_j^i \otimes W_l$$

is a $kC_{D_n}(a^i) = kC_n$-module. When $l = 0$, one can easily check that $S(a^i) \cong V_j^i$. When $l \neq 0$, the Brauer character of $S(a^i)$is

	1	a	a^2	$\ldots$	a^{n-1}
$\chi_{S(a^i)}$	2	$\omega^{j+l} + \omega^{j-l}$	$\omega^{2(j+l)} + \omega^{2(j-l)}$	$\ldots$	$\omega^{(n-1)(j+l)} + \omega^{(n-1)(j-l)}$

where $\omega \in \mathbb{C}$ is the n-th primitive root of unity. Hence $\chi_{S(a^i)} = \chi_{[j+l]_n} + \chi_{\{j-l\}_n}$, where $\chi_{[j+l]_n}$, $\chi_{\{j-l\}_n}$ are irreducible Brauer characters of kC_n. Thus $[D(V_j^i)][D(W_l)] = [D(V_{[j+l]_n}^i)] + [D(V_{\{j-l\}_n}^i)]$.

(4) Take $w \in D_n$. Consider the subspace

$$S(w) = \oplus_{j,k}(a^j \otimes_{k\langle b\rangle} U) \otimes (a^k \otimes_{k\langle b\rangle} U)$$

of $D(U) \otimes D(U)$, where the sum is taken over all indices j, k such that $(a^k b a^{-k}) \times (a^j b a^{-j}) = a^{2(k-j)} = w$. If $w = 1$, then

$$S(1) = \oplus_j (a^j \otimes_{k\langle b\rangle} U) \otimes (a^j \otimes_{k\langle b\rangle} U)$$

is a kD_n-module. The Brauer character of $S(1)$ is

	1	a	a^2	$\ldots$	a^s
$\chi_{S(1)}$	2	$\omega^i + \omega^{-i}$	$\omega^{2i} + \omega^{-2i}$	$\ldots$	$\omega^{si} + \omega^{-si}$

where $\omega \in \mathbb{C}$ is the n-th primitive root of unity. Hence $\chi_{S(1)} = \sum\limits_{i=0}^{s} \chi_i$, where χ_i $(0 \leqslant i \leqslant s)$ is the irreducible Brauer character of kD_n. Thus the multiplicity of $D(W_i)$ $(0 \leqslant i \leqslant s)$ in $D(U) \otimes D(U)$ is 1. If $w = a$, then

$$S(a) = \oplus_j (a^j \otimes_{k\langle b\rangle} U) \otimes (a^{j+s+1} \otimes_{k\langle b\rangle} U)$$

is a $kC_{D_n}(a) = kC_n$-module. The Brauer character of $S(a)$ is

	1	a	a^2	$\ldots$	a^{n-1}
$\chi_{S(a)}$	n	0	0	$\ldots$	0

Hence $\chi_{S(a)} = \sum\limits_{j=0}^{n-1} \chi_j$, where χ_j $(0 \leqslant j \leqslant n-1)$ is the irreducible Brauer character of kC_n. Thus the multiplicity of $D(V_j^1)$ $(0 \leqslant j \leqslant n-1)$ in $D(U) \otimes D(U)$ is 1.

Similarly, we can show that the multiplicity of $D(V_j^i)$ ($2 \leqslant i \leqslant s$, $0 \leqslant j \leqslant n-1$) in $D(U) \otimes D(U)$ is also 1.

(5) We only prove the first equation, since the others can be done similarly. Take $w \in D_n$. Consider the subspace

$$S(w) = \oplus_{i,j}(b^i \otimes_{kC_n} V_{j_1}^{i_1}) \otimes (b^j \otimes_{kC_n} V_{j_2}^{i_2})$$

of $D(V_{j_1}^{i_1}) \otimes D(V_{j_2}^{i_2})$, where the sum is taken over all indices i, j such that $(b^j a^{i_2} b^{-j}) \times (b^i a^{i_1} b^{-i}) = w$. If $w = a^{i_2 - i_1}$, then

$$S(a^{i_2-i_1}) = (b \otimes_{kC_n} V_{j_1}^{i_1}) \otimes (1 \otimes_{kC_n} V_{j_2}^{i_2}) \cong V_{\{j_2-j_1\}_n}^{i_2-i_1}$$

is a $kC_{D_n}(a^{i_2-i_1}) = kC_n$-module. Hence the multiplicity of $D(V_{\{j_2-j_1\}_n}^{i_2-i_1})$ in $D(V_{j_1}^{i_1}) \otimes D(V_{j_2}^{i_2})$ is 1. If $w = a^{i_1+i_2}$, then

$$S(a^{i_1+i_2}) = (1 \otimes_{kC_n} V_{j_1}^{i_1}) \otimes (1 \otimes_{kC_n} V_{j_2}^{i_2}) \cong V_{[j_1+j_2]_n}^{[i_1+i_2,s,n]}$$

is a $kC_{D_n}(a^{i_1+i_2}) = kC_n$-module. Hence, the multiplicity of $D(V_{[j_1+j_2]_n}^{[i_1+i_2,s,n]})$ in $D(V_{j_1}^{i_1}) \otimes D(V_{j_2}^{i_2})$ is 1. Thus, $[D(V_{j_1}^{i_1})][D(V_{j_2}^{i_2})] = [D(V_{\{j_2-j_1\}_n}^{i_2-i_1})] + [D(V_{[j_1+j_2]_n}^{[i_1+i_2,s,n]})]$, $i_1 < i_2$.

(6) Take $w \in D_n$. Consider the subspace

$$S(w) = \oplus_{l,k}(b^l \otimes_{kC_n} V_j^i) \otimes (a^k \otimes_{k\langle b\rangle} U)$$

of $D(V_j^i) \otimes D(U)$, where the sum is taken over all indices l, k such that $(a^k b a^{-k}) \times (b^l a^i b^{-l}) = w$. If $w = b$ and i is odd, then

$$S(b) = (1 \otimes_{kC_n} V_j^i) \otimes (a^{(n+i)/2} \otimes_{k\langle b\rangle} U) \oplus (b \otimes_{kC_n} V_j^i) \otimes (a^{(n-i)/2} \otimes_{k\langle b\rangle} U)$$

is a $kC_{D_n}(b) = k\langle b\rangle$-module. One can easily check that the multiplicity of U in $S(b)$ is 2. If $w = b$ and i is even, then

$$S(b) = (1 \otimes_{kC_n} V_j^i) \otimes (a^{i/2} \otimes_{k\langle b\rangle} U) \oplus (b \otimes_{kC_n} V_j^i) \otimes (a^{(2n-i)/2} \otimes_{k\langle b\rangle} U)$$

is a $kC_{D_n}(b) = k\langle b\rangle$-module. It is also easy to check that the multiplicity of U in $S(b)$ is 2. Hence the multiplicity of $D(U)$ in $D(V_j^i) \otimes D(U)$ is 2. This completes the proof. $\square$

Remark 3.3. I have computed the Grothendieck ring of $D(kD_n)$ in the cases $n = 3, 5, 7$ and found out that their generators are $[D(V_0^1)]$, $[D(V_1^1)]$, $[D(U)]$. Hence I guess the generators of $G_0(D(kD_n))$ are $[D(V_0^1)]$, $[D(V_1^1)]$, $[D(U)]$ for arbitrary n.

Remark 3.4. In case n is even, we can proceed similarly. However, the computation is very tedious. So I omit this case.

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