Martin Doležal
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A NOTE ON THE THREE-SEGMENT PROBLEM

Martin Doležal, Praha

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Abstract. We improve a theorem of C. L. Belna (1972) which concerns boundary behaviour of complex-valued functions in the open upper half-plane and gives a partial answer to the (still open) three-segment problem.

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Consider a function f defined in an open set G in the complex plane $\mathbb{C}$ with values in the Riemann sphere $\mathbb{W}$. For an arbitrary set $A \subset G$ and for all $p \in \bar{A} \setminus A$, the *cluster set* $C(f, A, p)$ *of* f *relative to the set* A *at the point* p is the set of all points $w \in \mathbb{W}$ for which there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset A$ such that $\lim_{k\to\infty} z_k = p$ and $\lim_{k\to\infty} f(z_k) = w$. If there exist three rectilinear segments S_1, S_2 and S_3 in G that have a common endpoint p such that $C(f, S_1, p) \cap C(f, S_2, p) \cap C(f, S_3, p) = \emptyset$, we say that f has the *three-segment property at* p. The following problem was posed in [1, Open question 1].

P r o b l e m 1. Does there exist a continuous complex-valued function in the open unit disk $\mathbb{D}$ having the three-segment property at each point of a set of positive one-dimensional measure or of second category in the unit circle?

It seems to be very probable that this problem is equivalent to the following one.

P r o b l e m 2. Does there exist a continuous function from the open upper half plane $\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$ into the Riemann sphere $\mathbb{W}$ having the three-segment property at each point of a set of positive one-dimensional measure or of second category in $\mathbb{R}$?

In this form, the 'three-segment problem' is stated in [2]. (Another formulation can be found in [4].)

First, let us introduce some terminology, slightly changing the terminology of [2]. A *ray at* $p \in \mathbb{R}$ *with the direction* $s \in (0, \pi)$ is the set $\{z \in \mathbb{C}\colon \arg(z - p) = s\}$. If λ_1, λ_2 and λ_3 are arbitrary functions from $\mathbb{R}$ into the open interval $(0, \pi)$, then $S_j(p)$ is the ray at $p \in \mathbb{R}$ with the direction $\lambda_j(p)$, $j = 1, 2, 3$. Whenever $C(f, S_1(p), p) \cap C(f, S_2(p), p) \cap C(f, S_3(p), p) = \emptyset$ for some function $f\colon \mathbb{H} \to \mathbb{W}$, we say that f has the *three-segment property at p relative to the functions* λ_1, λ_2 *and* λ_3.

Now, we can equivalently reformulate Problem 2 as follows: Does there exist a continuous function f from the open upper half plane $\mathbb{H}$ into the Riemann sphere $\mathbb{W}$ and functions λ_1, λ_2 and λ_3 from $\mathbb{R}$ into the open interval $(0, \pi)$ such that f has the three-segment property relative to the functions λ_1, λ_2 and λ_3 at each point of a set of positive one-dimensional measure or of second category in $\mathbb{R}$?

The theorem of [2] gives a partial answer to this problem. This theorem says that for $f\colon \mathbb{H} \to \mathbb{W}$ continuous and λ_1, λ_2 monotone and absolutely continuous on finite intervals, the set of all points at which f has the three-segment property relative to λ_1, λ_2 and λ_3 is of first category and measure zero in $\mathbb{R}$ for arbitrary λ_3. However, the proof of this theorem contains a gap. The claim (which can be found on page 240, four lines from below) that g_j satisfies the hypotheses of the lemma is not proved. Moreover, an easy example (see Remark below) shows that this claim is incorrect. Nevertheless, using the ideas of [2] but changing and refining the arguments, we show that the result of [2] is correct. Furthermore, we generalize this result, proving the following theorem. In particular, we prove that the assumption of absolute continuity of λ_1 and λ_2 on finite intervals can be removed since every monotone function $\lambda\colon \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere and has at most countably many discontinuities.

Theorem. *Let* $f\colon \mathbb{H} \to \mathbb{W}$ *be continuous. Let* λ_1, λ_2 *and* λ_3 *be functions from* $\mathbb{R}$ *into the open interval* $(0, \pi)$. *Let* λ_1 *and* λ_2 *be approximately differentiable a.e. on* $\mathbb{R}$.

(1) *Then the set* $Q(f; \lambda_1, \lambda_2, \lambda_3)$ *of all points at which* f *has the three-segment property relative to* λ_1, λ_2 *and* λ_3 *is of measure zero in* $\mathbb{R}$.
(2) *If there exists a measure zero set* $M \subset \mathbb{R}$ *of first category such that* $\lambda_1|(\mathbb{R} \setminus M)$ *and* $\lambda_2|(\mathbb{R} \setminus M)$ *are continuous, then the set* $Q(f; \lambda_1, \lambda_2, \lambda_3)$ *is also of first category in* $\mathbb{R}$.

P r o o f. (1) Let us denote by m the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{B}$ be a countable basis for the usual topology on $\mathbb{W}$, let $\mathcal{S}$ be the (countable) collection of all finite unions of the sets $B \in \mathcal{B}$ and let $\mathcal{S}^*$ be the set of all 3-tuples (G_1, G_2, G_3) of sets in $\mathcal{S}$ for which $\overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3 = \emptyset$. For each $(G_1, G_2, G_3) \in \mathcal{S}^*$ and all rational numbers α, β satisfying $0 < \alpha < \beta < \pi$ and each rational $r > 0$, let $Q(G_1, G_2, G_3; \alpha, \beta; r)$ be the set of all points $p \in \mathbb{R}$ at which there exists a ray $\tilde{S}_3(p)$ with a direction $\tilde{\lambda}_3(p)$ such that

(i) $\alpha \leqslant \tilde{\lambda}_3(p) \leqslant \beta$,

(ii) $\lambda_j(p) \notin (\alpha - r, \beta + r)$, $j = 1, 2$,

(iii) $f(S_j(p, r)) \subset \overline{G}_j$, $j = 1, 2$, where $S_j(p, r) := S_j(p) \cap \{z \in \mathbb{H} \colon \operatorname{Im}(z) \leqslant r\}$,

(iv) $f(\tilde{S}_3(p, r)) \subset \overline{G}_3$, where $\tilde{S}_3(p, r) := \tilde{S}_3(p) \cap \{z \in \mathbb{H} \colon \operatorname{Im}(z) \leqslant r\}$.

It is easy to see that $Q(f; \lambda_1, \lambda_2, \lambda_3)$ is a subset of the countable union of all $Q(G_1, G_2, G_3; \alpha, \beta; r)$. Denote by Q_0 one of the sets $Q(G_1, G_2, G_3; \alpha, \beta; r)$. The functions λ_1 and λ_2 are approximately continuous a.e. on $\mathbb{R}$ and thus measurable. Hence we can find open sets V_n, $n \in \mathbb{N}$, such that $m(V_n) < 1/n$ and both $\lambda_1|(\mathbb{R} \setminus V_n)$, $\lambda_2|(\mathbb{R} \setminus V_n)$ are continuous. Using the continuity of f, we can easily see that $Q_0^n := Q_0 \setminus V_n$ is closed and Q_0 is measurable because $m\Big(Q_0 \setminus \bigcup\limits_{n=1}^{\infty} Q_0^n\Big) = 0$. Let us assume that $m(Q_0) > 0$. Applying [3, Theorem 3.1.16.] to the function $\lambda = (\lambda_1|Q_0, \lambda_2|Q_0) \colon Q_0 \to \mathbb{R}^2$ (which is approximately differentiable a.e. on its domain) we obtain that there exists a continuously differentiable function $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2) \colon \mathbb{R} \to \mathbb{R}^2$ such that

$$m(\{p \in Q_0 \colon \lambda_j(p) = \tilde{\lambda}_j(p), \ j = 1, 2\}) > 0.$$

Denote $A := \{p \in Q_0 \colon \lambda_j(p) = \tilde{\lambda}_j(p), \ j = 1, 2\}$ and choose a point $p_0 \in A$ such that p_0 is a point of density of A. Without loss of generality we may assume that $p_0 = 0$. Let $\tilde{S}_3(0)$ be a ray with a direction $\tilde{\lambda}_3(0)$ given by $0 \in Q_0$. Let us fix $j \in \{1, 2\}$ and assume first that $\tilde{\lambda}_3(0) < \lambda_j(0)$. Whenever $\tilde{\lambda}_j(p) \in (0, \pi)$, we will denote by $\tilde{S}_j(p)$ the ray at p given by the direction $\tilde{\lambda}_j(p)$ and $\tilde{S}_j(p, r) := \tilde{S}_j(p) \cap \{z \in \mathbb{H} \colon \operatorname{Im}(z) \leqslant r\}$. The continuity of $\tilde{\lambda}_j$ implies that there exists $a_j > 0$ such that $\tilde{\lambda}_j(p) \in (0, \pi)$ and $\tilde{S}_3(0) \cap \tilde{S}_j(p) \neq \emptyset$ for all $p \in (0, a_j)$. Hence we can define a function $\mu_j \colon (0, a_j) \to \tilde{S}_3(0)$ by

$$\{\mu_j(p)\} = \tilde{S}_3(0) \cap \tilde{S}_j(p).$$

Next we define a function $g_j \colon [0, a_j) \to [0, +\infty)$ by $g_j(0) = 0$ and $g_j(p) = |\mu_j(p)|$ for $p \in (0, a_j)$. It is easy to verify that for all $p \in [0, a_j)$,

$$g_j(p) = p \, \frac{\sin(\tilde{\lambda}_j(p))}{\sin(\tilde{\lambda}_j(p) - \tilde{\lambda}_3(0))}$$

and

$$g_j'(p) = \frac{\sin(\tilde{\lambda}_j(p))}{\sin(\tilde{\lambda}_j(p) - \tilde{\lambda}_3(0))} - p \, \frac{\tilde{\lambda}_j'(p) \sin(\tilde{\lambda}_3(0))}{\sin^2(\tilde{\lambda}_j(p) - \tilde{\lambda}_3(0))}.$$

The continuity of $\tilde{\lambda}_j$ and $\tilde{\lambda}_j'$ implies that there exists $0 < b_j < a_j$ such that if $q_j^1 = \inf\{g_j'(p) \colon p \in [0, b_j]\}$, $q_j^2 = \sup\{g_j'(p) \colon p \in [0, b_j]\}$, then $0 < q_j^1 \leqslant q_j^2 \leqslant \frac{3}{2} q_j^1$

and $g_j(p) < r$ for all $p \in [0, b_j]$. Moreover, we can assume that $m(A \cap (0,b)) > \frac{3}{4}b$ for all $b \in (0, b_j]$ because 0 is a point of density of A. Then clearly $m(g_j((0,b))) = g_j(b) \leqslant q_j^2 b \leqslant \frac{3}{2} q_j^1 b$ for all $b \in (0, b_j]$. Since g_j is strictly increasing on $[0, b_j]$ we can use [5, Chapter VIII, 2., Lemma 3] obtaining

$$m(g_j(A \cap (0,b))) \geqslant q_j^1 m(A \cap (0,b)) > \tfrac{3}{4} q_j^1 b \geqslant \tfrac{1}{2} m(g_j((0,b)))$$

for all $b \in (0, b_j]$. In the case $\tilde{\lambda}_3(0) > \lambda_j(0)$ we can define functions μ_j and g_j on the left neighbourhood of 0 by the same formulas as above and we similarly obtain that there exists $b_j > 0$ such that

$$m(g_j(A \cap (-b,0))) > \tfrac{1}{2} m(g_j((-b,0)))$$

for all $b \in (0, b_j]$.

Now, we denote the domain of g_j by $D(g_j)$, $j = 1, 2$, and find $b_1' \in (-b_1, b_1) \cap D(g_1)$ and $b_2' \in (-b_2, b_2) \cap D(g_2)$ such that $b_1' \neq 0$, $b_2' \neq 0$ and $g_1(b_1') = g_2(b_2')$. Then $g_1((-b_1', b_1') \cap D(g_1)) = g_2((-b_2', b_2') \cap D(g_2))$ and it follows from the above estimates that the sets $g_1(A \cap (-b_1', b_1') \cap D(g_1))$ and $g_2(A \cap (-b_2', b_2') \cap D(g_2))$ are not disjoint. Thus there exist points $p_1 \in A \cap (-b_1', b_1') \cap D(g_1)$ and $p_2 \in A \cap (-b_2', b_2') \cap D(g_2)$ such that $\mu_1(p_1) = \mu_2(p_2) \in \tilde{S}_3(0, r)$. But we also have $\mu_j(p_j) \in \tilde{S}_j(p_j, r) = S_j(p_j, r)$, $j = 1, 2$. Therefore

$$f(\mu_1(p_1)) \in \overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3,$$

which contradicts $(G_1, G_2, G_3) \in \mathcal{S}^*$. Thus $m(Q_0) = 0$ and it follows that also $m(Q(f; \lambda_1, \lambda_2, \lambda_3)) = 0$.

(2) Let us assume now that there exists a measure zero set $M \subset \mathbb{R}$ of first category such that $\lambda_1 | (\mathbb{R} \setminus M)$ and $\lambda_2 | (\mathbb{R} \setminus M)$ are continuous. Let Q_0 have the same meaning as above. It is easy to verify that $Q_0 \cap (\mathbb{R} \setminus M)$ is closed in $\mathbb{R} \setminus M$ and thus $m(\overline{Q_0 \cap (\mathbb{R} \setminus M)}) = 0$. It follows that $Q_0 \cap (\mathbb{R} \setminus M)$ is nowhere dense. Therefore Q_0 is of first category and $Q(f; \lambda_1, \lambda_2, \lambda_3)$ is also of first category. □

R e m a r k. In [2], the functions $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are not introduced and near 0, the functions

$$g_j(p) = p \, \frac{\sin(\lambda_j(p))}{\sin(\lambda_j(p) - \operatorname{dir} S_3(0))}, \quad j = 1, 2,$$

(where $\operatorname{dir} S_3(0) = \lambda_3(0) < \lambda_j(0) = \operatorname{dir} S_j(0)$; there is a typo in [2] saying that $\operatorname{dir} S_3(0) > \operatorname{dir} S_j(0)$) are considered. It is claimed in [2, p. 240] that g_j satisfies the hypotheses of the lemma, in particular, g_j is monotone on $[0, a]$ for some $a > 0$. However, the properties of λ_j (λ_j is monotone, absolutely continuous on finite intervals and λ_j' is approximately continuous at 0) do not imply the existence of such

$a > 0$. Indeed, it is easy to construct a monotone function λ_j which is absolutely continuous on finite intervals and such that

(i) λ_j' is approximately continuous at 0,

(ii) there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset (0, +\infty)$ satisfying $\lim\limits_{n\to\infty} p_n = 0$ and $\lambda_j'(p_n) = \infty$, $n = 1, 2, \ldots$.

Then, whenever $\lambda_j'(p)$ exists, we have

$$g_j'(p) = \frac{\sin(\lambda_j(p))}{\sin(\lambda_j(p) - \operatorname{dir} S_3(0))} - p\,\frac{\lambda_j'(p)\sin(\operatorname{dir} S_3(0))}{\sin^2(\lambda_j(p) - \operatorname{dir} S_3(0))}$$

and it follows that $g_j'(0) > 0$ and $g_j'(p_n) = -\infty$, $n = 1, 2, \ldots$. Hence g_j is not monotone on $[0, a]$ for any $a > 0$.

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Author's address: *Martin Doležal*, Charles University in Prague, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8-Karlín, Czech Republic, e-mail: dolezal@karlin.mff.cuni.cz.