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MILDLY $(1,2)^*$-NORMAL SPACES AND SOME BITOPOLOGICAL FUNCTIONS

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Abstract. The aim of the paper is to introduce and study a new class of spaces called mildly $(1,2)^*$-normal spaces and a new class of functions called $(1,2)^*$-rg-continuous, $(1,2)^*$-R-map, almost $(1,2)^*$-continuous function and almost $(1,2)^*$-rg-closed function in bitopological spaces. Subsequently, the relationships between mildly $(1,2)^*$-normal spaces and the new bitopological functions are investigated. Moreover, we obtain characterizations of mildly $(1,2)^*$-normal spaces, properties of the new bitopological functions and preservation theorems for mildly $(1,2)^*$-normal spaces in bitopological spaces.

Keywords: mildly $(1,2)^*$-normal space, $(1,2)^*$-rg-closed set, $(1,2)^*$-rg-continuous function, almost $(1,2)^*$-continuous function, almost $(1,2)^*$-rg-closed function

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1. Introduction

The first step in normality was made by Vigilino [33] who defined semi-normal spaces. Then Singal and Arya [32] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a semi-normal space and an almost normal space. Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In recent years, many authors have studied several forms of normality [2], [3], [4], [10], [14], [16], [17], [19], [20], [22], [23], [31], [32]. In particular, the notion of mildly normal spaces was introduced by M. K. Singal and A. R. Singal [31]. Further it was widely studied and investigated by Noiri [16], [17] and J. K. Park and J. H. Park [22].

It is well known that the concept of closedness is fundamental as concerns the investigations of general topological spaces. Levine [13] initiated the study of generalized closed sets. The notion of regular g-closed sets as a generalization of g-closed sets due to Levine [13] was defined and investigated by Palaniappan and Rao [21]. Noiri [17] improved the characterization of mildly normal spaces by using regular g-closed sets and the preservation theorems for mildly normal spaces established in [1] and [16].

In this paper we introduce and investigate a generalization of mildly normal spaces by utilizing regular $(1,2)^*$-closed sets in bitopological spaces. The notions of $(1,2)^*$-rg-continuous functions, almost $(1,2)^*$-continuous functions, $(1,2)^*$-R-maps and almost $(1,2)^*$-rg-closed functions are introduced in bitopological spaces. Also, we obtain characterizations and properties of mildly $(1,2)^*$-normal spaces and their preservation theorems.

2. Preliminaries

Throughout the paper $(X,\ \tau_1,\ \tau_2)$, $(Y,\ \sigma_1,\ \sigma_2)$ and $(Z,\ \varrho_1,\ \varrho_2)$ (or simply X, Y and Z) denote bitopological spaces.

First we recall some definitions used in the sequel.

Definition 2.1 [24]**.** Let S be a subset of X. Then S is said to be $\tau_{1,2}$-open if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$.

The complement of a $\tau_{1,2}$-open set is $\tau_{1,2}$-closed.

Definition 2.2 [24]**.** Let S be a subset of X. Then

(i) the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-cl(S), is defined as $\bigcap\{F\colon\ S \subset F$ and F is $\tau_{1,2}$-closed$\}$;

(ii) the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-int (S), is defined as $\bigcup\{F\colon\ F \subset S$ and F is $\tau_{1,2}$-open$\}$.

Note 2.3 [24]. Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.4 [26]**.** Let S be a subset of X. Then S is said to be regular $(1,2)^*$-open if $S = \tau_{1,2}$-int $(\tau_{1,2}$-cl$(S))$.

The complement of a regular $(1,2)^*$-open set is regular $(1,2)^*$-closed.

The families of regular $(1,2)^*$-open and regular $(1,2)^*$-closed sets of X are denoted by $(1,2)^*$-RO(X) and $(1,2)^*$-RC(X) respectively.

Definition 2.5 [25]**.** Let S be a subset of X. Then S is said to be generalized $(1,2)^*$-closed (briefly $(1,2)^*$-g-closed) if $\tau_{1,2}$-cl$(S) \subset U$ whenever $S \subset U$ and U is $\tau_{1,2}$-open in X.

The complement of a $(1,2)^*$-g-closed set is $(1,2)^*$-g-open.

We now introduce a new set as follows.

Definition 2.6. Let S be a subset of X. Then S is said to be regular $(1,2)^*$-g-closed (briefly $(1,2)^*$-rg-closed) if $\tau_{1,2}$-cl$(S) \subset U$ whenever $S \subset U$ and $U \in (1,2)^*$-RO(X).

The complement of a $(1,2)^*$-rg-closed set is $(1,2)^*$-rg-open.

Remark 2.7 [26]. Every regular $(1,2)^*$-closed set is $\tau_{1,2}$-closed but not conversely.

Remark 2.8 [25]. Every $\tau_{1,2}$-closed set is $(1,2)^*$-g-closed but not conversely.

Theorem 2.9. *If A is $(1,2)^*$-g-closed, then A is $(1,2)^*$-rg-closed.*

Proof. Suppose that $A \subset U$, where U is regular $(1,2)^*$-open. Now U being regular $(1,2)^*$-open implies U is $\tau_{1,2}$-open. Thus $A \subset U$ and U is $\tau_{1,2}$-open. Since A is $(1,2)^*$-g-closed, $\tau_{1,2}$-cl$(A) \subset U$. Therefore A is $(1,2)^*$-rg-closed. □

Example 2.10. The converse of Theorem 2.9 is not true in general.

Let $X = \{a,b,c,d\}$ with $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ and $\tau_2 = \{\emptyset, X, \{a,b,d\}\}$. Clearly $A = \{a,c\}$ is $(1,2)^*$-rg-closed but it is not $(1,2)^*$-g-closed.

Remark 2.11. We have the following implications for properties of subsets.

Regular $(1,2)^*$-closed $\rightleftarrows$ $\tau_{1,2}$-closed $\rightleftarrows$ $(1,2)^*$-g-closed $\rightleftarrows$ $(1,2)^*$-rg-closed.

3. Characterizations of $(1,2)^*$-rg-closed sets

Example 3.1. The union of two $(1,2)^*$-rg-closed sets need not be $(1,2)^*$-rg-closed.

Let $X = \{a,b,c,d\}$ with $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}, \{a,c,d\}\}$. Then the sets in $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$ are $\tau_{1,2}$-closed. Clearly $\{a\}$ and $\{b\}$ are $(1,2)^*$-rg-closed. However, their union $\{a,b\}$ is not $(1,2)^*$-rg-closed.

Example 3.2. The intersection of two $(1,2)^*$-rg-closed sets need not be $(1,2)^*$-rg-closed.

Refer to Example 2.10, Clearly $\{a,b\}$ and $\{a,c\}$ are $(1,2)^*$-rg-closed but their intersection $\{a\}$ is not $(1,2)^*$-rg-closed.

Theorem 3.3. *If A is $(1,2)^*$-rg-closed, then $[\tau_{1,2}\text{-cl}(A)\setminus A]$ contains no non-empty regular $(1,2)^*$-closed set.*

Proof. Suppose that A is $(1,2)^*$-rg-closed. Let F be a regular $(1,2)^*$-closed subset of $\tau_{1,2}\text{-cl}(A)\setminus A$. Then $F \subset [\tau_{1,2}\text{-cl}(A) \cap (X \setminus A)]$ and so $A \subset [X \setminus F]$. But A is $(1,2)^*$-rg-closed. Therefore $\tau_{1,2}\text{-cl}(A) \subset [X \setminus F]$. Consequently, $F \subset [X \setminus \tau_{1,2}\text{-cl}(A)]$. We already have $F \subset \tau_{1,2}\text{-cl}(A)$. Hence $F \subset [\tau_{1,2}\text{-cl}(A) \cap X \setminus \tau_{1,2}\text{-cl}(A)] = \emptyset$. Thus $F = \emptyset$. Therefore $\tau_{1,2}\text{-cl}(A) \setminus A$ contains no non-empty regular $(1,2)^*$-closed set. $\square$

Example 3.4. The converse of Theorem 3.3 is not true.

Refer to Example 2.10. Let $A = \{a\}$. We have that $\tau_{1,2}\text{-cl}(A)\setminus A = \{a,c,d\}\setminus\{a\} = \{c,d\}$ does not contain any non-empty regular $(1,2)^*$-closed sets. However, A is $(1,2)^*$-rg-closed in X.

Theorem 3.5. *Let A be $(1,2)^*$-rg-closed set. Then A is regular $(1,2)^*$-closed if and only if $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A]$ is regular $(1,2)^*$-closed.*

Proof. Let A be a $(1,2)^*$-rg-closed. If A is regular $(1,2)^*$-closed, then $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A] = \emptyset$. We know $\emptyset$ is always regular $(1,2)^*$-closed. Therefore $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A]$ is regular $(1,2)^*$-closed. Conversely, suppose that $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A]$ is regular $(1,2)^*$-closed. Since A is $(1,2)^*$-rg-closed, $[\tau_{1,2}\text{-cl}(A) \setminus A]$ contains the regular $(1,2)^*$-closed set $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A]$. By Theorem 3.3, $[\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) \setminus A] = \emptyset$. Hence $\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}\,(A)) = A$. Therefore A is regular $(1,2)^*$-closed. $\square$

Theorem 3.6. *A set A is $(1,2)^*$-rg-open if and only if the following condition holds:*

$$F \subset \tau_{1,2}\text{-int}\,(A) \textit{ whenever } F \textit{ is regular } (1,2)^*\textit{-closed and } F \subset A.$$

Proof. Suppose the condition holds. Put $[X \setminus A] = B$. Suppose that $B \subset U$ where $U \in (1,2)^*\text{-RO}(X)$. Now $X \setminus A \subset U$ implies $F = [X \setminus U] \subset A$ and F is regular $(1,2)^*$-closed, which implies $F \subset \tau_{1,2}\text{-int}\,(A)$. Also $F \subset \tau_{1,2}\text{-int}\,(A)$ implies $[X \setminus \tau_{1,2}\text{-int}\,(A)] \subset [X \setminus F] = U$. This implies $[X \setminus (\tau_{1,2}\text{-int}\,(X \setminus B))] \subset U$. Therefore $[X \setminus (\tau_{1,2}\text{-int}\,(X \setminus B))] \subset U$ or equivalently $\tau_{1,2}\text{-cl}(B) \subset U$. Thus B is $(1,2)^*$-rg-closed. Hence A is $(1,2)^*$-rg-open. Conversely, suppose that A is $(1,2)^*$-rg-open, $F \subset A$ and F is regular $(1,2)^*$-closed. Then $[X \setminus F]$ is regular $(1,2)^*$-open. Then $(X\setminus A) \subset (X\setminus F)$. Hence $\tau_{1,2}\text{-cl}(X\setminus A) \subset (X\setminus F)$ because $(X\setminus A)$ is $(1,2)^*$-rg-closed. Therefore $F \subset (X \setminus \tau_{1,2}\text{-cl}(X \setminus A)) = \tau_{1,2}\text{-int}\,(A)$. $\square$

Theorem 3.7. *If A is $(1,2)^*$-rg-closed, then $[\tau_{1,2}\text{-cl}(A) \setminus A]$ is $(1,2)^*$-rg-open.*

Proof. Suppose that A is $(1,2)^*$-rg-closed and that $F \subset [\tau_{1,2}\text{-cl}(A) \setminus A]$, F being regular $(1,2)^*$-closed. By Theorem 3.3, $F = \emptyset$ and hence $F \subset \tau_{1,2}\text{-int}[\tau_{1,2}\text{-cl}(A) \setminus A]$. By Theorem 3.6, $[\tau_{1,2}\text{-cl}(A) \setminus A]$ is $(1,2)^*$-rg-open. $\square$

Example 3.8. The converse of Theorem 3.7 is not true.

Refer to Example 2.10. Let $A = \{a\}$. Then $[\tau_{1,2}\text{-cl}(A) \setminus A] = \{a,c,d\} \setminus \{a\} = \{c,d\}$ which is $(1,2)^*$-rg-open. But $A = \{a\}$ is not a $(1,2)^*$-rg-closed set.

4. Characterizations of mildly $(1,2)^*$-normal spaces

Definition 4.1. A space X is said to be mildly $(1,2)^*$-normal if for every pair of disjoint $H, K \in (1,2)^*\text{-RC}(X)$, there exist disjoint $\tau_{1,2}$-open sets U, V of X such that $H \subset U$ and $K \subset V$.

Theorem 4.2. *The following are equivalent for a space X.*

(i) *X is mildly $(1,2)^*$-normal;*

(ii) *for any disjoint $H, K \in (1,2)^*\text{-RC}(X)$, there exist disjoint $(1,2)^*$-g-open sets U, V such that $H \subset U$ and $K \subset V$;*

(iii) *for any disjoint $H, K \in (1,2)^*\text{-RC}(X)$, there exist disjoint $(1,2)^*$-rg-open sets U, V such that $H \subset U$ and $K \subset V$;*

(iv) *for any disjoint $H \in (1,2)^*\text{-RC}(X)$ and any $V \in (1,2)^*\text{-RO}(X)$ containing H, there exists a $(1,2)^*$-rg-open set U of X such that $H \subset U \subset \tau_{1,2}\text{-cl}(U) \subset V$.*

Proof. It is obvious that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (iv). Let $H \in (1,2)^*\text{-RC}(X)$ and $H \subset V \in (1,2)^*\text{-RO}(X)$. There exist disjoint $(1,2)^*$-rg-open sets U, W such that $H \subset U$ and $(X \setminus V) \subset W$. By Theorem 3.6, we have $(X \setminus V) \subset \tau_{1,2}\text{-int}\,(W)$ and $[U \cap \tau_{1,2}\text{-int}\,(W)] = \emptyset$. Therefore, we obtain $[\tau_{1,2}\text{-cl}(U) \cap \tau_{1,2}\text{-int}\,(W)] = \emptyset$ and hence $H \subset U \subset \tau_{1,2}\text{-cl}(U) \subset [X \setminus \tau_{1,2}\text{-int}\,(W)] \subset V$.

(iv) $\Rightarrow$ (i). Let H, K be disjoint regular $(1,2)^*$-closed sets of X. Then $H \subset (X \setminus K) \in (1,2)^*\text{-RO}(X)$ and there exists a $(1,2)^*$-rg-open set G of X such that $H \subset G \subset \tau_{1,2}\text{-cl}(G) \subset (X \setminus K)$. Put $U = \tau_{1,2}\text{-int}\,(G)$ and $V = X \setminus \tau_{1,2}\text{-cl}(G)$. Then U and V are disjoint $\tau_{1,2}$-open sets of X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly $(1,2)^*$-normal. $\square$

5. Some bitopological functions

We shall recall the definitions of some functions used in the sequel.

Definition 5.1. A function $f\colon X \to Y$ is said to be

(i) $(1,2)^*$-g-continuous [9] if $f^{-1}(F)$ is $(1,2)^*$-g-closed in X for every $\sigma_{1,2}$-closed set F of Y;

(ii) a $(1,2)^*$-R-map [12] if $f^{-1}(F) \in (1,2)^*$-RO(X) for every $F \in (1,2)^*$-RO(Y);

(iii) completely $(1,2)^*$-continuous [11] if $f^{-1}(F) \in (1,2)^*$-RO(X) for every $\sigma_{1,2}$-open set F of Y.

We now introduce a new class of functions.

Definition 5.2. A function $f\colon X \to Y$ is said to be

(i) $(1,2)^*$-rg-continuous if $f^{-1}(F)$ is $(1,2)^*$-rg-closed in X for every $\sigma_{1,2}$-closed set F of Y;

(ii) almost $(1,2)^*$-continuous if $f^{-1}(F)$ is $\tau_{1,2}$-open in X for every $F \in (1,2)^*$-RO(Y);

(iii) almost $(1,2)^*$-g-continuous if $f^{-1}(F)$ is $(1,2)^*$-g-closed in X for every $F \in (1,2)^*$-RC(Y);

(iv) almost $(1,2)^*$-rg-continuous if $f^{-1}(F)$ is $(1,2)^*$-rg-closed in X for every $F \in (1,2)^*$-RC(Y).

Example 5.3. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a,c\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}\}$. Let $Y = \{a,b,c\}$, $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b\}\}$. Then (a) define $f\colon X \to Y$ as $f(a) = c$; $f(b) = b$; $f(c) = a$. Clearly f is almost $(1,2)^*$-rg-continuous but it is neither almost $(1,2)^*$-g-continuous nor $(1,2)^*$-rg-continuous. (b) Define $f\colon X \to Y$ as $f(a) = a$, $f(b) = c$, $f(c) = b$. Clearly f is both $(1,2)^*$-continuous and a $(1,2)^*$-R-map but it is not completely $(1,2)^*$-continuous.

Example 5.4. Let $X = Y = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a,b\}\}$. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a,c\}\}$. Then (a) define $f\colon X \to Y$ as $f(a) = b$; $f(b) = c$; $f(c) = a$. Clearly f is both $(1,2)^*$-rg-continuous and almost $(1,2)^*$-g-continuous but it is not $(1,2)^*$-g-continuous. (b) Define $f\colon X \to Y$ as $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is both $(1,2)^*$-g-continuous and almost $(1,2)^*$-continuous but it is not $(1,2)^*$-continuous.

Example 5.5. Let $X = Y = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}, \{a,b\}\}$. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b\}\}$. Define $f\colon X \to Y$ as $f(a) = b$; $f(b) = a$; $f(c) = c$. Clearly f is almost a $(1,2)^*$-continuous but it is not $(1,2)^*$-R-map.

E x a m p l e 5.6. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c\}\}$. Let $\sigma_1 = \{\emptyset, Y, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a, b\}\}$. Define $f\colon X \to Y$ as $f(a) = a$; $f(b) = c$; $f(c) = b$. Clearly f is almost $(1,2)^*$-g-continuous but it is not almost $(1,2)^*$-continuous.

R e m a r k 5.7. From the definitions stated above and the examples given above, we obtain the following diagram.

$$
\begin{array}{ccc}
\text{Complete } (1,2)^*\text{-continuity} & \not\rightleftarrows & (1,2)^*\text{-R-map} \\
\downarrow\not\uparrow & & \downarrow\not\uparrow \\
(1,2)^*\text{-continuity} & \not\rightleftarrows & \text{almost } (1,2)^*\text{-continuity} \\
\downarrow\not\uparrow & & \downarrow\not\uparrow \\
(1,2)\text{-g-continuity} & \not\rightleftarrows & \text{almost } (1,2)^*\text{-g-continuity} \\
\downarrow\not\uparrow & & \downarrow\not\uparrow \\
(1,2)^*\text{- rg -continuity} & \not\rightleftarrows & \text{almost } (1,2)^*\text{- rg -continuity}
\end{array}
$$

Definition 5.8. A space X is said to be regular $(1,2)^*$-$T_{1/2}$ if every $(1,2)^*$-rg-closed set of X is regular $(1,2)^*$-closed in X.

Proposition 5.9. *If a function $f\colon X \to Y$ is $(1,2)^*$-rg-continuous and X is regular $(1,2)^*$-$T_{1/2}$, then f is completely $(1,2)^*$-continuous.*

P r o o f. Let F be any $\sigma_{1,2}$-closed set of Y. Since f is $(1,2)^*$-rg-continuous, $f^{-1}(F)$ is $(1,2)^*$-rg-closed in X and hence $f^{-1}(F) \in (1,2)^*$-RC(X). Therefore, f is completely $(1,2)^*$-continuous. □

Definition 5.10. A function $f\colon X \to Y$ is said to be $(1,2)^*$-rg-irresolute if $f^{-1}(F)$ is $(1,2)^*$-rg-closed in X for every $(1,2)^*$-rg-closed set F of Y.

R e m a r k 5.11. Every $(1,2)^*$-rg-irresolute function is $(1,2)^*$-rg-continuous but not conversely as shown by the following example.

E x a m p l e 5.12. Let $X = Y = \{a, b, c\}$. Let $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}\}$. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b, c\}\}$. Define $f\colon X \to Y$ as $f(a) = c$; $f(b) = b$; $f(c) = a$. Clearly f is $(1,2)^*$-rg-continuous but it is not $(1,2)^*$-rg-irresolute.

Proposition 5.13. *If $f\colon X \to Y$ is almost $(1,2)^*$-rg-continuous and X is regular $(1,2)^*$-$T_{1/2}$, then f is a $(1,2)^*$-R-map.*

P r o o f. Let $V \in (1,2)^*$-RC(Y). Since f is almost $(1,2)^*$-rg-continuous, $f^{-1}(V)$ is $(1,2)^*$-rg-closed in X. But X is regular $(1,2)^*$-$T_{1/2}$. Therefore $f^{-1}(V) \in (1,2)^*$-RC(X). Hence f is a $(1,2)^*$-R-map. □

R e m a r k 5.14. The composition of two $(1,2)^*$-rg-continuous functions need not be $(1,2)^*$-rg-continuous as shown by the following example.

E x a m p l e 5.15. Let $X = Y = Z = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}\}$. Let $\sigma_1 = \{\emptyset, Y, \{c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a,b\}\}$. Let $\varrho_1 = \{\emptyset, Z, \{c\}, \{b,c\}\}$ and $\varrho_2 = \{\emptyset, Z, \{a,b\}\}$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be the two identity functions. Then f and g are $(1,2)^*$-rg-continuous but $g \circ f$ is not $(1,2)^*$-rg-continuous.

Corollary 5.16. *The composition $g \circ f\colon X \to Z$ is $(1,2)^*$-rg-continuous if $f\colon X \to Y$ is $(1,2)^*$-rg-continuous and $g\colon Y \to Z$ is $(1,2)^*$-continuous.*

P r o o f. Let V be a $\varrho_{1,2}$-closed set in Z. Since g is $(1,2)^*$-continuous, $g^{-1}(V)$ is a $\sigma_{1,2}$-closed set in Y. Since f is $(1,2)^*$-rg-continuous, $f^{-1}(g^{-1}(V))$ is $(1,2)^*$-rg-closed. Therefore $g \circ f$ is $(1,2)^*$-rg-continuous. □

Definition 5.17. A function $f\colon X \to Y$ is said to be

(i) regular $(1,2)^*$-closed if $f(F)$ is regular $(1,2)^*$-closed in Y for every $\tau_{1,2}$-closed set F of X;

(ii) $(1,2)^*$-g-closed if $f(F)$ is $(1,2)^*$-g-closed in Y for every $\tau_{1,2}$-closed set F of X;

(iii) $(1,2)^*$-rg-closed if $f(F)$ is $(1,2)^*$-rg-closed in Y for every $\tau_{1,2}$-closed set F of X.

Definition 5.18. A function $f\colon X \to Y$ is said to be

(i) $(1,2)^*$-rc-preserving if $f(F)$ is regular $(1,2)^*$-closed in Y for every $F \in (1,2)^*$-RC(X);

(ii) almost $(1,2)^*$-closed if $f(F)$ is $\sigma_{1,2}$-closed in Y for every $F \in (1,2)^*$-RC(X);

(iii) almost $(1,2)^*$-g-closed if $f(F)$ is $(1,2)^*$-g-closed in Y for every $F \in (1,2)^*$-RC(X);

(iv) almost $(1,2)^*$-rg-closed if $f(F)$ is $(1,2)^*$-rg-closed in Y for every $F \in (1,2)^*$-RC(X).

R e m a r k 5.19. From the definitions stated above, we obtain the following diagram.

$$\begin{array}{ccc}
\text{regular } (1,2)^*\text{-closed} & \rightarrow & (1,2)^*\text{-rc-preserving} \\
\downarrow & & \downarrow \\
(1,2)^*\text{-closed} & \rightarrow & \text{almost } (1,2)^*\text{-closed} \\
\downarrow & & \downarrow \\
(1,2)^*\text{-g-closed} & \rightarrow & \text{almost } (1,2)^*\text{-g-closed} \\
\downarrow & & \downarrow \\
(1,2)^*\text{-rg-closed} & \rightarrow & \text{almost } (1,2)^*\text{-rg-closed}
\end{array}$$

R e m a r k 5.20. The following examples enable us to realize that none of the implications in the above diagram is reversible.

Example 5.21. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$, $\sigma_1 = \{\emptyset, Y, \{a, b\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a\}\}$. Then (a) define $f\colon X \to Y$ as $f(a) = b$; $f(b) = a$; $f(c) = c$. Clearly f is both $(1,2)^*$-g-closed and almost $(1,2)^*$-g-closed, but it is neither $(1,2)^*$-closed nor almost $(1,2)^*$-closed. (b) Define $f\colon X \to Y$ as $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is both $(1,2)^*$-rg-closed and almost $(1,2)^*$-rg-closed, but it is neither $(1,2)^*$-g-closed nor almost $(1,2)^*$-g-closed.

Example 5.22. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{a, c\}\}$, $\sigma_1 = \{\emptyset, Y, \{a, b\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a\}\}$. Then define $f\colon X \to Y$ as $f(a) = b$; $f(b) = a$; $f(c) = c$. Clearly f is both almost $(1,2)^*$-closed and almost $(1,2)^*$-g-closed, but it is neither $(1,2)^*$-g-closed nor $(1,2)^*$-closed.

Example 5.23. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$, $\sigma_1 = \{\emptyset, Y, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a, b\}\}$. Define $f\colon X \to Y$ as $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly f is both $(1,2)^*$-closed and almost $(1,2)^*$-closed, but it is neither regular $(1,2)^*$-closed nor $(1,2)^*$-rc-preserving.

Example 5.24. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$, $\sigma_1 = \{\emptyset, Y, \{a\}, \{a, c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{c\}\}$. Then (i) define $f\colon X \to Y$ as $f(a) = a$; $f(b) = c$; $f(c) = b$. Clearly f is $(1,2)^*$-rc-preserving but it is not regular $(1,2)^*$-closed. (ii) Define $f\colon X \to Y$ as $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost $(1,2)^*$-rg-closed but it is not $(1,2)^*$-rg-closed.

Proposition 5.25. *Let $f\colon X \to Y$ be a function. Then*

(i) *if f is $(1,2)^*$-rg-continuous, $(1,2)^*$-rc-preserving, then it is $(1,2)^*$-rg-irresolute;*

(ii) *if f is a $(1,2)^*$-R-map and $(1,2)^*$-rg-closed, then $f(A)$ is $(1,2)^*$-rg-closed in Y for every $(1,2)^*$-rg-closed set A of X.*

Proof. (i) Let B be any $(1,2)^*$-rg-closed set of Y and let $U \in (1,2)^*$-RO(X) contain $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus U)$, then we have $B \subset V$, $f^{-1}(V) \subset U$ and $V \in (1,2)^*$-RO(Y) since f is $(1,2)^*$-rc-preserving. Hence we obtain $\sigma_{1,2}$-cl$(B) \subset V$ and hence $f^{-1}(\sigma_{1,2}\text{-cl}(B)) \subset U$. By the $(1,2)^*$-rg-continuity of f we have $\tau_{1,2}$-cl$(f^{-1}(B)) \subset \tau_{1,2}$-cl$(f^{-1}\ (\sigma_{1,2}\text{-cl}(B))) \subset U$. This shows that $f^{-1}(B)$ is $(1,2)^*$-rg-closed in X. Therefore f is $(1,2)^*$-rg-irresolute.

(ii) Let A be any $(1,2)^*$-rg-closed set of X and let $V \in (1,2)^*$-RO(X) contain $f(A)$. Since f is a $(1,2)^*$-R-map, $f^{-1}(V) \in (1,2)^*$-RO(X) and $A \subset f^{-1}(V)$. Therefore, we have $\tau_{1,2}$-cl$(A) \subset f^{-1}(V)$ and hence $f(\tau_{1,2}\text{-cl}(A)) \subset V$. Since f is $(1,2)^*$-rg-closed, $f(\tau_{1,2}\text{-cl}(A))$ is $(1,2)^*$-rg-closed in Y and hence we obtain $\sigma_{1,2}$-cl$(f(A)) \subset \sigma_{1,2}$-cl$(f(\tau_{1,2}\text{-cl}(A))) \subset V$. This shows that $f(A)$ is $(1,2)^*$-rg-closed in Y. □

Corollary 5.26. *Let $f\colon X \to Y$ be a function.*

(i) *If f is $(1,2)^*$-continuous, regular $(1,2)^*$-closed, then $f^{-1}(B)$ is $(1,2)^*$-rg-closed in X for every $(1,2)^*$-rg-closed set B of Y.*

(ii) *If f is a $(1,2)^*$-R-map and $(1,2)^*$-closed, then $f(A)$ is $(1,2)^*$-rg-closed in Y for every $(1,2)^*$-rg-closed set A of X.*

Proof. This is an immediate consequence of Proposition 5.25. □

Proposition 5.27. *A surjection $f\colon X \to Y$ is almost $(1,2)^*$-rg-closed or almost $(1,2)^*$-g-closed if and only if for each subset S of Y and each $U \in (1,2)^*$-RO(X) containing $f^{-1}(S)$ there exists respectively an $(1,2)^*$-rg-open or $(1,2)^*$-g-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. We prove only the first case, the proof of the other being entirely analogous.

Necessity. Suppose that f is almost $(1,2)^*$-rg-closed. Let S be a subset of Y and let $U \in (1,2)^*$-RO(X) contain $f^{-1}(S)$. Put $V = Y \setminus f\ (X \setminus U)$, then V is a $(1,2)^*$-rg-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular $(1,2)^*$-closed set of X. Then $f^{-1}\ (Y \setminus f(F)) \subset (X \setminus F)$ and $(X \setminus F) \in (1,2)^*$-RO$(X)$. There exists a $(1,2)^*$-rg-open set V of Y such that $(Y \setminus f(F)) \subset V$ and $f^{-1}(V) \subset (X \setminus F)$. Therefore, we have $f(F) \supset Y \setminus V$ and $F \subset f^{-1}\ (Y \setminus V)$. Hence we obtain $f(F) = Y \setminus V$, and $f(F)$ is $(1,2)^*$-rg-closed in Y. This shows that f is almost $(1,2)^*$-rg-closed. □

6. Preservation theorems

In this section we investigate preservation theorems concerning mildly normal spaces in topological spaces.

Theorem 6.1. *If $f\colon X \to Y$ is an almost $(1,2)^*$-rg-continous $(1,2)^*$-rc-preserving or almost $(1,2)^*$-closed injection and Y is mildly $(1,2)^*$-normal or $(1,2)^*$-normal respectively, then X is mildly $(1,2)^*$-normal.*

Proof. Let A and B be any disjoint regular $(1,2)^*$-closed sets of X. Since f is an $(1,2)^*$-rc-preserving (almost $(1,2)^*$-closed) injection, $f(A)$ and $f(B)$ are disjoint regular $(1,2)^*$-closed ($\sigma_{1,2}$-closed) sets of Y. By the mild $(1,2)^*$-normality ($(1,2)^*$-normality) of Y, there exist disjoint $\sigma_{1,2}$-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Now, put $G = \sigma_{1,2}$-int ($\sigma_{1,2}$-cl(U)) and $H = \sigma_{1,2}$-int ($\sigma_{1,2}$-cl(V)), then G and H are disjoint regular $(1,2)^*$-open sets such that $f(A) \subset G$ and $f(B) \subset H$. Since f is almost $(1,2)^*$-rg-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint

$(1,2)^*$-rg-open sets containing A and B, respectively. It follows from Theorem 4.2 that X is mildly $(1,2)^*$-normal. $\square$

Theorem 6.2. *If $f\colon X \to Y$ is a completely $(1,2)^*$-continuous almost $(1,2)^*$-g-closed surjection and X is mildly $(1,2)^*$-normal, then Y is $(1,2)^*$-normal.*

Proof. Let A and B be any disjoint $\sigma_{1,2}$-closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular $(1,2)^*$-closed sets of X. Since X is mildly $(1,2)^*$-normal, there exist disjoint $\tau_{1,2}$-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \tau_{1,2}$-int $(\tau_{1,2}$-cl$(U))$ and $H = \tau_{1,2}$-int $(\tau_{1,2}$-cl$(V))$, then G and H are disjoint regular $(1,2)^*$-open sets such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Proposition 5.27, there exist $(1,2)^*$-g-open sets K and L of Y such that $A \subset K, B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. Since K and L are $(1,2)^*$-g-open, we obtain $A \subset \sigma_{1,2}$-int (K), $B \subset \sigma_{1,2}$-int(L) and $[\sigma_{1,2}$-int$(K) \cap \sigma_{1,2}$-int$(L)] = \emptyset$. This shows that Y is $(1,2)^*$-normal. $\square$

Corollary 6.3. *If $f\colon X \to Y$ is a completely $(1,2)^*$-continuous $(1,2)^*$-closed surjection and X is mildly $(1,2)^*$-normal, then Y is $(1,2)^*$-normal.*

Theorem 6.4. *Let $f\colon X \to Y$ be an $(1,2)^*$-R-map (almost $(1,2)^*$-continuous) and almost $(1,2)^*$-rg-closed surjection. If X is mildly $(1,2)^*$-normal ($(1,2)^*$-normal), then Y is mildly $(1,2)^*$-normal.*

Proof. Let A and B be any disjoint regular $(1,2)^*$-closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular $(1,2)^*$-closed or $\tau_{1,2}$-closed sets of X. Since X is respectively mildly $(1,2)^*$-normal or $(1,2)^*$-normal, there exist disjoint $\tau_{1,2}$-open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \tau_{1,2}$-int$(\tau_{1,2}$-cl$(U))$ and $H = \tau_{1,2}$-int $(\tau_{1,2}$-cl$(V))$, then G and H are disjoint regular $(1,2)^*$-open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Proposition 5.27, there exist $(1,2)^*$-rg-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from Theorem 4.2 that Y is mildly $(1,2)^*$-normal. $\square$

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