Nils Svanstedt
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STOCHASTIC HOMOGENIZATION OF A CLASS OF MONOTONE EIGENVALUE PROBLEMS

Nils Svanstedt, Göteborg

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Abstract. Stochastic homogenization (with multiple fine scales) is studied for a class of nonlinear monotone eigenvalue problems. More specifically, we are interested in the asymptotic behaviour of a sequence of realizations of the form

$$-\operatorname{div}\Big(a\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, \nabla u_\varepsilon^\omega\Big)\Big) = \lambda_\varepsilon^\omega \mathcal{C}(u_\varepsilon^\omega).$$

It is shown, under certain structure assumptions on the random map $a(\omega_1,\omega_2,\xi)$, that the sequence $\{\lambda_\varepsilon^{\omega,k}, u_\varepsilon^{\omega,k}\}$ of kth eigenpairs converges to the kth eigenpair $\{\lambda^k, u^k\}$ of the homogenized eigenvalue problem

$$-\operatorname{div}(b(\nabla u)) = \lambda\overline{\mathcal{C}}(u).$$

For the case of p-Laplacian type maps we characterize b explicitly.

Keywords: stochastic, homogenization, eigenvalue

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1. Introduction

In this paper we consider the stochastic homogenization problem for the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}\Big(a\Big(T_1\Big(\dfrac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\dfrac{x}{\varepsilon_2}\Big)\omega_2, \nabla u_\varepsilon^\omega\Big)\Big) = \lambda_\varepsilon \mathcal{C}(u_\varepsilon^\omega) \quad \text{in } Q, \\ u_\varepsilon^\omega = 0 \quad \text{in } \partial Q. \end{cases} \tag{1}$$

Here and throughout the paper we will write Q for an open bounded set in $\mathbb{R}^n$. The maps a and $\mathcal{C}$ are of the forms $a(\omega_1,\omega_2,\xi) = \alpha(\omega_1,\omega_2)|\xi|^{p-2}\xi$ and $\mathcal{C}(\eta) = |\eta|^{p-2}\eta$ for real valued $p \geqslant 1$. For each fixed $\omega_i \in \Omega_i$, $i = 1, 2$, the realization (1) is an eigenvalue problem. Following the framework in [13], see also [14] and [10], we associate two probability spaces $(\Omega_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2$. Each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated countably additive non-negative probability measure on $\mathcal{F}_k$ normalized by $\mu_k(\Omega_k) = 1$. With every $x \in \mathbb{R}^n$ we associate the dynamical system

$$T_k(x)\colon\ \Omega_k \to \Omega_k.$$

For the random field

$$a(\omega_1, \omega_2, \xi)$$

we can then, for fixed ω_1 and ω_2, consider the realization

$$a(T_1(x)\omega_1, T_2(x)\omega_2, \xi)$$

and the "speeded up" realization

$$a\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, \xi\Big).$$

With this construction which will be precisely defined in Section 2 the random fields become stationary due to the invariance properties of the associated probability measure and therefore, the Birkhoff ergodic theorem applies and we can define limits of the speeded up realizations in terms of expectations (mean values) over the probability spaces.

In our analysis we will have to assume that ε_1 and ε_2 are two well separated functions (scales) of $\varepsilon > 0$ which converge to zero as ε tends to zero. We say that ε_1 and ε_2 are well separated if

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_2}{\varepsilon_1} = 0.$$

This means that ε_2 is a finer scale than ε_1. For instance if $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \varepsilon^2$, then ε_1 and ε_2 are well separated scales.

The homogenization problem for monotone p-Laplacian operators have been studied recently in [2] and in [8]. In [2] the homogenization of the first eigenvalue λ^1 is proved by G-convergence methods. In [8] Champion and de Pascale employ Γ-convergence to prove the homogenization for every eigenvalue λ^k. In the present work we extend the result to a stochastic setting and allow multiple scales in the elliptic operator. The present work is much inspired by both [2] and [8]. For a nice introduction to eigenvalues of the p-Laplacian we recommend the lecture notes [11] by Lindqvist.

Homogenization problems with more than one oscillating scale in the periodic setting was first introduced in [3] for linear elliptic problems. The multiscale monotone stochastic elliptic and parabolic cases have been recently studied in [13].

In the present work we will use the classical framework of G-convergence, which can be thought of as a non-periodic "homogenization" or stabilization of sequences of operator equations. We refer to [4], [5], and [12] concerning G-convergence results for elliptic operators needed in this report. Here we show that the general theory also applies to the situation of multiple scales and multiscale stochastic homogenization of a class of nonlinear eigenvalue problems.

The homogenization problem for monotone operators in the random setting has been studied by Efendiev and Pankov, see [10] and the references therein. They consider single spatial and temporal scales but consider oscillations also in time. The corresponding multiscale situation is studied in [13].

2. Some basic notation

Let $(\Omega, \mathcal{F}, \mu)$ denote a probability space, where $\mathcal{F}$ is a complete σ-algebra and μ is a probability measure. With every $x \in \mathbb{R}^n$ we associate the dynamical system

$$T(x)\colon\ \Omega \to \Omega,$$

where both $T(x)$ and $T(x)^{-1}$ are assumed to be μ-measurable. Moreover, we assume that the following (measure preserving) properties are satisfied:

- $T(0)\omega = \omega$ for each $\omega \in \Omega$.
- $T(x+y) = T(x)T(y)$ for $x, y \in \mathbb{R}^n$.
- The set $\{(x,\omega) \in \mathbb{R}^n \times \Omega\colon\ T(x)\omega \in F\}$ is a $\mathrm{d}x \times \mathrm{d}\mu(\omega)$ measurable subset of $\mathbb{R}^n \times \Omega$ for each $F \in \mathcal{F}$, where $\mathrm{d}x$ denotes the Lebesgue measure.
- For any measurable function $f(\omega)$ defined on Ω, the function $f(T(x)\omega)$ defined on $\mathbb{R}^n \times \Omega$ is also measurable when $\mathbb{R}^n$ is endowed with the Lebesgue measure.

The dynamical system T is said to be *ergodic* if every invariant function f (i.e. a function f which satisfies $f(T(x)\omega) = f(\omega)$) is constant almost everywhere in Ω.

Example 1 (periodic case). As a special case we recover the periodic functions by letting

$$\Omega = \{\omega \in \mathbb{R}^n\colon\ 0 \leqslant \omega_k \leqslant 1,\ k = 1,\dots,n\} \quad \text{and} \quad T(x)\colon\ \Omega \to \Omega$$

given by

$$T(x)\omega = x + \omega \pmod 1.$$

For a random field $f(x,\omega)$ the "periodic" realization is given by $f(x+\omega)$.

Definition 1. We say that a vector field f is a potential field if there exists a function $g \in W_0^{1,p}(\mathbb{R}^n)$ such that $f = \nabla g$.

By $L^p(\Omega)$ we denote the equivalence class of all μ-integrable functions (with exponent $p \geqslant 1$).

Definition 2. We say that a random vector field $f \in [L^p(\Omega)]^n$ is a potential field if almost all its realizations are potential fields. We denote this field by L^p_{pot}.

Definition 3. We also define the space of vector fields with mean value zero:

$$V_{\mathrm{pot}}(\Omega) = \left\{ f \in [L^p(\Omega)]^n \colon \langle f \rangle = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = 0 \right\}.$$

We observe that by the Fubini Theorem it follows that if $f \in L^p(\Omega)$ then almost all realizations $f(T(x)\omega) \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The number $M(f)$ is called the mean value of f if

$$\lim_{\varepsilon \to 0} \int_K f(x/\varepsilon)\,\mathrm{d}x = |K| M(f)$$

for any Lebesgue measurable bounded set $K \in \mathbb{R}^n$. Alternatively, the mean value can be expressed in terms of weak convergence. If the family $\{f(\cdot/\varepsilon)\}$ is in $L^p(Q)$, $p \geqslant 1$ then $M(f)$ is called the mean value of f if

$$\{f(\cdot/\varepsilon)\} \rightharpoonup M(f) \quad \text{in } L^p(Q).$$

We can now formulate (see [9], pp. 685–698):

Theorem 1 (Birkhoff Ergodic Theorem). *Let $f \in L^p(\Omega)$, $p \geqslant 1$. Then for almost all $\omega \in \Omega$ the realization $f(T(x)\omega)$ possesses a mean value $M(f(T(x)\omega))$. Moreover, as a function of $\omega \in \Omega$, this mean value $M(f(T(x)\omega))$ is invariant and*

$$\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega M(f(T(x)\omega))\,\mathrm{d}\mu(\omega).$$

If the system $T(x)$ is ergodic then

$$\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = M(f(T(x)\omega)).$$

Now let $\{(\Omega_k, \mathcal{F}_k, \mu_k)\}_{k=1}^M$ denote a family of probability spaces, where each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated probability measure. With every $x \in \mathbb{R}^n$ we also associate the dynamical system

$$T_k(x) \colon \Omega_k \to \Omega_k.$$

We also introduce

$$\mathbf{T} = (T_1, \ldots, T_M)$$

as a dynamical system on the product space $(\Omega_1 \times \ldots \times \Omega_M)$. We can now state a multidimensional extension of the Birkhoff ergodic theorem (see [9], pp. 685–698).

Theorem 2. *Let $f \in L^p(\Omega_1 \times \ldots \times \Omega_M)$, $p \geqslant 1$. Then for almost all $\omega_k \in \Omega_k$ the realization $f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)$ possesses a mean value*

$$M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)).$$

Moreover, as a function of $\omega_k \in \Omega_k$, this mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$ is invariant and

$$\begin{aligned}\langle f \rangle &\equiv \int_{\Omega_1} \ldots \int_{\Omega_M} f(\omega_1, \ldots, \omega_M)\,\mathrm{d}\mu_1(\omega_1)\ldots\,\mathrm{d}\mu_M(\omega_M)\\ &= \int_{\Omega_1} \ldots \int_{\Omega_M} M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))\,\mathrm{d}\mu_1(\omega_1)\ldots\,\mathrm{d}\mu_M(\omega_M).\end{aligned}$$

If in addition the system $\mathbf{T}$ is ergodic on $(\Omega_1 \times \ldots \times \Omega_M)$ then

$$\langle f \rangle = M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)).$$

We continue by setting the appropriate structure conditions:

Definition 5. We assume that $1 < p < \infty$ and that $0 < \beta_1 < \beta_2 < \infty$ and define the class S of maps

$$a\colon\ Q \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfying

(i) $a(\cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$,
(ii) $a(x, \cdot)$ is continuous a.e. in Q,
(iii) $|(a(x, \xi)| \leqslant \beta_2 |\xi|^{p-1}$ a.e. in Q for all $\xi_1, \xi_2 \in \mathbb{R}^n$,
(iv) $(a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geqslant \beta_1 |\xi_1 - \xi_2|^p$ a.e. in Q for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$.

Example 2. The p-Laplacian map $\alpha(x)|\xi|^{p-2}\xi$ belongs to the class S if $\alpha \in L^\infty(Q)$ and is strictly positive. This map is in addition cyclically monotone. See [7] for a complete study of G-convergence of cyclically monotone operators, like the p-Laplacian, and its relation to Γ-convergence of the associated convex lower semicontinuous functionals.

Let us introduce some function spaces related to the differential equations studied in this paper. Let V be a reflexive real Banach space, with the dual V'. We will denote $V = W_0^{1,p}(Q)$ with the norm $\|u\|_V^p = \int_Q |\nabla u|^p \, dx$ and $V' = W^{-1,q}(Q)$, where $1/p + 1/q = 1$. We also define the spaces

$$U = L^p(Q; \mathbb{R}^n) \quad \text{and} \quad U' = L^q(Q; \mathbb{R}^n).$$

For the readers' convenience we recall briefly the definitions of G-convergence of operators and Γ-convergence of functionals.

2.1. G-convergence

For a complete treatment of a large class of (possibly multivalued) elliptic operators we refer to [5] and [4]. However, in the present work we will only consider single-valued operators. Consider the sequence of elliptic Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a_h(x, \nabla u_h)) = f_h & \text{in } Q, \\ u_h \in V. \end{cases} \tag{2}$$

Definition 6. The sequence $\{a_h\} \subset S$ is said to G-converge to $a \in S$ if, for every $f \in V'$, the sequence $\{u_h\}$ of solutions of (2) satisfies

$$\begin{aligned} u_h &\rightharpoonup u \quad \text{in } V, \\ a_h(\cdot, \nabla u_h) &\rightharpoonup a(\cdot, \nabla u) \quad \text{in } U', \end{aligned}$$

where u is the unique solution to the problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f & \text{in } Q, \\ u \in V. \end{cases}$$

The following compactness result is proved in [4]:

Theorem 3. *For every sequence $\{a_h\} \subset S$ there exists a subsequence, still denoted by $\{a_h\}$, and a map $a \in S$ such that $\{a_h\}$ G-converges to a.*

2.2. Γ-convergence

Let X be a topological space and let $\mathcal{N}(x)$ denote the set of all open neighborhoods of $x \in X$. Further, let $\{F_h\}$ be a sequence of functions from X into $\overline{\mathbb{R}}$.

Definition 7. The Γ-lower and Γ-upper limits of the sequence $\{F_h\}$ are the functions from X into $\overline{\mathbb{R}}$ defined by

$$F'(x) = \Gamma - \liminf_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{z\in\omega} F_h(z)$$

and

$$F''(x) = \Gamma - \limsup_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{z\in\omega} F_h(z),$$

respectively. If these two limits coincide, i.e. if there exists a unique function $F\colon X\to\overline{\mathbb{R}}$ such that

$$F = \Gamma - \liminf_{h\to\infty} F_h(x) = \Gamma - \limsup_{h\to\infty} F_h(x),$$

we say that the sequence $\{F_h\}$ Γ-converges to F.

Remark 1. By the definition it is obvious that $\{F_h\}$ Γ-converges to F if and only if

$$\Gamma - \limsup_{h\to\infty} F_h \leqslant F \leqslant \Gamma - \liminf_{h\to\infty} F_h.$$

This means that Γ-convergence and lower semicontinuity are closely related concepts.

We have the following sequential characterization of Γ-convergence, see [6], Proposition 8.1:

Theorem 4. *Let X be a separable metric space and let $\{F_h\}$ be a sequence of functionals from X into $\overline{\mathbb{R}}$. Then*

(i) *for every $x\in X$ and for every sequence $\{x_h\}$ converging to x,*

$$F'(x) \leqslant \liminf_{h\to\infty} F_h(x_h);$$

(ii) *for every $x\in X$ there exists a sequence $\{x_h\}$ converging to x such that*

$$F'(x) = \liminf_{h\to\infty} F_h(x_h);$$

(iii) *for every $x\in X$ and for every sequence $\{x_h\}$ converging to x,*

$$F''(x) \leqslant \limsup_{h\to\infty} F_h(x_h);$$

(iv) *for every $x\in X$ there exists a sequence $\{x_h\}$ converging to x such that*

$$F''(x) = \limsup_{h\to\infty} F_h(x_h).$$

Consequently, $\{F_h\}$ Γ-converges to a function $F \in X$ if and only if

(v) *for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,*

$$F(x) \leqslant \liminf_{h\to\infty} F_h(x_h)$$

and

(vi) *for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that*

$$F(x) = \lim_{h\to\infty} F_h(x_h).$$

Moreover, Γ-convergence enjoys the following compactness property, see [6], Theorem 8.5:

Theorem 5. *Let X be a separable metric space. Then every sequence $\{F_h\}$ of functionals from X into $\overline{\mathbb{R}}$ has a Γ-convergent subsequence.*

3. The nonlinear eigenvalue problem

We will consider the eigenvalue problem for p-Laplacian operators $\mathcal{A}\colon V \to V'$ of the type

$$\mathcal{A}(u) = -\operatorname{div}(\alpha(x)|\nabla u|^{p-2}\nabla u).$$

More precisely, we consider the nonlinear monotone elliptic eigenvalue problem

$$\begin{cases} -\operatorname{div}(\alpha(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u \quad \text{in } Q, \\ u \in V. \end{cases} \tag{3}$$

An eigenvalue to (3) is a real number λ such that there exists a non-trivial solution (eigenfunction) $u \in V$ to (3).

Let us assume that α is strictly positive and bounded in $L^\infty(Q)$. Then, see Theorem 3.1 and Proposition 3.2 in [2], there exists a function $h\colon Q \times \mathbb{R}^n \to \mathbb{R}_+$ such that

(i) $h(\cdot,\xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$,
(ii) $h(x,\cdot)$ is continuous a.e. in Q,
(iii) $h(x,\cdot)$ is convex and Gâteaux differentiable with Gâteaux derivative $a(x,\cdot)$,
(iv) $h(x,\cdot)$ is positively homogeneous of degree p,
(v) $h(x,\cdot)$ is even,
(vi) $\beta_1|\xi|^p \leqslant h(x,\xi) \leqslant \beta_2|\xi|^p$ a.e. in Q for all $\xi \in \mathbb{R}^n$.

E x a m p l e 3. For $a(x,\xi) = \alpha(x)|\xi|^{p-2}\xi$ we have $h(x,\xi) = p^{-1}\alpha(x)|\xi|^p$. G-convergence is the asymptotic property of solutions u_h to Euler equations (see above)

$$\begin{cases} -\operatorname{div}(a_h(x,\nabla u_h)) = f_h \quad \text{in } Q, \\ u_h \in V. \end{cases}$$

Γ-convergence is the asymptotic property of the corresponding functionals

$$F_h(v) = \int_Q h_h(x,\nabla v)\,\mathrm{d}x.$$

We will make use of the strong connection between G- and Γ-convergence for p-Laplacian operators in the proof of the main result (Theorem 8). Recall also that the solution to the Euler equation is also the minimum to the minimization problem

$$\min_{v\in V}\{F_h(v) - \langle f,v\rangle\}.$$

The kth eigenvalue $\lambda^k \in \mathbb{R}$ to (3) is constructed by using the classical Ljusternik-Schnirelman theory and is defined via

$$\lambda^k = \inf_{\gamma(G)\geqslant k}\ \sup_{g(\varphi)\in G}\Big\{\frac{F(\varphi)}{g(\varphi)}\Big\}, \tag{4}$$

where $\gamma(G)$ is the Krasnosel'skii genus of the set $G \subset L^p$. We refer e.g. to [1] and the references therein for more details. The functionals F and g are given by

$$F(\varphi) = \int_Q \alpha(x)|\nabla\varphi|^p\,\mathrm{d}x$$

and

$$g(\varphi) = \int_Q |\varphi|^p\,\mathrm{d}x.$$

By the properties of the p-Laplacian map $a(x,\xi) = \alpha(x)|\xi|^{p-2}\xi$ it follows that the functional F is convex and lower semicontinuous and hence also weakly lower semicontinuous. For $\alpha = 1$ the following existence theorem is proved in [1].

Theorem 6. *Let $k\in\mathbb{N}$. There exists a non-trivial $u^k \in V$ which solves* (3) *with $\lambda^k > 0$ defined as in* (4).

R e m a r k 2. A full understanding of the spectrum is not yet available. It is known that

- The first eigenvalue is strictly positive and isolated.
- The first eigenvalue is the reciprocal of the constant $C(p,Q)$ in the Poincaré inequality
$$\int_Q |u|^p \,\mathrm{d}x \leqslant C(p,Q) \int_Q |\nabla u|^p \,\mathrm{d}x.$$
- There are infinitely many eigenvalues λ^k, $k \in \mathbb{N}$ to (3) and $\lambda^k \to +\infty$ as $k \to +\infty$.

R e m a r k 3. For $a \in S$ the homogenization of the first eigenvalue and eigenfunction is due to Baffico et. al. [2]. They employ G-convergence techniques and the weak lower semicontinuity of the functional F to prove the existence of a minimizer to the Rayleigh quotient. The extension to higher eigenvalues is due to Champion and De Pascale [8]. They base their proof on Γ-convergence.

R e m a r k 4. In the present work we will use G-convergence techniques to prove a homogenization result for a sequence of p-Laplacian type eigenvalue problems in the random stationary case and a comparison result for G- and Γ-convergence. Observe that by the properties of the class S, the G-convergence for operators in the class S is equivalent to the Γ-convergence of functionals like F. This is due to a comparison result, Theorem 3.3, by Defranceschi [7].

The corresponding random montone elliptic eigenvalue problem can now be studied by means of realizations of stationary auxiliary fields. For fixed $\omega_i \in \Omega_i$, $i = 1, 2$, we consider the monotone eigenvalue problem

$$\begin{cases} -\operatorname{div}(\alpha(T_1(x)\omega_1, T_2(x)\omega_2)|\nabla u^\omega|^{p-2}\nabla u^\omega) = \lambda^\omega |u^\omega|^{p-2}u^\omega \quad \text{in } Q, \\ u \in V. \end{cases} \tag{5}$$

As above we now define the kth eigenvalue $\lambda^{\omega,k} \in \mathbb{R}$ to the realization (5) via

$$\lambda^{\omega,k} = \inf_{\gamma(G)\geqslant k} \sup_{g(\varphi)\in G} \left\{ \frac{F^\omega(\varphi)}{g(\varphi)} \right\}, \tag{6}$$

where F^ω and g are given by

$$F^\omega(\varphi) = \int_Q \alpha(T_1(x)\omega_1, T_2(x)\omega_2)|\nabla\varphi|^p \,\mathrm{d}x$$

and

$$g(\varphi) = \int_Q |\varphi|^p \,\mathrm{d}x.$$

The aim of this paper is to prove a homogenization result for a sequence

$$(7)\qquad \begin{cases} -\operatorname{div}\Bigl(\alpha\Bigl(T_1\Bigl(\dfrac{x}{\varepsilon_1}\Bigr)\omega_1, T_2\Bigl(\dfrac{x}{\varepsilon_2}\Bigr)\omega_2\Bigr)|\nabla u_\varepsilon^\omega|^{p-2}\nabla u_\varepsilon^\omega\Bigr) = \lambda_\varepsilon^\omega |u_\varepsilon^\omega|^{p-2}u_\varepsilon^\omega \quad \text{in } Q,\\ u_\varepsilon^\omega \in V. \end{cases}$$

For every $k \in \mathbb{N}$ and $\varepsilon > 0$ the kth eigenvalue $\lambda_\varepsilon^{\omega,k} \in \mathbb{R}$ to (7) is given by

$$(8)\qquad \lambda_\varepsilon^{\omega,k} = \inf_{\gamma(G)\geqslant k}\ \sup_{g(\varphi)\in G} \Bigl\{\frac{F_\varepsilon^\omega(\varphi)}{g(\varphi)}\Bigr\},$$

where F_ε^ω and g are given by

$$F_\varepsilon^\omega(\varphi) = \int_Q \alpha\Bigl(T_1\Bigl(\frac{x}{\varepsilon_1}\Bigr)\omega_1, T_2\Bigl(\frac{x}{\varepsilon_2}\Bigr)\omega_2\Bigr)|\nabla\varphi|^p \,\mathrm{d}x$$

and

$$g(\varphi) = \int_Q |\varphi|^p \,\mathrm{d}x.$$

We will prove below that for every $k \in \mathbb{N}$ the eigenpair $\{\lambda_\varepsilon^{\omega,k}, u_\varepsilon^{\omega,k}\} \in \mathbb{R}\times V$ satisfies

$$\lambda_\varepsilon^{\omega,k} \to \lambda^k$$

and

$$u_\varepsilon^{\omega,k} \rightharpoonup u^k \quad \text{in } V,$$

where the pair $\{\lambda^k, u^k\} \in \mathbb{R}\times V$ is the kth eigenpair to a homogenized monotone eigenvalue problem

$$(9)\qquad \begin{cases} -\operatorname{div}(b(\nabla u)) = \lambda\overline{C}(u) \quad \text{in } Q,\\ u \in V. \end{cases}$$

The basic underlying tool for proving this will be the G-convergence of elliptic operators.

We define $A_\varepsilon^\omega\colon V \to U'$ as

$$A_\varepsilon^\omega(x,\xi) = a\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, \xi\Big). \tag{10}$$

Theorem 7. *Let us consider the sequence of elliptic boundary value problems*

$$\begin{cases} -\operatorname{div}(A_\varepsilon^\omega(x,\nabla u_\varepsilon^\omega)) = f_\varepsilon^\omega & \text{in } Q, \\ u_\varepsilon^\omega \in V. \end{cases} \tag{11}$$

Assume that $A_\varepsilon^\omega \in S$, where the constants in Definition 5 are independent of ε. Further assume that the sequence $\{f_\varepsilon^\omega\}$ is compact in V' and that the dynamical system $\mathbf{T}(x) = (T_1(x), T_2(x))$ is ergodic on the product space $\Omega_1 \times \Omega_2$.

For every $(\omega_1,\omega_2) \in \Omega_1 \times \Omega_2$ a passage $\varepsilon \to 0$ yields

$$u_\varepsilon^\omega \rightharpoonup u \quad \text{in } V,$$
$$A_\varepsilon^\omega(\cdot, \nabla u_\varepsilon^\omega) \rightharpoonup b(\nabla u) \quad \text{in } U',$$

where u solves the homogenized problem

$$\begin{cases} -\operatorname{div}(b(\nabla u)) = \langle f \rangle & \text{in } Q, \\ u \in V. \end{cases} \tag{12}$$

The homogenized operator b is given by

$$b(\xi) = \int_{\Omega_1} b_1(\omega_1, \xi + z_1^\xi(\omega_1))\, \mathrm{d}\mu_1(\omega_1),$$

where $z_1^\xi(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$ is the solution to the auxiliary ε_1-scale local problem

$$\langle b_1(\omega_1, \xi + z_1^\xi(\omega_1), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$. The operator b_1 is given by

$$b_1(\omega_1,\xi) = \int_{\Omega_2} a(\omega_1,\omega_2,\xi + z_2^\xi(\omega_1,\omega_2))\, \mathrm{d}\mu_2(\omega_2),$$

where $z_2^\xi(\omega_1,\omega_2) \in V_{\mathrm{pot}}(\Omega_2)$ is the solution to the auxiliary ε_2-scale local problem

$$\langle a(\omega_1,\omega_2,\xi + z_2^\xi(\omega_1,\omega_2), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{\mathrm{pot}}(\Omega_2)$ a.e. $\omega_1 \in \Omega_1$.

Proof. Let us first consider the case with a fixed right-hand side, say $f \in V'$ in (11). By the structure conditions it follows that for every $(\varepsilon_1, \varepsilon_2)$ there exists a unique solution $u_\varepsilon^\omega \in V$ for a.e. $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$.

Again by the structure conditions and by the reflexivity of V and U it follows that (up to subsequences)

$$u_\varepsilon^\omega \rightharpoonup u^* \quad \text{in } V$$

and

$$A_\varepsilon^\omega(\cdot, \nabla u_\varepsilon^\omega) \rightharpoonup \xi^* \quad \text{in } U'.$$

The rest of the proof amounts to verifying that $\xi^* = b(\nabla u^*)$ and at the same time characterizing b explicitly. We start out by the ε_2-process. Let us fix $\xi \in \mathbb{R}^n$. For a.e. $\omega_1 \in \Omega_1$ we let $z_2^\xi(\omega_1, \omega_2) \in V_{\text{pot}}(\Omega_2)$ be the solution to

$$\langle a(\omega_1, \omega_2, \xi + z_2^\xi(\omega_1, \omega_2), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{\text{pot}}(\Omega_2)$. The existence and uniqueness of the solution $z_2^\xi(\omega_1, \omega_2) \in V_{\text{pot}}(\Omega_2)$ follow by a Weyl decomposition type argument, see [15], pp. 228–229. Consider the realization

$$v_2^{\xi,\omega_2}(\omega_1, x) = z_2^\xi(\omega_1, T_2(x)\omega_2).$$

By the definition of $V_{\text{pot}}(\Omega_2)$ there exists a function $q_2^{\xi,\omega_2} \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ such that $v_2^{\xi,\omega_2} = \nabla q_2^{\xi,\omega_2}$ for a.e. $\omega_2 \in \Omega_2$. Let us now define

$$w_{\varepsilon_2}^{\xi,\omega_2} = \langle \xi, x\rangle + \varepsilon_2 q_2^{\xi,\omega_2}\Big(\omega_1, \frac{x}{\varepsilon_2}\Big).$$

By construction

$$\nabla w_{\varepsilon_2}^{\xi,\omega_2} = \xi + \nabla q_2^{\xi,\omega_2}\Big(\omega_1, \frac{x}{\varepsilon_2}\Big).$$

By the Birkhoff ergodic theorem (Theorem 1) and by the properties of $V_{\text{pot}}(\Omega_2)$, keeping in mind that T_2 is ergodic, we have

$$\int_{\Omega_2} z_2^\xi \, \mathrm{d}\mu_2(\omega_2) = 0$$

and hence, as $\varepsilon \to 0$,

$$\nabla w_{\varepsilon_2}^{\xi,\omega_2} \rightharpoonup \xi \quad \text{in } U.$$

Next we consider the realizations

$$a(\omega_1, T_2(x)\omega_2, \xi + z_2^\xi(\omega_1, T_2(x)\omega_2)).$$

By the structure conditions

$$a(\omega_1, \omega_2, \xi + z_2^\xi(\omega_1, \omega_2)) \in L^p_{\mathrm{loc}}(\Omega_1 \times \Omega_2)$$

and an application of the Birkhoff ergodic theorem yields

$$a\Big(\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, \xi + z_2^\xi\Big(\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2\Big)\Big) \rightharpoonup b_1(\omega_1, \xi) \quad \text{in } U',$$

where

$$b_1(\omega_1, \xi) = \int_{\Omega_2} a(\omega_1, \omega_2, \xi + z_2^\xi(\omega_1, \omega_2))\, \mathrm{d}\mu_2(\omega_2).$$

We proceed by solving the ε_1-process. Let us again fix $\xi \in \mathbb{R}^n$. Let $z_1^\xi(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$ be the solution to

$$\langle b_1(\omega_1, t, \xi + z_1^\xi(\omega_1), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$ for a.e. $\omega_1 \in \Omega_1$. The existence and uniqueness of the solution $z_1^\xi(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$ follow by the same argument as for z_2 above. Consider now the realization

$$v_1^{\xi,\omega_1}(x) = z_1^\xi(T_1(x)\omega_1).$$

By the definition of $V_{\mathrm{pot}}(\Omega_1)$ there exists a function $q_1^{\xi,\omega_1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $v_1^{\xi,\omega_1} = \nabla q_1^{\xi,\omega_1}$ for a.e. $\omega_1 \in \Omega_1$. Let us now define

$$w_{\varepsilon_1}^{\xi,\omega_1} = \langle \xi, x\rangle + \varepsilon_1 q_1^{\xi,\omega_1}\Big(\frac{x}{\varepsilon_1}\Big).$$

By construction

$$\nabla w_{\varepsilon_1}^{\xi,\omega_1} = \xi + \nabla q_1^{\xi,\omega_1}\Big(\frac{x}{\varepsilon_1}\Big).$$

By the Birkhoff ergodic theorem and by the properties of $V_{\mathrm{pot}}(\Omega_1)$, keeping in mind that T_1 is ergodic, we obtain

$$\int_{\Omega_1} z_1^\xi \, d\mu_1(\omega_1) = 0$$

and hence, as $\varepsilon \to 0$,

$$\nabla w_{\varepsilon_1}^{\xi,\omega_1} \rightharpoonup \xi \quad \text{in } U.$$

Next we consider the realizations

$$b_1(T_1(x)\omega_1, \xi + z_1^\xi(T_1(x)\omega_1)).$$

By the structure conditions

$$b_1(\omega_1, \xi + z_1^\xi(\omega_1)) \in L^p_{\mathrm{loc}}(\Omega_1).$$

An application of the Birkhoff ergodic theorem yields

$$b_1\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, \xi + z_1^\xi\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1\Big)\Big) \rightharpoonup b(\xi) \quad \text{in } U',$$

where

$$b(\xi) = \int_{\Omega_1} b_1(\omega_1, \xi + z_1^\xi(\omega_1))\, \mathrm{d}\mu_1(\omega_1).$$

Let us now combine the two steps and define the perturbed test function

$$w_\varepsilon^\omega = \langle \xi, x\rangle + \varepsilon_1 q_1^\xi\Big(\frac{x}{\varepsilon_1}\Big) + \varepsilon_2 q_2^{\xi,\omega_1}\Big(\omega_1, \frac{x}{\varepsilon_2}\Big).$$

By construction

$$\nabla w_\varepsilon^\omega = \xi + \nabla q_1^\xi\Big(\frac{x}{\varepsilon_1}\Big) + \nabla q_2^{\xi,\omega_1}\Big(\frac{x}{\varepsilon_2}\Big)$$

and thus by the Birkhoff ergodic theorem

$$\nabla w_\varepsilon^\omega \rightharpoonup \xi \quad \text{in } U.$$

We now apply the multiscale version of the Birkhoff ergodic theorem (Theorem 2) which yields

$$a\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, \nabla w_\varepsilon^\omega\Big) \rightharpoonup b(\xi) \quad \text{in } U'.$$

Let us now show that $\xi^* = b(\nabla u^*)$. The uniqueness of the solution to the homogenized problem (12) will then imply that the whole sequence converges. First we recall that, by the general G-convergence results for monotone operators in [12], $b \in S$. Further, by the monotonicity, we have

$$\int_Q \langle A_\varepsilon^\omega(x, \nabla u_\varepsilon^\omega) - A_\varepsilon^\omega(x, \nabla w_\varepsilon^\omega), \nabla u_\varepsilon^\omega - \nabla w_\varepsilon^\omega\rangle \Phi(x)\, \mathrm{d}x \geqslant 0$$

for every $\Phi \in C_0^\infty(Q)$, $\Phi \geqslant 0$. By using the standard compensated compactness argument, the monotonicity and the multiscale Birkhoff Theorem, we obtain after a limit passage ($\varepsilon \to 0$)

$$\int_Q \langle \xi^* - b(\xi), \nabla u^* - \xi\rangle\, \Phi(x)\, \mathrm{d}x \geqslant 0.$$

This implies that

$$\langle \xi^* - b(\xi), \nabla u^* - \xi\rangle \geqslant 0$$

for a.e. $x \in Q$. Finally, by the continuity and the maximal monotonicity of b, the Minty trick yields

$$\xi^* = b(\nabla u^*)$$

and by the uniqueness of the solution to the homogenized problem the whole sequences converge and we can put $u = u^*$. If we now replace the fixed $f \in V'$ by the sequence $\{f_\varepsilon^\omega\}$ that is compact in V' we can argue exactly as in Theorem 4.1 in [12] and the proof is complete. $\square$

5. Homogenization of the elliptic eigenvalue problem

We define

$$A_\varepsilon^\omega(x,\xi) = \alpha_\varepsilon^\omega(x)|\xi|^{p-2}\xi$$

where

$$\alpha_\varepsilon^\omega(x) = \alpha\Bigl(T_1\Bigl(\frac{x}{\varepsilon_1}\Bigr)\omega_1, T_2\Bigl(\frac{x}{\varepsilon_2}\Bigr)\omega_2\Bigr).$$

Theorem 8. *Consider the sequence of eigenvalue problems*

$$\begin{cases} -\operatorname{div}(A_\varepsilon^\omega(x,\nabla u_\varepsilon^\omega)) = \lambda_\varepsilon^\omega |u_\varepsilon^\omega|^{p-2}u_\varepsilon^\omega & \text{in } Q, \\ u_\varepsilon^\omega \in V. \end{cases} \tag{13}$$

Assume that α is strictly positive and bounded in $L^\infty(Q)$ and that $\mathbf{T}(x) = (T_1(x), T_2(x))$ is ergodic on the product space $\Omega_1 \times \Omega_2$. Also assume that $\alpha_\varepsilon^\omega$ is strictly positive and bounded in $L^\infty(\Omega)$. Then for every $k \in \mathbb{N}$ the eigenpair $\{\lambda_\varepsilon^{\omega,k}, u_\varepsilon^{\omega,k}\} \in \mathbb{R} \times V$ satisfies

$$\lambda_\varepsilon^{\omega,k} \to \lambda^k,$$
$$u_\varepsilon^{\omega,k} \rightharpoonup u^k \quad \text{in } V,$$

where the pair $\{\lambda^k, u^k\} \in \mathbb{R} \times V$ is the kth eigenpair to the homogenized monotone eigenvalue problem

$$\begin{cases} -\operatorname{div}(b(\nabla u)) = \lambda |u|^{p-2}u & \text{in } Q, \\ u \in V. \end{cases} \tag{14}$$

The operator b is given by

$$\begin{aligned} b(\xi) &= \int_{\Omega_1} b_1(\omega_1, \xi + z_1^\xi(\omega_1))\,\mathrm{d}\mu_1(\omega_1) \\ &= \int_{\Omega_1}\int_{\Omega_2} \alpha(\omega_1,\omega_2)|\xi + z_1^\xi + z_2^\xi|^{p-2}(\xi + z_1^\xi + z_2^\xi)\,\mathrm{d}\mu_2(\omega_2)\,\mathrm{d}\mu_1(\omega_1), \end{aligned} \tag{15}$$

where $z_1^\xi(\omega_1) \in V_{\rm pot}(\Omega_1)$ is the solution to the auxiliary ε_1-scale local problem

$$\langle b_1(\omega_1, \xi + z_1^\xi(\omega_1), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{\rm pot}(\Omega_1)$. The operator b_1 is defined as

$$\begin{aligned} b_1(\omega_1, \xi) &= \int_{\Omega_2} a(\omega_1, \omega_2, \xi + z_2^\xi(\omega_1, \omega_2))\,\mathrm{d}\mu_2(\omega_2) \\ &= \int_{\Omega_2} \alpha(\omega_1, \omega_2)|\xi + z_2^\xi|^{p-2}(\xi + z_2^\xi)\,\mathrm{d}\mu_2(\omega_2), \end{aligned}$$

where $z_2^\xi(\omega_1, \omega_2) \in V_{\rm pot}(\Omega_2)$ is the solution to the auxiliary ε_2-scale local problem

$$\langle a(\omega_1, \omega_2, \xi + z_2^\xi(\omega_1, \omega_2), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{\rm pot}(\Omega_2)$ a.e. $\omega_1 \in \Omega_1$.

Proof. By referring to the classical Ljusternik-Schnirelman theory, we define for every $k \in \mathbb{N}$ and $\varepsilon > 0$ the kth eigenvalue $\lambda_\varepsilon^{\omega,k} \in \mathbb{R}$ to (13) via

$$\lambda_\varepsilon^{\omega,k} = \inf_{\gamma(G)\geqslant k} \sup_{g(\varphi)\in G} \Big\{\frac{F_\varepsilon^\omega(\varphi)}{g(\varphi)}\Big\}, \tag{16}$$

where F_ε^ω and g are given by

$$F_\varepsilon^\omega(\varphi) = \int_Q \alpha_\varepsilon^\omega(x)|\nabla\varphi|^p\,\mathrm{d}x$$

and

$$g(\varphi) = \int_Q |\varphi|^p\,\mathrm{d}x.$$

We actually have the following bounds:

$$0 < \lambda_-^k \leqslant \lambda_\varepsilon^{\omega,k} \leqslant \lambda_+^k < \infty, \tag{17}$$

where

$$\lambda_-^k = \inf_{\gamma(G)\geqslant k} \sup_{g(\varphi)\in G} \Big\{\frac{F^-(\varphi)}{g(\varphi)}\Big\}$$

and

$$\lambda_+^k = \inf_{\gamma(G)\geqslant k} \sup_{g(\varphi)\in G} \Big\{\frac{F^+(\varphi)}{g(\varphi)}\Big\},$$

where

$$F^-(\varphi) = \int_Q C_1 |\nabla \varphi|^p \, \mathrm{d}x$$

and

$$F^+(\varphi) = \int_Q C_2 |\nabla \varphi|^p \, \mathrm{d}x$$

for constants $0 < C_1 < C_2 < \infty$. This is a consequence of the assumptions made on $\alpha_\varepsilon^\omega$. From the existence theory for the eigenvalue problem (for every $\varepsilon > 0$ and for every realization) the corresponding sequence of eigenfunctions $\{u_\varepsilon^{\omega,k}\}$ is bounded in V. This implies that (up to a subsequence)

$$u_\varepsilon^{\omega,k} \rightharpoonup u^k \quad \text{in } V.$$

By the assumptions made on $\alpha_\varepsilon^\omega$ it follows that $A_\varepsilon^\omega \in S$. Therefore, by Theorem 7, the sequence $\{A_\varepsilon^\omega\}$ G-converges to the map b defined as above and, taking (17) into account, the oscillating source term is

$$f_\varepsilon^\omega = \lambda_\varepsilon^{\omega,k} |u_\varepsilon^{\omega,k}|^{p-2} u_\varepsilon^{\omega,k} \to \langle f \rangle \quad \text{in } V'.$$

Using the definition of the G-convergence the limit problem corresponding to the kth eigenvalue problem can now be written as

$$\begin{cases} -\operatorname{div}(b(\nabla u^k)) = \langle f \rangle & \text{in } Q, \\ u^k \in V. \end{cases} \tag{18}$$

It remains to show that

$$\langle f \rangle = \lambda^k |u^k|^{p-2} u^k$$

and thus that λ^k is the kth eigenvalue to the homogenized monotone eigenvalue problem (14). We can without loss of generality assume that the sequence $\{u_\varepsilon^{\omega,k}\}$ is normalized and define the kth eigenvalue as

$$\lambda_\varepsilon^{\omega,k} = \inf_{\gamma(G) \geqslant k} \sup_{\varphi \in G} \{F_\varepsilon^\omega(\varphi)\}. \tag{19}$$

Now we apply the comparison Theorem 3.3 by Defranceschi in [7] and conclude by the G-convergence of the p-Laplacian operators $\{A_\varepsilon^\omega\}$ that the sequence $\{\lambda_\varepsilon^{\omega,k}\}$ of eigenvalues defined as in (19) Γ-converges to a limit eigenvalue λ^k given by

$$\lambda^k = \Gamma - \lim_{\varepsilon \to 0} \inf_{\gamma(G) \geqslant k} \sup_{\varphi \in G} \{F_\varepsilon^\omega(\varphi)\}. \tag{20}$$

Again by Theorem 3.3 in [7] the Euler-Lagrange operator corresponding to the limit functional in (20) is precisely the G-limit operator $-\operatorname{div}(b(\cdot))$ in (18). In order to identify the limit eigenvalue problem we observe that since

$$u_\varepsilon^{\omega,k} \rightharpoonup u^k \quad \text{in } V,$$

the compact embedding of V into $L^p(Q)$ implies that

$$u_\varepsilon^{\omega,k} \to u^k \quad \text{in } L^p(Q).$$

Up to a subsequence we then have $u_\varepsilon^{\omega,k} \to u^k$ a.e. in Q. Therefore, by the Egoroff theorem there exists a Lebesgue measurable set Q_μ with $|Q \setminus Q_\mu| < \mu$ such that $u_\varepsilon^{\omega,k} \to u^k$ uniformly in Q_μ. Now let χ_{Q_μ} be the characteristic function of Q_μ. By the uniform convergence of $u_\varepsilon^{\omega,k}$ a limit passage ($\varepsilon \to 0$) in

$$\int_Q \chi_{Q_\mu} \lambda_\varepsilon^{\omega,k} |u_\varepsilon^{\omega,k}|^{p-2} u_\varepsilon^{\omega,k} \psi \, \mathrm{d}x$$

yields

$$\int_Q \chi_{Q_\mu} \lambda^k |u^k|^{p-2} u^k \psi \, \mathrm{d}x.$$

A passage to the limit ($\mu \to 0$), using that $|Q \setminus Q_\mu| \to 0$, yields the limit

$$\int_Q \lambda^k |u^k|^{p-2} u^k \psi \, \mathrm{d}x.$$

As already said, we can without loss of generality assume that the sequence $\{u_\varepsilon^{\omega,k}\}_{\varepsilon>0}$ of eigenfunctions is normalized. By the strong convergence in $L^p(Q)$ and the normalization we conclude that also u^k is normalized. □

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Author's address: *N. Svanstedt*, Department of Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden, e-mail: `nilss@chalmers.se`.