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THE LOCAL METRIC DIMENSION OF A GRAPH

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Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of k distinct vertices in a nontrivial connected graph G, the metric code of a vertex v of G with respect to W is the k-vector

$$\operatorname{code}(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

where $d(v, w_i)$ is the distance between v and w_i for $1 \leqslant i \leqslant k$. The set W is a local metric set of G if $\operatorname{code}(u) \neq \operatorname{code}(v)$ for every pair u, v of adjacent vertices of G. The minimum positive integer k for which G has a local metric k-set is the local metric dimension $\operatorname{lmd}(G)$ of G. A local metric set of G of cardinality $\operatorname{lmd}(G)$ is a local metric basis of G. We characterize all nontrivial connected graphs of order n having local metric dimension 1, $n-2$, or $n-1$ and establish sharp bounds for the local metric dimension of a graph in terms of well-known graphical parameters. Several realization results are presented along with other results on the number of local metric bases of a connected graph.

Keywords: distance, local metric set, local metric dimension

MSC 2010: 05C12

1. Introduction

A research area in graph theory that has increased in popularity during the past few decades is that of studying various methods that can be used to distinguish all of the vertices in a connected graph or to distinguish the two vertices if they are adjacent. Many of these methods involve graph colorings or distance in graphs.

If all of the vertices of a graph G of order n are distinguished as a result of being assigned distinct colors, then of course n colors are needed to accomplish this. On the other hand, if the goal is to distinguish every two adjacent vertices in G by a vertex coloring, then this can be accomplished by means of a proper coloring of G and the minimum number of colors needed to do this is the *chromatic number* $\chi(G)$ of G. There are, however, other methods that have been used to distinguish every two adjacent vertices in G by means of vertex colorings which may require fewer than $\chi(G)$ colors (see [12], for example).

With a given edge coloring c (proper or not) of a graph G, each vertex of G can be labeled with the set of colors of its incident edges. If distinct vertices have distinct labels, then c is a vertex-distinguishing edge coloring (see [3], [14] for example); while if every two adjacent vertices have distinct labels, then c is a neighbor-distinguishing edge coloring (see [2], [16] for example). With a given vertex coloring c (proper or not) of a graph G, each vertex of G can also be labeled with the set of colors of its neighboring vertices. Again, if distinct vertices have distinct labels, then c is a vertex-distinguishing vertex coloring (see [1], [7], [18] for example); while if every two adjacent vertices have distinct labels, then c is a neighbor-distinguishing vertex coloring (see [8], [9], [11] for example).

Distance in graphs has also been used to distinguish all of the vertices of a graph. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest path between these two vertices. Suppose that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered set of vertices of a nontrivial connected graph G. For each vertex v of G, there is associated a k-vector called the *metric code*, or simply the *code* of v (with respect to W), which is denoted by $\mathrm{code}_W(v)$ and defined by

$$\mathrm{code}_W(v) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$$

(or simply $\mathrm{code}(v)$ if the set W under consideration is clear). If $\mathrm{code}(u) \neq \mathrm{code}(v)$ for every pair u, v of distinct vertices of G, then W is called a *metric set* or a *resolving set*. The minimum k for which G has a metric k-set is the *global metric dimension*, or simply the *metric dimension* of G, which is denoted by $\dim(G)$. Resolving sets and metric dimensions of graphs were introduced, independently, by Harary and Melter [13] and Slater [20], [21], although, as indicated in [4], these concepts were studied earlier for hypercubes under the guise of a coin weighing problem. In recent years, this concept has been studied widely (see [4], [5], [13], [15], [17], [19], [20], [21], for example) with a variety of applications.

Consequently, the major problem dealing with resolving sets is to minimize the number of vertices in a subset W of the vertex set of a connected graph G so that the distances to the vertices of W are not the same for any two vertices of G. When using colorings to distinguish the vertices of G, the goal is to minimize the number of colors needed so that every two vertices of G can be distinguished in some way by the type of coloring being used.

A more common problem in graph theory concerns distinguishing every two neighbors in a graph G by means of some coloring rather than distinguishing all the vertices of G by a graph coloring. Since distinguishing all the vertices of a connected graph

G has been studied with the aid of distances in G, this suggests the topic of using distances in G to distinguish the two vertices in each pair of neighbors.

The foregoing discussion then gives rise to a local version of resolving sets. In this case, we consider those ordered sets W of vertices of G for which any two vertices of G having the same code with respect to W are not adjacent in G. If $\mathrm{code}(u) \neq \mathrm{code}(v)$ for every pair u, v of *adjacent* vertices of G, then W is called a *local metric set* of G. The minimum k for which G has a local metric k-set is the *local metric dimension* of G, which is denoted by $\mathrm{lmd}(G)$. A local metric set of cardinality $\mathrm{lmd}(G)$ in G is a *local metric basis* of G. The local metric dimension exists for every nontrivial connected graph G. In fact, $V(G)$ is always a local metric set of G. Indeed, for each independent set U of vertices in G, the set $V(G) - U$ is a local metric set. Thus we have the following observation. The independence number of a graph G is denoted by $\alpha(G)$.

Observation 1.1. *For every nontrivial connected graph G of order n,*

$$\mathrm{lmd}(G) \leqslant n - \alpha(G).$$

While each metric set of a nontrivial connected graph G is *vertex-distinguishing* (since every two vertices of G have distinct codes), each local metric set is *neighbor-distinguishing* (since every two *adjacent* vertices of G have distinct codes). Thus every metric set is also a local metric set and so if G is a nontrivial connected graph of order n, then

$$1 \leqslant \mathrm{lmd}(G) \leqslant \dim(G) \leqslant n - 1. \tag{1}$$

To illustrate these concepts, consider the graph G of Figure 1. In this case, $W_1 = \{v_1, v_4\}$ is a local metric 2-set and $W_2 = \{v_1, v_3, v_5\}$ is a metric 3-set. The corresponding codes for the vertices of G with respect to the sets W_1 and W_2, respectively, are shown in Figure 1. In fact, $\mathrm{lmd}(G) = 2$ and $\dim(G) = 3$.

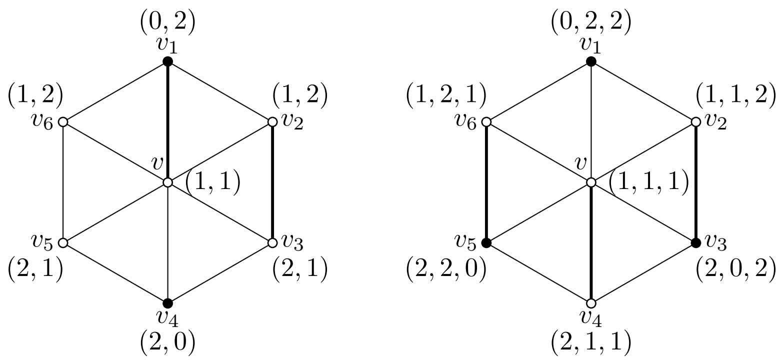

Figure 1: A graph with local metric dimension 2 and metric dimension 3

These examples illustrate a useful observation. When determining whether a given set W of vertices of a nontrivial connected graph G is a local metric set of G, one need only investigate the pairs of adjacent vertices in $V(G) - W$ since $w \in W$ is the only vertex of G whose distance from w is 0. Furthermore, if W is a subset of the vertex set of a graph G containing a local metric set of G, then W is also a local metric set of G.

2. Graphs with prescribed order and local metric dimension

We refer to the book [6] for graph theory notation and terminology not described in this paper. It is known that if G is a nontrivial connected graph of order n, then $\dim(G) = n - 1$ if and only if $G = K_n$ and $\dim(G) = 1$ if and only if $G = P_n$. In the case of local metric dimensions, there is an analogous result. Before stating this result, we present some additional terminology. Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *true twins*; while if u and v are nonadjacent, they are *false twins*. If u and v are true twins and v and w are true twins, then so too are u and w. Hence two vertices being true twins produces an equivalence relation on $V(G)$. If the resulting *true twin equivalence classes* are $U_1, U_2, \ldots, U_l$, then every local metric set of G must contain at least $|U_i| - 1$ vertices from U_i for each i with $1 \leqslant i \leqslant l$. Thus we have the following observation.

Observation 2.1. *If G is a nontrivial connected graph of order n having l true twin equivalence classes, then* $\mathrm{lmd}(G) \geqslant n - l$.

Observe also that there is no connected graph having exactly two true twin equivalence classes. To see this, suppose that G is a nontrivial connected graph having U_1 and U_2 as its only true twin equivalence classes. Since G is connected, there exist two adjacent vertices $x \in U_1$ and $y \in U_2$. However, this implies that every vertex in U_1 is adjacent to y, which in turn implies that every vertex in U_2 is adjacent to every vertex in U_1. Therefore, G is a complete graph, which is impossible. We state this observation below.

Observation 2.2. *There is no nontrivial connected graph having exactly two true twin equivalence classes.*

For a vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The *diameter* $\mathrm{diam}(G)$ of G is the largest eccentricity among all vertices of G. If G is a nontrivial connected graph of order n with diameter d

and $v_0, v_1, \ldots, v_d$ is a path of length d in G, then $V(G) - \{v_1, v_2, \ldots, v_d\}$ is a local metric set of G. This yields another observation.

Observation 2.3. *If G is a nontrivial connected graph of order n and diameter d, then $\operatorname{lmd}(G) \leqslant n - d$.*

Theorem 2.4. *Let G be a nontrivial connected graph of order n. Then $\operatorname{lmd}(G) = n - 1$ if and only if $G = K_n$ and $\operatorname{lmd}(G) = 1$ if and only if G is bipartite.*

Proof. Since the complete graph K_n has only one true twin equivalence class, $\operatorname{lmd}(K_n) \geqslant n - 1$ (by Observation 2.1). It then follows, by (1), that $\operatorname{lmd}(K_n) = n - 1$. On the other hand, if $G \neq K_n$, then $\alpha(G) \geqslant 2$ and so $\operatorname{lmd}(G) \leqslant n - 2$ by Observation 1.1.

It remains to show that $\operatorname{lmd}(G) = 1$ if and only if G is bipartite. Suppose first that G is a bipartite graph with partite sets U and V. Let $W = \{w\}$, where $w \in U$ say. Since $d(u, w)$ is even for each $u \in U$ and $d(u, v)$ is odd for each $v \in V$, it follows that W is a local metric basis and so $\operatorname{lmd}(G) = 1$. To verify the converse, let G be a nontrivial connected graph having local metric dimension 1 and let $W = \{w\}$ be a local metric basis of G. For $0 \leqslant i \leqslant e(w)$, let $N_i = \{v \in V(G)\colon\ d(v, w) = i\}$. (Therefore, $N_0 = W$ and $N_1 = N(w)$.) Since W is a local metric basis, each set N_i is an independent set. Furthermore, if i and j are integers with $0 \leqslant i, j \leqslant e(w)$ and $|i - j| \geqslant 2$, then no vertex in N_i is adjacent to any vertex in N_j. Therefore, G is a bipartite graph with partite sets $U = N_0 \cup N_2 \cup \ldots \cup N_{2\lfloor e(w)/2\rfloor}$ and $V = N_1 \cup N_3 \cup \ldots \cup N_{2\lceil e(w)/2\rceil - 1}$. □

Next, we characterize all nontrivial connected graphs of order $n \geqslant 3$ having local metric dimension $n - 2$. The *clique number* $\omega(G)$ of a graph G is the order of a largest complete subgraph (clique) in G.

Theorem 2.5. *A connected graph G of order $n \geqslant 3$ has local metric dimension $n - 2$ if and only if $\omega(G) = n - 1$.*

Proof. First, let G be a connected graph of order $n \geqslant 3$ with clique number $n-1$. Since $G \neq K_n$, it follows that $\operatorname{lmd}(G) \leqslant n-2$. Let $H = K_{n-1}$ be a clique in G and let $v \in V(G) - V(H)$ with $d = \deg v$. Then $1 \leqslant d \leqslant n-2$. Observe that $U_1 = \{v\}$, $U_2 = N(v) = \{v_1, v_2, \ldots, v_d\}$, and $U_3 = V(G) - N[v] = \{v_{d+1}, v_{d+2}, \ldots, v_{n-1}\}$ are the true twin equivalence classes. Therefore, $\operatorname{lmd}(G) \geqslant n - 3$ by Observation 2.1.

Assume, to the contrary, that G contains a local metric set W' consisting of $n - 3$ vertices. Then again, by Observation 2.1, $V(G) - W' = \{v, x_2, x_3\}$, where $x_i \in U_i$ for $i = 2, 3$. However, this implies that x_2 and x_3 are adjacent and $d(x_2, w) = d(x_3, w)$

for all $w \in W'$. Thus $\operatorname{code}(x_2) = \operatorname{code}(x_3)$, which is a contradiction. Therefore, $\operatorname{lmd}(G) = n - 2$.

For the converse, let G be a connected graph of order $n \geqslant 3$ with $\operatorname{lmd}(G) = n - 2$. We show that $\omega(G) = n - 1$. Since $G \neq K_n$ (by Theorem 2.4), it follows that $\omega(G) \leqslant n - 1$. The result follows immediately for $n = 3$ since $G = P_3$. Since every connected graph G of order 4 with $\omega(G) \leqslant 2$ is bipartite, it follows that $\operatorname{lmd}(G) = 1$ (by Theorem 2.4). Therefore, $\omega(G) = 3$ for all connected graphs G of order 4 with $\operatorname{lmd}(G) = 2$. Hence we may now assume that $n \geqslant 5$. Suppose that there is some graph G of order $n \geqslant 5$ for which $\operatorname{lmd}(G) = n - 2$ and $\omega(G) \leqslant n - 2$. By Observation 2.3, $\operatorname{diam}(G) = 2$. Also, there exists a set $X = \{x_1, x_2, x_3, x_4\}$ consisting of four distinct vertices such that $x_1x_2, x_3x_4 \notin E(G)$. Consider the induced subgraph $H = \langle X \rangle$.

If $\Delta(H) = 2$, say $\deg_H(x_1) = 2$, then let $W_1 = V(G) - \{x_2, x_3, x_4\}$. Since $d(x_1, x_3) = d(x_1, x_4) = 1$ while $d(x_1, x_2) = 2$ and x_3 and x_4 are not adjacent in G, it follows that W_1 is a local metric set. Similarly, if $\delta(H) = 0$, say $\deg_H(x_1) = 0$, then $W_2 = V(G) - \{x_1, x_3, x_4\}$ is a local metric set since the three vertices x_1, x_3, and x_4 are mutually nonadjacent. Therefore, $\operatorname{lmd}(G) \leqslant n - 3$ in each case, which cannot occur.

Therefore, we consider the case where $\Delta(H) = \delta(H) = 1$ and we may assume that $E(H) = \{x_1x_3, x_2x_4\}$. If there exists a vertex $v^* \in V(G) - X$ such that $d(x_1, v^*) \neq d(x_3, v^*)$ or $d(x_2, v^*) \neq d(x_4, v^*)$, say the former, then $\operatorname{code}_{W_2}(x_1) \neq \operatorname{code}_{W_2}(x_3)$ since $v^* \in W_2$. Since x_4 is adjacent to neither x_1 nor x_3, it follows that W_2 is a local metric set. On the other hand, if $d(x_1, v) = d(x_3, v)$ and $d(x_2, v) = d(x_4, v)$ for every $v \in V(G) - X$, then there exists a vertex v' in $V(G) - X$ which is adjacent to every vertex in X since $\operatorname{diam}(G) = 2$. Then let $W_3 = V(G) - \{x_3, x_4, v'\}$ and observe that $d(v', x_1) = d(v', x_2) = 1$ while $d(x_4, x_1) = d(x_3, x_2) = 2$. Therefore, $\operatorname{code}_{W_3}(v') \notin \{\operatorname{code}_{W_3}(x_3), \operatorname{code}_{W_3}(x_4)\}$. Since x_3 and x_4 are not adjacent, it follows that W_3 is a local metric set. Hence, $\operatorname{lmd}(G) \leqslant n - 3$, which is again a contradiction. $\square$

We have seen that if G is a nontrivial connected graph of order n with $\operatorname{lmd}(G) = k$, then $1 \leqslant k \leqslant n-1$. In fact, every pair k, n of integers with $1 \leqslant k \leqslant n-1$ is realizable as the local metric dimension and order of a connected graph, respectively, as we show next.

Theorem 2.6. *For each pair k, n of integers with $1 \leqslant k \leqslant n - 1$, there exists a connected graph G of order n with $\operatorname{lmd}(G) = k$.*

P r o o f. By Theorems 2.4 and 2.5, there exists a connected graph G of order n with $\operatorname{lmd}(G) = k$ for $k \in \{1, n - 2, n - 1\}$. Thus we may assume that $2 \leqslant k \leqslant n - 2$.

Let G be the graph obtained from K_{k+1} with vertex set $V = \{v_1, v_2, \ldots, v_{k+1}\}$ and the path $v_{k+2}, v_{k+3}, \ldots, v_n$ by joining v_{k+1} and v_{k+2}. Since $V - \{v_{k+1}\}$ is a local metric set, $\operatorname{lmd}(G) \leqslant k$. Assume, to the contrary, that $\operatorname{lmd}(G) \leqslant k-1$ and let W' be a local metric set of $k-1$ vertices. Since every local metric set must contain at least $k-1$ vertices from $V - \{v_{k+1}\}$ by Observation 2.1, we may assume that $W' = \{v_1, v_2, \ldots, v_{k-1}\}$. However then $\operatorname{code}_{W'}(v_k) = \operatorname{code}_{W'}(v_{k+1}) = (1, 1, \ldots, 1)$, which is a contradiction. Thus $\operatorname{lmd}(G) = k$. □

We noted that if G is a nontrivial connected graph with $\operatorname{lmd}(G) = a$ and $\dim(G) = b$, then $a \leqslant b$. On the other hand, every pair a, b of positive integers with $a \leqslant b$ can be realized as the local metric dimension and metric dimension, respectively, of some connected graph. In order to verify this, we state (without proofs) two useful lemmas which provide the metric dimension and local metric dimension of all complete multipartite graphs.

Lemma 2.7. *Let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete k-partite graph of order n, where $k \geqslant 2$, $n = n_1 + n_2 + \ldots + n_k$, and $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$. If $n_2 = 1$, then let p be the largest integer such that $n_p = 1$; otherwise let $p = 1$. Then* $\dim(G) = n - k + p - 1$.

Lemma 2.8. *For each complete k-partite graph G, where $k \geqslant 2$,* $\operatorname{lmd}(G) = k - 1$.

Theorem 2.9. *For each pair a, b of positive integers with $a \leqslant b$, there is a nontrivial connected graph G with* $\operatorname{lmd}(G) = a$ *and* $\dim(G) = b$.

P r o o f. Consider the complete $(a+1)$-partite graph $G = K_{1,1,\ldots,1,b-a+2}$ of order $b+2$. Then $\operatorname{lmd}(G) = (a+1) - 1 = a$ by Lemma 2.8 and $\dim(G) = (b+2) - (a+1) + a - 1 = b$ by Lemma 2.7. □

3. Bounds for the local metric dimension of a graph

In this section we establish bounds for the local metric dimension of a nontrivial connected graph in terms of its order and other well-known graphical parameters. Other results involving the local metric dimension and the number of true twin equivalence classes of a graph are also presented.

Theorem 3.1. *If G is a nontrivial connected graph with clique number ω, then*

$$\operatorname{lmd}(G) \geqslant \lceil \log_2 \omega \rceil. \tag{2}$$

Furthermore, for each integer $\omega \geqslant 2$, there exists a connected graph G_ω with clique number ω such that $\operatorname{lmd}(G_\omega) = \lceil \log_2 \omega \rceil$.

P r o o f. Let $F = K_\omega$ be a clique in G with $V(F) = \{v_1, v_2, \ldots, v_\omega\}$. Suppose that $\operatorname{lmd}(G) = k$ and let W be a local metric basis. For each $v_i \in V(F)$, let $\operatorname{code}(v_i) = (a_{1,i}, a_{2,i}, \ldots, a_{k,i})$. Since $|d(v_i, x) - d(v_{i'}, x)| \leqslant 1$ for every two vertices $v_i, v_{i'} \in V(F)$ and every vertex x in G, it follows that $|\{a_{j,i} \colon 1 \leqslant i \leqslant \omega\}| \leqslant 2$ for $1 \leqslant j \leqslant k$. Therefore, there are at most 2^k possible codes for the ω vertices in F with respect to W. Since every vertex in F must have a distinct code, it follows that $\omega \leqslant 2^k$ or $k \geqslant \log_2 \omega$. Therefore, $\operatorname{lmd}(G) \geqslant \lceil \log_2 \omega \rceil$.

We now construct a connected graph G_ω with clique number ω such that $\operatorname{lmd}(G_\omega) = \lceil \log_2 \omega \rceil$ for each integer $\omega \geqslant 2$. If $\omega = 2$, then let G_2 be a nontrivial tree and so $\operatorname{lmd}(G_2) = 1 = \lceil \log_2 \omega \rceil$. Thus we may assume that $\omega \geqslant 3$. Then there exists a unique integer $k \geqslant 2$ such that $2^{k-1} + 1 \leqslant \omega \leqslant 2^k$. Let $\omega = 2^{k-1} + p$, where p is an integer with $1 \leqslant p \leqslant 2^{k-1}$. Construct the graph G_ω from the complete graph K_ω with vertex set $V(K_\omega) = \{v_1, v_2, \ldots, v_\omega\}$ by adding the k new vertices in the set $W = \{w_1, w_2, \ldots, w_k\}$ as follows: Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2^k}\}$ be the set of the 2^k distinct ordered k-tuples whose coordinates are elements in $\{1, 2\}$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2^k}$ are listed in the lexicographic order. Thus $\mathbf{x}_1 = (1, 1, \ldots, 1)$ and $\mathbf{x}_{2^k} = (2, 2, \ldots, 2)$. For each j with $1 \leqslant j \leqslant k$, the vertex w_j is joined to the vertex v_i $(1 \leqslant i \leqslant 2^{k-1} + p)$ if and only if the j-th coordinate of $\mathbf{x}_i$ is 1. Thus v_1 is adjacent to every vertex in W while v_ω is adjacent to none of the vertices in W if $\omega = 2^k$. Since G_ω is a connected graph with $\omega(G_\omega) = \omega = 2^{k-1} + p$, it follows that $\operatorname{lmd}(G_\omega) \geqslant k$ by (2). On the other hand, $\operatorname{code}(v_i) = \mathbf{x}_i$ for $1 \leqslant i \leqslant 2^{k-1} + p$ and so W is a local metric set of G_ω. Therefore, $\operatorname{lmd}(G_\omega) = k$. $\square$

The graph G_ω constructed in the proof of Theorem 3.1 illustrates an interesting feature of the local metric dimension, namely if H is a subgraph of a graph G, then it is possible that $\operatorname{lmd}(H) > \operatorname{lmd}(G)$. For example, let $H = K_\omega$ be the complete subgraph of order ω in the graph G_ω. Then $\operatorname{lmd}(H) = \omega - 1 > \lceil \log_2 \omega \rceil = \operatorname{lmd}(G_\omega)$ for each $\omega \geqslant 4$.

We now present another lower bound for the local metric dimension of a graph in terms of its order and clique number. This lower bound is particularly useful when n is large and $n - \omega$ is small.

Theorem 3.2. *If G is a nontrivial connected graph of order n with $\omega = \omega(G)$, then*

$$\operatorname{lmd}(G) \geqslant n - 2^{n-\omega}.$$

Furthermore, for each pair n, ω of integers with $2^{n-\omega} \leqslant \omega \leqslant n$, there exists a connected graph G of order n whose clique number is ω such that $\operatorname{lmd}(G) = n - 2^{n-\omega}$.

P r o o f. The result is immediate if $\omega \in \{n-1, n\}$ and so assume that $2 \leqslant \omega \leqslant n-2$. Suppose that $H = K_\omega$ is a clique in G and let $X = V(H)$ and $Y = V(G) - X$. Consider an arbitrary local metric set W. We show that $|W| \geqslant n - 2^{n-\omega}$.

Let $p = |Y \cap W|$. Therefore, $0 \leqslant p \leqslant n-\omega$. Since $d(x,w) = 1$ for all $x \in X - X \cap W$ and $w \in X \cap W$, there are at most 2^p possible codes for the vertices in $X - X \cap W$. Hence, $|X - X \cap W| \leqslant 2^p$ and so $|X \cap W| \geqslant \omega - 2^p$. Therefore, $|W| \geqslant p + (\omega - 2^p)$. If $p = 0$, then

$$|W| \geqslant \omega - 1 = n - (n - \omega + 1) > n - 2^{n-\omega}$$

since $n - \omega \geqslant 2$. If $1 \leqslant p \leqslant n - \omega$, then consider the function f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x + (\omega - 2^x)$. Observe that $f'(x) = 1 - 2^x \ln 2 < 0$ for $x \geqslant 1$ and so $f(x) \geqslant f(n-\omega) = n - 2^{n-\omega}$ for $1 \leqslant x \leqslant n - \omega$. Therefore,

$$|W| \geqslant f(p) \geqslant n - 2^{n-\omega}.$$

Next, let n and ω be positive integers with $2^{n-\omega} \leqslant \omega \leqslant n$. Clearly $G = K_n$ possesses the desired property for $\omega = n$. Therefore, suppose that $2^{n-\omega} \leqslant \omega < n$. Let X, Y, and Z be pairwise disjoint sets of vertices, where $X = \{x_1, x_2, \ldots, x_{n-\omega}\}$, $Y = \{y_1, y_2, \ldots, y_{2^{n-\omega}}\}$, and $Z = \emptyset$ if $2^{n-\omega} = \omega$ and $Z = \{z_1, z_2, \ldots, z_{\omega - 2^{n-\omega}}\}$ otherwise. Also, let $\mathcal{P}(A) = \{S_1, S_2, \ldots, S_{2^{n-\omega}}\}$ be the power set of the set $A = \{1, 2, \ldots, n-\omega\}$. We construct a graph G with $V(G) = X \cup Y \cup Z$ such that $\langle X \cup Z \rangle \cong K_{n-2^{n-\omega}}$, $\langle Y \cup Z \rangle \cong K_\omega$, and $x_i y_j \in E(G)$ if and only if $i \in S_j$ for $1 \leqslant i \leqslant n - \omega$ and $1 \leqslant j \leqslant 2^{n-\omega}$. Therefore, the order of G is n and since $n - \omega \geqslant 1$, it follows that $\omega(G) = \max\{\omega, n - 2^{n-\omega} + 1\} = \omega$. Furthermore, $X \cup Z$ is a local metric basis of G and so $\operatorname{lmd}(G) = |X \cup Z| = n - 2^{n-\omega}$. □

The following is an immediate consequence of Theorem 3.2.

Corollary 3.3. *If G is a nontrivial connected graph of order n and $\operatorname{lmd}(G) = n-k$, then $\omega(G) \leqslant n - \lceil \log_2 k \rceil$.*

We have seen in Observations 1.1 and 2.3 that if G is a nontrivial connected graph of order n having independence number α and diameter d, then $\operatorname{lmd}(G) \leqslant n - \alpha$ and $\operatorname{lmd}(G) \leqslant n - d$. In fact, more can be said.

Theorem 3.4. *For each pair α, n of integers with $1 \leqslant \alpha \leqslant n-1$, there exists a nontrivial connected graph G of order n and independence number α such that $\operatorname{lmd}(G) = n - \alpha$.*

P r o o f. For integers α and n with $1 \leqslant \alpha \leqslant n-1$, let $G = K_{n-\alpha} + \overline{K}_\alpha$. Since $V(K_{n-\alpha})$ is a local metric set, it follows that $\operatorname{lmd}(G) \leqslant n - \alpha$. By Observation 2.1, every local metric set must contain at least $n - \alpha - 1$ vertices from the set $V(K_{n-\alpha})$.

Let W be any subset of $V(K_{n-\alpha})$ with $|W| = n - \alpha - 1$. If $V(K_{n-\alpha}) - W = \{v\}$, then $\mathrm{code}_W(v) = \mathrm{code}_W(x)$ for each vertex x in $\overline{K}_\alpha$. Thus W is not a local metric set and so $\mathrm{lmd}(G) = n - \alpha$. $\square$

Theorem 3.5. *For each pair d, n of integers with $1 \leqslant d \leqslant n - 1$, there exists a nontrivial connected graph G of order n and diameter d such that* $\mathrm{lmd}(G) = n - d$.

Proof. Let d and n be integers with $1 \leqslant d \leqslant n - 1$. For $d = 1$, the complete graph K_n has the desired property. Also, consider the path P_n for $d = n - 1$. Hence suppose that $2 \leqslant d \leqslant n - 2$. Let G be the graph obtained from a complete graph $H = K_{n-d}$ of order $n - d$ and a path $P\colon v_1, v_2, \ldots, v_d$ of order d by joining every vertex in H to v_1. Then $\mathrm{diam}(G) = d$ and $\mathrm{lmd}(G) \leqslant n - d$ by Observation 2.3. Also, observe that $U_i = \{v_i\}$ for $1 \leqslant i \leqslant d$ and $U_{d+1} = V(K_{n-d})$ are the $d + 1$ true twin equivalence classes. Hence, $\mathrm{lmd}(G) \geqslant n - d - 1$ by Observation 2.1 and furthermore, every local metric set contains at least $n - d - 1$ vertices in U_{d+1}. Assume, to the contrary, that $\mathrm{lmd}(G) = n - d - 1$ and let W be a local metric basis. Then there exists a vertex $x \in U_{d+1} - W$. However then, $\mathrm{code}(x) = \mathrm{code}(v_1)$, which contradicts the fact that W is a local metric set. Therefore, $\mathrm{lmd}(G) = n - d$. $\square$

By Observations 2.1 and 2.2, if G is a nontrivial connected graph of order n having l true twin equivalence classes, then $l \neq 2$ and $\mathrm{lmd}(G) \geqslant n - l$. For a fixed integer $n \geqslant 2$, we next determine all possible values of l for which there is a connected graph G of order n with l true twin equivalence classes such that $\mathrm{lmd}(G) = n - l$.

Theorem 3.6. *Let n and l be integers with $1 \leqslant l \leqslant n-1$. There exists a connected graph G of order n with l true twin equivalence classes such that $\mathrm{lmd}(G) = n - l$ if and only if $l = 1$ or $3 \leqslant l \leqslant n - 2$.*

Proof. Let G be a connected graph of order n with l true twin equivalence classes $U_1, U_2, \ldots, U_l$. Then $l = 1$ or $3 \leqslant l \leqslant n - 1$ by Observation 2.2. If $l = n - 1$, then we may assume, without loss of generality, that $|U_i| = 1$ for $1 \leqslant i \leqslant n - 2$ and $U_{n-1} = \{x, y\}$. Then x and y are adjacent. Since G is connected and $d(x, v) = d(y, v)$ for every $v \in V(G) - \{x, y\}$, there exists a vertex z that is adjacent to both x and y. This implies that G contains a triangle and so G is not bipartite. By Theorem 2.4, $\mathrm{lmd}(G) \geqslant 2$ and so $\mathrm{lmd}(G) \neq n - (n - 1) = n - l$.

To verify the converse, first observe that $G = K_n$ has the desired property for $l = 1$. If $3 \leqslant l \leqslant n - 2$, then let G be the graph obtained from vertex-disjoint complete graphs $H_1 = K_{n-l}$ and $H_2 = K_2$ of orders $n - l$ and 2, respectively, and a path $P\colon v_1, v_2, \ldots, v_{l-2}$ of order $l - 2$ by joining (i) every vertex in H_1 to v_1 and (ii) the two vertices in H_2 to v_{l-2}. Then $U_i = \{v_i\}$ for $1 \leqslant i \leqslant l - 2$, $U_{l-1} = V(H_1)$, and $U_l = V(H_2)$ are the true twin equivalence classes. Let $x \in V(H_1)$ and $y \in V(H_2)$

and observe that the set $W = V(G) - [V(P) \cup \{x, y\}]$ is a local metric set containing $n-l$ vertices. Therefore, $\operatorname{lmd}(G) \leqslant n-l$. Since $\operatorname{lmd}(G) \geqslant n-l$ by Observation 2.1, we obtain the desired result. $\square$

The following result presents a sharp upper bound for the local metric dimension of a nontrivial connected graph in terms of its order, the number of true twin equivalence classes, and the number of singleton true twin equivalence classes.

Theorem 3.7. *Let G be a nontrivial connected graph of order n having l true twin equivalence classes. If p of these l true twin equivalence classes consist of a single vertex, then*

$$\operatorname{lmd}(G) \leqslant n - l + p.$$

Proof. If $l = 1$ or $p \in \{l-1, l\}$, then the result immediately follows by (1). Since $l \neq 2$ by Observation 2.2, we may assume that $l \geqslant 3$ and $0 \leqslant p \leqslant l-2$. Suppose that $U_1, U_2, \ldots, U_l$ are the true twin equivalence classes and $|U_i| \geqslant 2$ for $p+1 \leqslant i \leqslant l$. For each i with $p+1 \leqslant i \leqslant l$, let $u_i \in U_i$. Now let $U = \{u_{p+1}, u_{p+2}, \ldots, u_l\}$ and $W = V(G) - U$. Then $|W| = n - (l-p) = n - l + p$. We show that W is a local metric set of G. Let x and y be two adjacent vertices of U. Then x and y belong to distinct true twin equivalence classes. Therefore, there exists $z \in V(G) - \{x, y\}$ such that $d(x, z) \neq d(y, z)$. If $z \in W$, then $\operatorname{code}_W(x) \neq \operatorname{code}_W(y)$, as desired. Thus we may assume that $z \notin W$ and so $z \in U$. Then $z = u_j$ for some j with $p+1 \leqslant j \leqslant l$. Let $z' \in U_j - \{z\}$. Then $z' \in W$ and $d(x, z') = d(x, z) \neq d(y, z) = d(y, z')$. Thus W is a local metric set and so $\operatorname{lmd}(G) \leqslant |W| = n - l + p$. $\square$

The upper bound in Theorem 3.7 is sharp. To see this, let $k \geqslant 3$ be an integer, $A = \{1, 2, \ldots, k-2\}$, and let $\mathcal{P}(A) = \{S_1, S_2, \ldots, S_{2^{k-2}}\}$ be the power set of A. Define the sets $S_{2^{k-2}+1}, S_{2^{k-2}+2}, \ldots, S_{2^k}$ by

$$S_{i+2^{k-2}} = S_i \cup \{k-1\}, \quad S_{i+2^{k-1}} = S_i \cup \{k\}, \quad S_{i+2^{k-1}+2^{k-2}} = S_i \cup \{k-1, k\}$$

for $1 \leqslant i \leqslant 2^{k-2}$. Thus $\{S_1, S_2, \ldots, S_{2^k}\}$ is the power set of $A \cup \{k-1, k\} = \{1, 2, \ldots, k\}$. Let $H = K_{2^k}$ be a complete graph of order 2^k with $V(H) = \{u_1, u_2, \ldots, u_{2^k}\}$. We construct G from H by adding k new vertices in the set $W_0 = \{w_1, w_2, \ldots, w_k\}$ and joining u_i to w_j if and only if $j \in S_i$. Note that $G = G_\omega$ ($\omega = 2^k$) is described in the proof of Theorem 3.1. Hence W_0 is a local metric basis and $\operatorname{lmd}(G) = k$. Furthermore, $\deg w_i = 2^{k-1}$ for $1 \leqslant i \leqslant k$. We show that $\operatorname{lmd}(G - w_i) = k + 2^{k-1} - 1$ for $1 \leqslant i \leqslant k$. By symmetry, it suffices to show that $\operatorname{lmd}(G - w_k) = k + 2^{k-1} - 1$. Since the set $W = \{u_1, u_2, \ldots, u_{2^{k-1}}\} \cup (W_0 - \{w_k\})$ is a local metric set of $G - w_k$ containing $k + 2^{k-1} - 1$ vertices, $\operatorname{lmd}(G - w_k) \leqslant k + 2^{k-1} - 1$.

Observe that each set $U_i = \{u_i, u_{i+2^{k-1}}\}$ is a true twin equivalence class in $G - w_k$ for $1 \leqslant i \leqslant 2^{k-1}$. Thus, if there exists a local metric set W' in $G - w_k$ containing at most $k + 2^{k-1} - 2$ vertices, then we may assume that $\{u_1, u_2, \ldots, u_{2^{k-1}}\} \subseteq W'$ and $w_{k-1} \notin W'$. On the other hand, $d_{G-w_k}(u_{i+2^{k-1}}, v) \neq d_{G-w_k}(u_{i+2^{k-1}+2^{k-2}}, v)$ if and only if $v = w_{k-1}$ for $1 \leqslant i \leqslant 2^{k-2}$. Therefore, we may assume that $\{u_{2^{k-1}+1}, u_{2^{k-1}+2}, \ldots, u_{2^{k-1}+2^{k-2}}\} \subseteq W'$ as well. However then, $2^{k-1} + 2^{k-2} \leqslant |W'| < 2^{k-1} + k - 1$, which is impossible. Therefore, $\operatorname{lmd}(G - w_k) = k + 2^{k-1} - 1$ as claimed. Since the order of G is $n = 2^k + k - 1$ and G has $l = 2^{k-1} + k - 1$ true twin equivalence classes, namely the 2-sets U_i $(1 \leqslant i \leqslant 2^{k-1})$ and the singleton sets $\{w_j\}$ $(1 \leqslant j \leqslant k-1)$, it follows that $p = k - 1$ and so $\operatorname{lmd}(G - w_k) = n - l + p$, as desired.

The following is an immediate consequence of Observation 2.1 and Theorem 3.7.

Corollary 3.8. *If G is a nontrivial connected graph of order n with l true twin equivalence classes none of which is a singleton set, then* $\operatorname{lmd}(G) = n - l$.

Suppose that G is a nontrivial connected graph of order n having l true twin equivalence classes, p of which consist of a single vertex. Theorem 3.6 provides all possible values of l for which $\operatorname{lmd}(G) = n - l$. We now study the structures of such graphs. If $p = 0$, then $\operatorname{lmd}(G) = n - l$ by Corollary 3.8; while if $p = l$, then every true twin equivalence class is a singleton set and so $n = l$. Since G is nontrivial, $\operatorname{lmd}(G) > 0 = n - l$ and so $\operatorname{lmd}(G) \neq n - l$. Therefore, it remains to consider the case where $1 \leqslant p \leqslant l - 1$. We first establish some additional definitions. For two subsets X and Y of the vertex set $V(G)$ of a connected graph G, define the *distance* $d(X, Y)$ *between* X *and* Y by

$$d(X, Y) = \min\{d(x, y) \colon\ x \in X \text{ and } y \in Y\}.$$

Thus $d(X, Y) = 0$ if and only if $X \cap Y \neq \emptyset$ and $d(X, Y) = 1$ if and only if $X \cap Y = \emptyset$ and some vertex in X is adjacent a vertex in Y. Suppose that $\mathcal{S} = \{U_1, U_2, \ldots, U_l\}$ is the set of all true twin equivalence classes of a nontrivial connected graph G. For an ordered subset $X = \{X_1, X_2, \ldots, X_k\}$ of $\mathcal{S}$ and an element U of $\mathcal{S}$, the *code* $\operatorname{code}^*_X(U)$ *of* U (with respect to X) is defined as the ordered k-tuple

$$\operatorname{code}^*_X(U) = (a_1, a_2, \ldots, a_k),$$

where

$$a_i = \begin{cases} d(U, X_i) & \text{if } U \neq X_i, \\ 1 & \text{if } U = X_i. \end{cases}$$

In other words, $a_i = 1$ if and only if $d(U, X_i) \leqslant 1$ for $1 \leqslant i \leqslant k$. We are now prepared to present a necessary and sufficient condition for a nontrivial connected graph G of order n having l true twin equivalence classes (where p of these classes consist of a single vertex and $1 \leqslant p \leqslant l-1$) to have local metric dimension $n-l$.

Theorem 3.9. *Let G be a nontrivial connected graph of order n and let $\mathcal{S} = \{U_1, U_2, \ldots, U_l\}$ be the set of true twin equivalence classes of G. Suppose that $|U_i| \geqslant 2$ for $1 \leqslant i \leqslant l-p$ and $|U_i| = 1$ for $l-p+1 \leqslant i \leqslant l$, where $1 \leqslant p \leqslant l-1$. Let $X = \{U_1, U_2, \ldots, U_{l-p}\}$. Then $\operatorname{lmd}(G) = n-l$ if and only if $\operatorname{code}^*_X(U) \neq \operatorname{code}^*_X(U')$ for every two elements U, U' in $\mathcal{S}$ with $d(U, U') = 1$.*

Proof. Suppose first that $\operatorname{lmd}(G) = n-l$ and let W_0 be a local metric basis of G. Then $V(G) - W_0 = \{u_1, u_2, \ldots, u_l\}$, where $u_i \in U_i$ for $1 \leqslant i \leqslant l$ and $W_0 \subseteq \bigcup_{i=1}^{l-p} U_i$. Suppose that $U_i, U_j \in \mathcal{S}$ and $d(U_i, U_j) = 1$. Then $i \neq j$. Let $\operatorname{code}^*_X(U_i) = (a_1, a_2, \ldots, a_{l-p})$ and $\operatorname{code}^*_X(U_j) = (b_1, b_2, \ldots, b_{l-p})$. Since $u_i u_j \in E(G)$ and $\operatorname{code}_{W_0}(u_i) \neq \operatorname{code}_{W_0}(u_j)$, there exists a vertex $w \in W_0 \cap U_s$, where $1 \leqslant s \leqslant l-p$, such that $d(u_i, w) \neq d(u_j, w)$. Observe that $s \neq i, j$, since otherwise $d(u_i, w) = d(u_j, w) = 1$. Then $U_s \neq U_i, U_j$ and so $a_s = d(U_i, U_s) = d(u_i, w) \neq d(u_j, w) = d(U_j, U_s) = b_s$, implying that $\operatorname{code}^*_X(U_i) \neq \operatorname{code}^*_X(U_j)$.

For the converse, let $u_i \in U_i$ for $1 \leqslant i \leqslant l$ and consider the set $W_1 = V(G) - \{u_1, u_2, \ldots, u_l\}$. We show that W_1 is a local metric set of G. Suppose that u_i and u_j are adjacent in G. Let $\operatorname{code}^*_X(U_i) = (a_1, a_2, \ldots, a_{l-p})$ and $\operatorname{code}^*_X(U_j) = (b_1, b_2, \ldots, b_{l-p})$. These codes are different and so we may assume, without loss of generality, that $a_1 \neq b_1$. Since $|U_1| \geqslant 2$, there exists a vertex $w \in U_1 - \{u_1\}$. Then observe that $d(u_i, w) = a_1 \neq b_1 = d(u_j, w)$. Since $w \in W_1$, it follows that W_1 is a local metric set of G and so $\operatorname{lmd}(G) \leqslant |W_1| = n-l$. Therefore, $\operatorname{lmd}(G) = n-l$ by Observation 2.1. □

Corollary 3.10. *Let G be a nontrivial connected graph of order n having l true twin equivalence classes, p of which consist of a single vertex. Then $\operatorname{lmd}(G) = n-l$ if and only if* (i) $p = 0$ *or* (ii) $1 \leqslant p \leqslant l-1$ *and G satisfies the conditions described in Theorem* 3.9.

4. On the uniqueness and non-uniqueness of local metric bases in a graph

We now turn our attention to determining those positive integers k for which there exists a nontrivial connected graph G with local metric dimension k such that either (i) G has a single local metric basis or (ii) G contains two local metric bases that are arbitrarily far apart. We begin with (i).

Theorem 4.1. *There exists a nontrivial connected graph G with* $\mathrm{lmd}(G) = k$ *having a unique local metric basis if and only if* $k \geqslant 2$.

P r o o f. Let G be a nontrivial connected graph having local metric dimension k. If $k = 1$, then G is bipartite and any singleton set $W \subseteq V(G)$ is a local metric basis. Therefore, if G has a unique local metric basis, then $k \geqslant 2$.

To verify the converse, suppose that $k \geqslant 2$. Consider the set $A = \{1, 2, \ldots, k\}$ and let $\mathcal{P}(A) = \{S_1, S_2, \ldots, S_{2^k}\}$ be the power set of A. Let $H = K_{2^k}$ be a complete graph of order 2^k with $V(H) = \{u_1, u_2, \ldots, u_{2^k}\}$. We construct G from H by adding k new vertices in the set $W = \{w_1, w_2, \ldots, w_k\}$ and joining u_i to w_j if and only if $j \in S_i$. Note that $G = G_\omega$, where $\omega = 2^k$, described in the proof of Theorem 3.1. Hence $\mathrm{lmd}(G) = k$ and furthermore, W is a local metric basis.

We show that W is the only local metric basis of G. Let W' be a local metric basis and assume, to the contrary, that $W' \neq W$. By symmetry, we may assume that $w_k \notin W'$. Let $\mathcal{P}(B) = \{S'_1, S'_2, \ldots, S'_{2^{k-1}}\}$ be the power set of $B = A - \{k\}$. We may also assume that $S_i = S'_i$ and $S_{i+2^{k-1}} = S'_i \cup \{k\}$ for $1 \leqslant i \leqslant 2^{k-1}$. Since W' is a local metric set of G and $w_k \notin W'$, this implies that for each i ($1 \leqslant i \leqslant 2^{k-1}$), at least one of u_i and $u_{i+2^{k-1}}$ belongs to W'. Hence, $k = |W'| \geqslant 2^{k-1}$. This is impossible if $k \geqslant 3$. If $k = 2$, on the other hand, then $W' \subseteq V(H)$, say $W' = \{u_1, u_2\}$. However then, $\mathrm{code}_{W'}(u_3) = \mathrm{code}_{W'}(u_4)$, a contradiction. Therefore, if W' is a local metric basis of G, then $W' = W$. □

To describe a solution to the problem stated in (ii), we first present some preliminary information, beginning with a lemma, which gives the local metric dimension $\mathrm{lmd}(G \times H)$ of the Cartesian product $G \times H$ of two graphs G and H in terms of $\mathrm{lmd}(G)$ and $\mathrm{lmd}(H)$. For the metric dimension of graphs, it was shown in [5] that $\dim(G) \leqslant \dim(G \times K_2) \leqslant \dim(G) + 1$ for every connected graph G. In [4] bounds (or exact values) have been established on $\mathrm{lmd}(G \times H)$ for many well-known classes of graphs G and H.

Lemma 4.2. *For every two connected graphs G and H,*

$$\mathrm{lmd}(G \times H) = \max\{\mathrm{lmd}(G), \mathrm{lmd}(H)\}.$$

P r o o f. Suppose that G and H are connected graphs of orders p and q, respectively, with $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_q\}$. Also, let $\mathrm{lmd}(G) = k$ and $\mathrm{lmd}(H) = l$ and assume, without loss of generality, that $k \geqslant l$. Rename the vertices of G and H, if necessary, so that $W_G = \{u_1, u_2, \ldots, u_k\}$ and $W_H = \{v_1, v_2, \ldots, v_l\}$ are local metric bases of G and H, respectively. Construct

$G \times H$ from q disjoint copies $G_1, G_2, \ldots, G_q$ of G with $V(G_i) = \{u_{1,i}, u_{2,i}, \ldots, u_{p,i}\}$ for $1 \leqslant i \leqslant q$ by joining $u_{j,i}$ and $u_{j',i'}$ if and only if $j = j'$ and $v_i v_{i'} \in E(H)$.

Now consider the set $W = \{u_{1,1}, u_{2,2}, \ldots, u_{l,l}\} \cup W'$, where $W' = \emptyset$ if $k = l$ and $W' = \{u_{l+1,1}, u_{l+2,1}, \ldots, u_{k,1}\}$ if $k > l$. We show that W is a local metric set of $G \times H$. Let x and y be adjacent vertices in $G \times H$ not belonging to W. We consider two cases.

Case 1. *Both x and y belong to $V(G_\alpha)$ for some α $(1 \leqslant \alpha \leqslant q)$.* Since $x, y \in V(G_\alpha)$, observe that $x = u_{a,\alpha}$ and $y = u_{b,\alpha}$ for some a and b and $u_a u_b \in E(G)$. Since W_G is a local metric basis of G, there exists a vertex $u_c \in W_G$ such that $d_G(u_a, u_c) \neq d_G(u_b, u_c)$. Then the set W contains a vertex $w_1 = u_{c,\beta}$ for some β and observe that

$$d_{G\times H}(x, w_1) = d_G(u_a, u_c) + d_H(v_\alpha, v_\beta) \neq d_G(u_b, u_c) + d_H(v_\alpha, v_\beta) = d_{G\times H}(y, w_1).$$

Therefore, $\mathrm{code}_W(x) \neq \mathrm{code}_W(y)$.

Case 2. *$x \in V(G_\alpha)$ and $y \in V(G_\beta)$ for some α and β, where $\alpha \neq \beta$ $(1 \leqslant \alpha, \beta \leqslant q)$.* Then $x = u_{a,\alpha}$ and $y = u_{a,\beta}$ for some a and $v_\alpha v_\beta \in E(H)$. Since W_H is a local metric basis of H, there exists a vertex $v_\gamma \in W_H$ such that $d_H(v_\alpha, v_\gamma) \neq d_H(v_\beta, v_\gamma)$. Then the set W contains a vertex $w_2 = u_{b,\gamma}$ for some b and observe that

$$d_{G\times H}(x, w_2) = d_G(u_a, u_b) + d_H(v_\alpha, v_\gamma) \neq d_G(u_a, u_b) + d_H(v_\beta, v_\gamma) = d_{G\times H}(y, w_2).$$

Therefore, $\mathrm{code}_W(x) \neq \mathrm{code}_W(y)$.

Hence, every two adjacent vertices in $G \times H$ have distinct codes with respect to W and so $\mathrm{lmd}(G \times H) \leqslant |W| = k = \mathrm{lmd}(G)$.

To show that $\mathrm{lmd}(G) \leqslant \mathrm{lmd}(G \times H)$, let W be a local metric basis of $G \times H$ and let W_1 be the subset of $V(G_1)$ such that $u_{i,1} \in W_1$ if and only if $u_{i,j} \in W$. We show that W_1 is a local metric set of G_1. Let $x, y \in V(G_1) - W_1$ be two adjacent vertices in G_1. Since W is a local metric set of $G \times H$, there exists a vertex $w = u_{a,\alpha} \in W$ such that $d_{G\times H}(x, w) \neq d_{G\times H}(y, w)$. Therefore, W_1 contains a vertex $w' = u_{a,1}$. Then observe that

$$d_{G_1}(x, w') = d_{G\times H}(x, w) - d_H(v_1, v_\alpha) \neq d_{G\times H}(y, w) - d_H(v_1, v_\alpha) = d_{G_1}(y, w'),$$

implying that $\mathrm{code}_{W_1}(x) \neq \mathrm{code}_{W_1}(y)$. Hence $\mathrm{lmd}(G_1) \leqslant |W_1| \leqslant |W| = \mathrm{lmd}(G \times H)$. $\square$

By Observation 2.1, if $U_1, U_2, \ldots, U_l$ are the true twin equivalence classes of a graph G, then every local metric basis of G must contain at least $|U_i| - 1$ vertices from U_i for each i $(1 \leqslant i \leqslant l)$. Therefore, if G has two local metric bases W and

W' that are disjoint, then $1 \leqslant d(W, W') \leqslant \operatorname{diam}(G)$ and $|U_i| \leqslant 2$ for each i. Let $d(W, W') = t$. Then $|W| + |W'| + (t-1) \leqslant n$ and so $t \leqslant n - 2\operatorname{lmd}(G) + 1$. Also, if $|U_i| = 2$ for some i, say $U_1 = \{u, v\}$, then $u \in W$ and $v \in W'$, implying that $d(W, W') = 1$. Hence, if G contains two local metric bases W and W' such that $d(W, W') \geqslant 2$, then every true twin equivalence class of G is a singleton set.

Theorem 4.3. *For each pair k, t of positive integers, there exists a connected graph G with $\operatorname{lmd}(G) = k$ having two local metric bases W and W' such that $d(W, W') = s$ for each s with $1 \leqslant s \leqslant t$.*

Proof. Construct $G = K_{k+1} \times P_{t+1}$ from $t+1$ copies $H_1, H_2, \ldots, H_{t+1}$ of K_{k+1}, where $V(H_i) = \{u_{1,i}, u_{2,i}, \ldots, u_{k+1,i}\}$ for $1 \leqslant i \leqslant t+1$, by joining $u_{j,i}$ to $u_{j,i+1}$ for $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant k+1$. By Lemma 4.2 and Theorem 2.4, $\operatorname{lmd}(G) = k$. Furthermore, each set $W_i = V(H_i) - \{u_{k+1,i}\}$ is a local metric basis for $1 \leqslant i \leqslant t+1$. Since $d(W_1, W_{s+1}) = s$ for $1 \leqslant s \leqslant t$, we obtain the desired result. □

By the proofs of Theorems 4.1 and 4.3, there exists a connected graph with a local metric basis W for which the subgraph $\langle W \rangle$ induced by W is an empty graph and there also exists a connected graph with a local metric basis W' for which the subgraph $\langle W' \rangle$ induced by W' is a complete graph. In fact, for every graph H, there is a connected graph with a local metric basis W such that $\langle W \rangle = H$.

Theorem 4.4. *For every graph H, there exists a connected graph G having local metric basis W such that $\langle W \rangle = H$.*

Proof. Suppose that H is a graph of order $k \geqslant 1$ with $V(H) = \{w_1, w_2, \ldots, w_k\}$. Let $\mathcal{P}(A) = \{S_1, S_2, \ldots, S_{2^k}\}$ be the power set of the set $A = \{1, 2, \ldots, k\}$. Construct G from H by adding 2^k new vertices in the set $U = \{u_1, u_2, \ldots, u_{2^k}\}$ such that $\langle U \rangle = K_{2^k}$ and joining w_i to u_j if and only if $i \in S_j$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant 2^k$. Observe that $\omega(G) = 2^k$ and so $\operatorname{lmd}(G) \geqslant k$ by Theorem 3.1. Since $V(H)$ is a local metric set containing k vertices, it follows that $\operatorname{lmd}(G) = k$ and $V(H)$ is a local metric basis of G. □

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References

[1] *M. Anderson, C. Barrientos, R. C. Brigham, J. R. Carrington, M. Kronman, R. P. Vitray, J. Yellen*: Irregular colorings of some graph classes. Bull. Inst. Combin. Appl. *55* (2009), 105–119.

[2] *P. N. Balister, E. Győri, J. Lehel, R. H. Schelp*: Adjacent vertex distinguishing edge-colorings. SIAM J. Discrete Math. *21* (2007), 237–250.

[3] *A. C. Burris, R. H. Schelp*: Vertex-distinguishing proper edge colorings. J. Graph Theory. *26* (1997), 73–82.

[4] *J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood*: On the metric dimension of Cartesian products of graphs. SIAM J. Discr. Math. *21* (2007), 423–441.

[5] *G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann*: Resolvability in graphs and the metric dimension of a graph. Discrete Appl. Math. *105* (2000), 99–113.

[6] *G. Chartrand, L. Lesniak, P. Zhang*: Graphs & Digraphs: Fifth Edition. Chapman & Hall/CRC, Boca Raton, FL, 2010.

[7] *G. Chartrand, L. Lesniak, D. W. VanderJagt, P. Zhang*: Recognizable colorings of graphs. Discuss. Math. Graph Theory. *28* (2008), 35–57.

[8] *G. Chartrand, F. Okamoto, C. W. Rasmussen, P. Zhang*: The set chromatic number of a graph. Discuss. Math. Graph Theory. *29* (2009), 545–561.

[9] *G. Chartrand, F. Okamoto, E. Salehi, P. Zhang*: The multiset chromatic number of a graph. Math. Bohem. *134* (2009), 191–209.

[10] *G. Chartrand, F. Okamoto, P. Zhang*: The metric chromatic number of a graph. Australas. J. Combin. *44* (2009), 273–286.

[11] *G. Chartrand, F. Okamoto, P. Zhang*: Neighbor-distinguishing vertex colorings of graphs. J. Combin. Math. Combin. Comput. To appear.

[12] *G. Chartrand, P. Zhang*: Chromatic Graph Theory. Chapman & Hall/CRC Press, Boca Raton, FL, 2009.

[13] *F. Harary, R. A. Melter*: On the metric dimension of a graph. Ars Combin. *2* (1976), 191–195.

[14] *F. Harary, M. Plantholt*: The point-distinguishing chromatic index. Graphs and Applications. Wiley, New York, 1985, pp. 147–162.

[15] *C. Hernando, M. Mora, I. M. Pelayo, C. Seara, D. R. Wood*: Extremal graph theory for the metric dimension and diameter. Electronic Notes in Discrete Mathematics *29* (2007), 339–343.

[16] *M. Karoński, T. Łuczak, A. Thomason*: Edge weights and vertex colours. J. Combin. Theory Ser. B. *91* (2004), 151–157.

[17] *A. Khuller, B. Raghavachari, A. Rosenfeld*: Landmarks in graphs. Discr. Appl. Math. *70* (1996), 217–229.

[18] *M. Radcliffe, P. Zhang*: Irregular colorings of graphs. Bull. Inst. Combin. Appl. *49* (2007), 41–59.

[19] *V. Saenpholphat*: Resolvability in Graphs. Ph.D. Dissertation, Western Michigan University, 2003.

[20] *P. J. Slater*: Leaves of trees. Congress. Numer. *14* (1975), 549–559.

[21] *P. J. Slater*: Dominating and reference sets in graphs. J. Math. Phys. Sci. *22* (1988), 445–455.

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