Commentationes Mathematicae Universitatis Carolinae

Arina A. Arkhipova; Jana Stará

A priori estimates for quasilinear parabolic systems with quadratic nonlinearities in the gradient

Commentationes Mathematicae Universitatis Carolinae, Vol. 51 (2010), No. 4, 639--652

Persistent URL: http://dml.cz/dmlcz/140844

A priori estimates for quasilinear parabolic systems with quadratic nonlinearities in the gradient

Arina A. Arkhipova, Jana Stará

Abstract. We derive local a priori estimates of the Hölder norm of solutions to quasilinear elliptic systems with quadratic nonlinearities in the gradient. We assume higher integrability of solutions and smallness of its BMO norm but the Hölder norm is estimated in terms of BMO norm of the solution under consideration, only.

Keywords: quasilinear parabolic systems, quadratic nonlinearities, regularity, Morrey, VMO spaces

Classification: 35J60

1. Introduction

In this paper we study regularity of weak solutions of systems

$$u_t - \operatorname{div}\big(a(z,u)\nabla u\big) + b(z,u,\nabla u) = 0;\ z \in Q, \tag{1}$$

where $Q = \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $T > 0$ is an arbitrary fixed positive number. By u_t we denote the time derivative of a function $u : Q \to \mathbb{R}^N$, $N > 1$, and by $\nabla u = \{u^i_{x_\alpha}\}^{i\leq N}_{\alpha\leq n}$ its gradient with respect to the space variables.

We assume that $a(z,\eta)$ and $b(z,\eta,\xi)$ satisfy the following conditions:

- (H1) There are positive constants λ, Λ such that

$$\big(a(z,\eta)\xi,\xi\big) = a^{\alpha\beta}_{ij}(z,\eta)\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2, \quad \xi \in \mathbb{R}^{nN}, \tag{2}$$

$$|(a(z,\eta)p,q)| \leq \Lambda|p||q|, \quad p,q \in \mathbb{R}^{nN}, \tag{3}$$

  for all $(z,\eta) \in Q \times \mathbb{R}^N$.
- (H2) The coefficients $a^{\alpha\beta}_{ij}(.,u) \in \mathrm{VMO}(Q)$ for every $u \in \mathbb{R}^N$, $i,j \leq N$, $\alpha,\beta \leq n$ and, moreover, denoting

$$q^2(R) = \sup_{z^o\in Q, u\in\mathbb{R}^N, r\in(0,R]} \fint_{Q_r(z^o)} |a(z,u) - (a(z,u))_{r,z^o}|^2\,dz,$$

Both authors were partially supported by Centro Ennio de Giorgi, the first author by RFFI 08-01-00748 and the second author also by GAČR 201/09/0917.

we have

$$q^2(R) \to 0 \quad \text{for} \quad R \to 0. \tag{4}$$

- (H3) For every $z \in Q$ and $u, v \in \mathbb{R}^N$ it holds

$$|a(z,u) - a(z,v)| \le \omega(|u-v|^2), \tag{5}$$

where ω is a nondecreasing, bounded and concave function on $[0,\infty)$ with $\lim_{s\to 0+} \omega(s) = 0$.

- (H4) Functions $b(z,\eta,\xi)$ are measurable in z for $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$, and continuous in (η,ξ) for a.a. $z \in Q$. Moreover

$$|b(z,\eta,\xi)| \le b_o|\xi|^2 + b_1, \ (z,\eta) \in Q \times \mathbb{R}^N, \xi \in \mathbb{R}^{nN}, \tag{6}$$

with some constants $b_o > 0, b_1 \ge 0$.[1]

We consider a weak solution $u \in V = L_2((0,T); W_2^1(\Omega))$ of system (1) which satisfies the identity

$$\int_Q (-u\varphi_t + (a(z,u)\nabla u, \nabla\varphi) + b(z,u,\nabla u)\varphi)\, dz = 0; \quad \forall \varphi \in C_0^\infty(Q). \tag{7}$$

The regularity problem for elliptic and parabolic systems with strongly nonlinear terms in the gradient was studied in [1]–[6], [15], [17], [23], [24]. Partial regularity of *bounded* weak solutions of (1) was proved under the smallness assumption

$$2b_o\|u\|_{\infty,Q} < \lambda. \tag{8}$$

More precisely, Hölder continuity of u in a neighborhood of a point $z^o \in Q$ was proved provided that (8) is valid and

$$\liminf_{R\to 0+} \frac{1}{R^n} \int_{Q_R(z^o)} |\nabla u|^2\, dz < \epsilon_o^2 \tag{9}$$

for an ϵ_o small enough depending on the data only (here and later on $Q_R(z^o)$ is a parabolic cylinder, see the notation at the end of Introduction).

It follows from the above mentioned results that smoothness of *bounded* weak solution u in a neighborhood of z^o holds under the assumption

$$\operatorname{osc}_{Q_R(z^o)} u < \theta \tag{10}$$

for a small enough positive θ and $R = R(z^o) > 0$ (instead of (8), (9)).

Unfortunately, description (10) of regular points does not allow to obtain reasonable estimates of the set Σ of singular points of the considered solution u. As it is known, an appropriate estimate of Σ helps to study the solvability of systems (1) (see [9]).

[1] It is sufficient to assume that $b_1 \in L^q(Q)$ for a $q > \frac{n+2}{2}$.

Condition (10) was relaxed in [3], [4] to the assumption

$$[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} < \theta, \tag{11}$$

where $\theta \ll 1$ and where $[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))}$ denotes the seminorm of u in Campanato space $\mathcal{L}^{2,n+2}(Q_R(z^o))$. (Relaxations of the same type were obtained for corresponding elliptic systems in [2], [5], [6].)

A priori local L_p estimate (for $p > 2$) of the gradient of u was obtained in [3] under the assumption (11). In [4] Hölder continuity of u with respect to the parabolic metric δ and the estimate

$$\|u\|_{C^\alpha(\overline{Q_{\tau R}(z^o)},\delta)} \le C\left(\|u\|_{2,Q_R}, \|\nabla u\|_{2,Q_R}, R^{-1}, \alpha\right) \tag{12}$$

are proved for any $\alpha \in (0,1)$ and some $\tau \in (0,1)$ provided that condition (11) holds with sufficiently small θ depending on the data only.

Note that a sufficient condition that ensures (11) can be expressed by the requirement

$$\|\nabla u\|_{L^{2,n}(Q_R(z^o))} < \theta_1 \tag{13}$$

with a small parameter θ_1 (see Proposition 2.1 in [3]).

As it is known, the condition

$$\Phi(R, z^o) = \frac{1}{R^n} \int_{Q_R(z^o)} |\nabla u|^2 \, dz < \theta \tag{14}$$

with small enough θ describes regular points z^o of u for the simplest systems with $b = 0$ in (1). This condition allows to estimate the Hausdorff measure of a singular set of u. In this sense, condition (14) is an optimal assumption to estimate the Hölder norm of u in the neighborhood of z^o. Let us remark that the *monotonicity* of the function $\Phi(r, \xi^o)$ in the argument r for any $\xi^o \in Q_R(z^o)$ ensures estimate (13) with $\theta_1 = c\theta$ provided that assumption (14) holds. It means that description (13) of regular points of u transforms the problem of the optimal description of regular points to the possibility to state a monotonicity type estimate for the function $\Phi(r, \xi^o)$. We also remark that a priori estimates of C^α norm of u were obtained in [3], [4] under Dirichlet boundary conditions up to the lateral surface of Q.

A priori estimate (12) was obtained in [4] for solutions u of system (1) from the space $W^{1,2}(Q)$. The existence of $u_t \in L_2(Q)$ and strong energy estimate were essentially exploited in the proof of (12) by the so called direct method. Actually, parabolic systems with elliptic operators of variational structure were considered in [4].

In this paper, we derive a local a priori estimate (12) under smallness condition (11) for weak solutions $u \in V = L_2((0,T); W_2^1(\Omega))$ under assumptions (H1)–(H4). We also assume that the L_m-norm of the gradient of u, for some $m > 2$, is finite, but the constant C in (12) does not depend on this norm. For the proof we apply

the "$\mathcal{A}$-caloric approximation" method. This method was introduced for parabolic systems in [14] as a development of the "$\mathcal{A}$-harmonic approximation" method for elliptic systems (see [12], [13], [18]). (See [7], [25], [14] for its applications to parabolic problems.)

Moreover, we relax in this paper the assumptions on the smoothness of the matrix $a(z,v)$, namely we do not require the continuity of $a(z,v)$ in z. Recall that regularity of weak solutions to linear systems with principle matrices in VMO-space was studied by Chiarenza, Frasca, Longo (see [10]) or Huang (see [19]). It follows from [21] that some smoothness of the principal matrix of parabolic systems is necessary even in the linear case. Regularity of minimizers of certain functionals under VMO smoothness of matrices $a(z,v)$ in z was studied in [11], [26]. Thus, it seems reasonable to study regularity of weak solutions to systems satisfying assumption (H4).

We adopt the following notation:

$$\begin{aligned} B_R(x^0) &= \{x \in \mathbb{R}^n : |x - x^0| < R\},\ Q_R(z^0) = B_R(x^0) \times (t^0 - R^2, t^0), \\ (u)_{R,z^0} &= \frac{1}{\omega_n R^{n+2}} \int_{Q_R(z^0)} u(z)\,dz = \fint_{Q_R(z^0)} u(z)\,dz,\ \omega_n = \text{ meas}\, B_1, \\ V_R(z^0) &= L_2\left((t^0 - R^2, t^0); W_2^1(B_R(x^0))\right), z^0 = (x^0, t^0) \in \mathbb{R}^{n+1}. \end{aligned}$$

We will leave out $z^o$and write Q_R, V_R and $(u)_R$ if it does not cause misunderstanding.

2. Auxiliary results

We start with a reformulation of the $\mathcal{A}$-caloric lemma due to Duzaar and Mingione (see [14]). (Such a reformulation was proposed in the elliptic case by M. Giaquinta (see Appendix in [13]).

Definition ($\mathcal{A}$-caloric function)**.** Let $\mathcal{A}$ be an $nN \times nN$ constant positive definite matrix. A function $h \in V$ is $\mathcal{A}$-caloric if for any $\varphi \in C_0^1(Q)$

$$\int_Q \Big(h\varphi_t - (A\nabla h, \nabla\varphi)\Big)dz = 0.$$

Lemma 1 (A-caloric approximation)**.** *Let $0 < \lambda < \Lambda$ and $n, N \in \mathbb{N}$ (with $n, N \geq 2$) be fixed. Then for any $\varepsilon > 0$ there exists a $C(\varepsilon) = C(\varepsilon, n, N, \lambda, \Lambda) > 0$ such that the following holds: for any bilinear form A on $\mathbb{R}^{nN}$ such that*

$$(A\xi, \xi) \geq \lambda|\xi|^2, \tag{15}$$

$$\|A\| \leq \Lambda, \tag{16}$$

and for any $u \in V_R(z^o)$ there exist an A-caloric $h \in V_R(z^o)$ and $\varphi \in C_0^1(Q_R(z^o))$ so that

$$\|\nabla\varphi\|_{L_\infty} \leq 1, \tag{17}$$

$$\fint_{Q_R(z^o)} (|h|^2 + R^2|\nabla h|^2)\,dz \le \fint_{Q_R(z^o)} (|u|^2 + R^2|\nabla u|^2)\,dz, \tag{18}$$

$$\fint_{Q_R(z^o)} |u-h|^2 \le \varepsilon \fint_{Q_R(z^o)} \left(|u|^2 + R^2|\nabla u|^2\right) dz + C(\varepsilon)R^2 \left| \fint_{Q_R(z^o)} \left(u\varphi_t - (A\nabla u, \nabla\varphi)\right) dz \right|^2. \tag{19}$$

Proof: Let us prove Lemma 1 for $x^o = 0$, $t^o = 0$ and $R = 1$. The full assertion then follows by a standard homotopy argument. For $u \neq 0$ set $\tilde{u} = u\left(\fint_{Q_1} |u|^2 + |\nabla u|^2\,dz \right)^{-1/2}$, $Q_1 = Q_1(0)$, and fix an $\varepsilon > 0$. Then $\fint_{Q_1}(|\tilde{u}|^2 + |\nabla\tilde{u}|^2)\,dz \le 1$ and if, moreover,

$$\left| \fint_{Q_1} \left(\tilde{u}\tilde{\psi}_t - (A\nabla\tilde{u}, \nabla\tilde{\psi})\right) dz \right| \le \delta \|\nabla\tilde{\psi}\|_{L_\infty} \tag{20}$$

for all $\tilde{\psi} \in C_0^1(Q_1)$ and some $\delta = \delta(\epsilon)$, then $\tilde{u}$ satisfies all assumptions of Lemma 4.1 in [14]. Hence there is an $\mathcal{A}$-caloric $\tilde{h} \in V_1$ such that

$$\fint_{Q_1} \left(|\tilde{h}|^2 + |\nabla\tilde{h}|^2\right) dz \le 1, \ \fint_{Q_1} |\tilde{u} - \tilde{h}|^2\,dz \le \varepsilon.$$

Setting $h = \tilde{h}\left(\fint_{Q_1} (|u|^2 + |\nabla u|^2)\,dz \right)^{1/2}$ and $\varphi = 0$, we get the inequalities (17)–(19).

If, on the contrary, the inequality (20) is not satisfied, there is a nonzero $\tilde{\psi} \in C_0^1(Q_1)$ such that

$$\left| \fint_{Q_1} \left(\tilde{u}\tilde{\psi}_t - (A\nabla\tilde{u}, \nabla\tilde{\psi})\right) dz \right| > \delta \|\nabla\tilde{\psi}\|_{L_\infty}.$$

After changing (if necessary) the sign of $\tilde{\psi}$, we have

$$\fint_{Q_1(z^o)} \left(\tilde{u}\tilde{\psi}_t - (A\nabla\tilde{u}, \nabla\tilde{\psi})\right) dz > \delta \|\nabla\tilde{\psi}\|_{L_\infty},$$

and for $\psi = \frac{\tilde{\psi}}{\|\nabla\tilde{\psi}\|_{L_\infty}}$ the inequality (17) is satisfied. Thus, also the inequalities

$$\frac{1}{\delta} \fint_{Q_1} (\tilde{u}\psi_t - (A\nabla\tilde{u}, \nabla\psi))\,dz > 1,$$

$$\frac{1}{\delta} \fint_{Q_1} (u\psi_t - (A\nabla u, \nabla\psi))\,dz > \left(\fint_{Q_1} (|u|^2 + |\nabla u|^2)\,dz \right)^{1/2}$$

are valid. For this ψ and $h = 0$

$$\fint_{Q_1} |u-h|^2\,dz = \fint_{Q_1} |u|^2\,dz \le \fint_{Q_1} (|u|^2 + |\nabla u|^2)\,dz < \frac{1}{\delta^2}\Big|\fint_{Q_1} u\psi_t - (A\nabla u, \nabla\psi)\,dz\Big|^2,$$

and Lemma 1 holds with $C_\varepsilon = \frac{1}{\delta^2}$, $R = 1$, $z^0 = 0$. □

Corollary 1. *Let the assumptions of Lemma* 1 *be satisfied and* $(u)_{Q_R(z^o)} = 0$. *Then the assertion of Lemma* 1 *remains true and, moreover, the* $\mathcal{A}$*-caloric function* h *satisfying* (18), (19) *can be chosen so that* $(h)_{Q_R(z^o)} = 0$ *and* (17) *also holds.*

Proof: For given A, u and ε find $\tilde h, \varphi$ according to Lemma 1 and set $h = \tilde h - (\tilde h)_{Q_R(z^o)}$. Then h has zero mean value over $Q_R(z^o)$. Relation (17) is obvious, and (18), (19) follow from the inequalities

$$\fint_{Q_R(z^o)} |h|^2\,dz \le \fint_{Q_R(z^o)} |\tilde h|^2\,dz, \quad \fint_{Q_R(z^o)} |u-h|^2\,dz \le \fint_{Q_R(z^o)} |u-\tilde h|^2\,dz.$$

Here we used the fact that for any function $u \in L_2(Q)$ its integral mean value is the minimizer of $F(c) = \int_Q |u-c|^2\,dz$. □

Corollary 2. *Let the assumptions of Lemma* 1 *be satisfied. Then for any* $\varepsilon > 0$ *there exists a positive constant* $C(\varepsilon) = C(\varepsilon, n, N, \lambda, \Lambda)$ *such that the following holds: for any bilinear form* A *on* $\mathbb{R}^{nN}$ *satisfying* (15), (16) *and for any* $u \in V_R(z^o)$ *there exist an* A*-caloric* $h \in V_R(z^o)$ *and* $\varphi \in C_0^1(Q_R(z^o))$ *satisfying* (17) *so that*

$$\fint_{Q_R(z^o)} \big(|h-(h)_{R,z^o}|^2 + R^2|\nabla h|^2\big)\,dz \le \fint_{Q_R(z^o)} \big(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\big)\,dz, \tag{21}$$

$$\fint_{Q_R(z^o)} |u-h|^2\,dz \le \varepsilon \fint_{Q_R(z^o)} \big(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\big)\,dz + C(\varepsilon)R^2\Big|\fint_{Q_R(z^o)} (u\varphi_t - (A\nabla u, \nabla\varphi))\,dz\Big|^2. \tag{22}$$

Proof: Apply Corollary 1 to the function $\tilde u = u - (u)_{R,z^o}$ and find A-caloric $\tilde h, \varphi$ so that $\tilde h$ has mean value zero and they satisfy (17), (18), (19). Set $h = \tilde h + (u)_{R,z^o}$. Then h is A-caloric. As $(h)_{R,z^o} = (u)_{R,z^o}$, inequality (18) for $\tilde h$ and $\tilde u$ gives

$$\fint_{Q_R(z^o)} \big(|h-(h)_{R,z^o}|^2 + R^2|\nabla h|^2\big)\,dz = \fint_{Q_R(z^o)} \big(|\tilde h|^2 + R^2|\nabla\tilde h|^2\big)\,dz$$

$$\leq \fint_{Q_R(z^o)} \left(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\right) dz.$$

Further, (19) for $\tilde{h}$ and $\tilde{u}$ implies

$$\begin{aligned}\fint_{Q_R(z^o)} |u-h|^2\,dz &= \fint_{Q_R(z^o)} |\tilde{u}-\tilde{h}|^2\,dz\\ &\leq \varepsilon \fint_{Q_R(z^o)} \left(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\right) dz\\ &\quad + C(\varepsilon)\left|R\fint_{Q_R(z^o)} \left(u\varphi_t - (A\nabla u, \nabla\varphi)\right) dz\right|^2,\end{aligned}$$

which accomplishes the proof. □

3. Main theorem

We start with the following remark.

Remark 1. Let $b \equiv 0$ in system (1). It is well known that there exists $p_0 = p_0(\lambda, \Lambda, n) > 2$ such that $\nabla u \in L_p(Q')$, $, Q' \subset\subset Q$, $p \in [2, p_0)$, for any solution $u \in V(Q)$ (see [8], [20]). It appears that an L_p estimate ($p > 2$) for the gradient is also valid for system (1) with $b \neq 0$ provided that some smallness assumption holds.

To describe this assumption we introduce the following definition:

Definition. Let θ and R_o be fixed positive numbers. We say that "u satisfies condition S_{θ,R_o}" in a cylinder $Q_R(z^o) \subset Q$ provided that

- $\nabla u \in L_m(Q_R(z^o))$ with an exponent $m > 2$,
- $[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \leq \theta$ with some $R = R(z^o) \leq R_o$.

Theorem 1. *Let assumptions* (H1)–(H4) *hold and let $u \in V$ satisfy* (7). *Then there exist parameters θ, R_o such that if u fulfills condition S_{θ,R_o} in a cylinder $Q_R(z^o)$ with $R \leq R_o$ then $u \in C^\alpha(Q_{\tau R}(z^o))$ for some $\tau \in (0,1)$, any $\alpha \in (0,1)$ and*

$$\|u\|_{C^\alpha(\overline{Q_{\tau R}(z^o)})} \leq C_1\left(\|u\|_V, \alpha, R^{-1}\right). \tag{23}$$

The constant C_1 does not depend on the norm $\|\nabla u\|_{m,Q_R}$. It neither depends on the exponent m from condition S_{θ,R_0} provided that $m \geq p_0$ (see Remark 1).

Remark 2. Further, we consider u on $Q_R(z^o)$ satisfying S_{θ_o,R_o} with some θ_0 we specify below. Then

$$\int_Q \Big(- u\varphi_t + (a(z,u)\nabla u, \nabla\varphi) + b(z,u,\nabla u)\varphi\Big)\,dz = 0 \tag{24}$$

holds for any test function $\varphi \in W_2^1(Q_R) \cap \mathcal{L}^{2,n+2}(Q_R)$, spt $\varphi \subset Q_R$. Indeed, $\mathcal{L}^{2,n+2}(Q_R) \hookrightarrow L_s(Q_R)$ for any $s < \infty$, and we have

$$\int_{Q_R} |\nabla u|^2 |\varphi|\, dz \le \left(\int_{Q_R} |\nabla u|^m\, dz \right)^{2/m} \cdot \left(\int_{Q_R} |\varphi|^{\frac{m}{m-2}} \right)^{\frac{m-2}{m}} < \infty.$$

Moreover, $u \in V_R(z^o) \cap L_s(Q_R(z^o))$ for any $s < \infty$. Applying the Steklov average approximations for the test functions φ in (24), one can prove that u has finite norm

$$\sup_{(t^o - R^2, t^o)} \|u(\cdot,t)\|_{2,B_R(x^o)} + \|\nabla u\|_{2,Q_R(z^o)},$$

and u is a continuous in t function in $L_2(B_R(x^o))$-norm. (See the details in paragraph 4, Chapter 3, [22].)

Proposition 1. *There exist θ_o, R_o such that if u satisfies condition S_{θ_o,R_o} in $Q_R(z^o)$ and (24) then*

$$\left(\fint_{Q_r} |\nabla u|^p\, dz \right)^{2/p} \le C_2 \fint_{Q_{ar}} (1 + |\nabla u|)^2\, dz, \quad \forall Q_{ar} \subset Q_R(z^o), \tag{25}$$

with any $p \in (2, \min(p_o, m))$, where $p_o = p_o(\lambda, \Lambda, n) > 2$, $a > 2$ is an absolute constant. The constant C_2 depends on $\lambda, \Lambda, n, b_0, b_1$ and θ_o but does not depend on $\|\nabla u\|_{m,Q_R}$.

Proposition 1 is a consequence of Theorem 2.1 in [3]. We should remark that a priori estimate (25) was derived for smooth solutions in [3] but we take into account Remark 2 and assert that the proof of Theorem 2.1 is valid under the condition S_{θ_o,R_o}.

In what follows, θ_o is fixed by Proposition 1, and we assume further that

$$[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \le \theta \tag{26}$$

holds with some $\theta \le \theta_o$ that will be chosen later.

The smallness assumption (26) does not allow to derive Caccioppoli inequality for u, but with the help of (25) we can prove

Proposition 2. *Let u satisfy condition S_{θ,R_o} in $Q_R(z^o)$ and (24). Then the inequality*

$$\begin{aligned} \int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \le \frac{C_3}{r^2} \int_{Q_{a_1 r}(\zeta^o)} |u - (u)_{a_1 r}|^2\, dz \\ + C_4 \theta \int_{Q_{a_1 r}(\zeta^o)} |\nabla u|^2\, dz + c_5 r^{n+4}, \end{aligned} \tag{27}$$

holds for all $Q_{a_1 r}(\zeta^o) \subset Q_{R_1}(z^o)$. Here $R_1 = \frac{R}{\sqrt{n}}$, $a_1 = 2a$ and $a > 2$ is fixed in (25).

PROOF: We put $\varphi = (u-(u)_{2r,\zeta^o})h^2$ in (24), where h is a smooth cut-off function supported in $Q_{2r}(\zeta^o)$, $h=1$ in $Q_r(\zeta^o)$, and derive for u the inequality

$$\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \le \frac{c}{r^2}\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r}|^2\,dz + c\left(\int_{Q_{2r}(\zeta^o)} b_0|\nabla u|^2|u-(u)_{2r,\zeta^o}|\,dz + b_1\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|\,dz\right).$$

Using Hölder and Young inequality we estimate the last term on the right hand side:

$$\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|\,dz \le c\left(\frac{1}{r^2}\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|^2\,dz + r^{n+4}\right). \tag{28}$$

We explain now how to estimate the strongly nonlinear term I

$$I = \int_{Q_{2r}(\zeta^o)} |\nabla u|^2|u-(u)_{2r,\zeta^o}|\xi^2.$$

Estimates (25), (26) imply that the following chain of inequalities is valid:

$$\begin{aligned} I &\le \left(⨍_{Q_{2r}(\zeta^o)} |\nabla u|^p\,dz\right)^{2/p}\left(⨍_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}|Q_{2r}| \\ &\le C\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{p',n+2}(D_{R/\sqrt{n}}(z^o))} \\ &\le C(p)\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{2,n+2}(D_{R/\sqrt{n}}(z^o))} \\ &\le C(p)\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \\ &\le C(p)\theta\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right). \end{aligned} \tag{29}$$

Here $p' = \frac{p}{p-2}$ and the parabolic cube $D_{R_1}(z^0) \subset Q_R(z^o)$. The inequality is valid due to isomorphism of $\mathcal{L}^{s,n+2}(D_{R_1})$ and $\mathcal{L}^{2,n+2}(D_{R_1})$ for different $s>1$. Due to (28), (29) we arrive at (27). □

Now we can exclude in (27) the term with θ.

Proposition 3. *There exists $\theta_1 \le \theta_o$ such that the assumptions of Proposition 2 with $\theta \le \theta_1$ ensure that the inequality*

$$\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \le \frac{C_6}{r^2}\int_{Q_{lr}(\zeta^o)} |u-(u)_{lr,\zeta^o}|^2\,dz + c_7 r^{n+4} \tag{30}$$

holds with an absolute constant $l > 2$.

Proof: We rewrite (27) changing cylinders Q_r by parabolic cubes D_r and apply parabolic version of Lemma 0.5 of [16]. □

Remark 3. From (30) and (26) it follows that

$$\frac{1}{r^n}\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \le C_8 \fint_{Q_{lr}(\zeta^o)} |u-(u)_{lr,\zeta^o}|^2\,dz + C_8 r^4, \tag{31}$$

and thus also

$$\|\nabla u\|^2_{L^{2,n}(Q_{R_2}(z^0))} \le C_8(\theta^2 + R^4),\ R_2 = \tau R, \tag{32}$$

with a $\tau \in (0,1)$. We put

$$C_8(\theta^2 + R_o^4) = \kappa^2(\theta, R_o) \to 0 \quad \text{for} \quad \theta, R_o \to 0. \tag{33}$$

Hence

$$\|\nabla u\|_{L^{2,n}(Q_{R_2}(z^0))} \le \kappa(\theta, R_o),\ R_2 = \tau R < R_o. \tag{34}$$

Proof of Theorem 1: Let $R_3 = \frac{R_2}{2}$, $\zeta^o \in Q_{R_3}(z^o)$, $r \le \frac{R_3}{l}$, where $l > 2$ is fixed in (30). Then $Q_r(\zeta^o) \subset Q_{R_3}(z^o)$. Put $A = (a(z,u^o))_{r,\zeta^o}$, $u^o = (u)_{r,\zeta^o}$ and estimate

$$L_r(\zeta^o) = \left| \int_{Q_r(\zeta^o)} \big(u\varphi_t - (A\nabla u, \nabla\varphi)\big)\,dz \right|$$

for all $\varphi \in C_0^1(Q_r(\zeta^o))$ with $C_\varphi = \|\nabla\varphi\|_{\infty, Q_r(\zeta^o)} \le 1$. (Note that it implies that $\max_{Q_r(\zeta^o)} |\varphi| \le C_\varphi r$.) Hence, (6) and (34) supply the inequalities

$$\begin{aligned} L_r(\zeta^o) &\le C_\varphi \bigg\{ \int_{Q_r(\zeta^o)} |a(z,u) - a(z,u^o)||\nabla u|\,dz \\ &\quad + \int_{Q_r(\zeta^o)} |a(z,u^o) - A||\nabla u|\,dz \\ &\quad + b_o r^{1+n/2} \bigg(\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \bigg)^{1/2} \bigg(\frac{1}{r^n}\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \bigg)^{1/2} + b_1 r^{n+3} \bigg\} \\ &\le C_\varphi \bigg\{ \Big[\omega^{1/2}(\theta) + q(R_o) + b_o\kappa(\theta, R_o)\Big] r^{1+n/2} \bigg(\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \bigg)^{1/2} + C r^{n+3} \bigg\}. \end{aligned}$$

We put

$$T(\theta, R_o) = cC_\varphi\big[\omega^{1/2}(\theta) + q(R_o) + b_o\kappa(\theta, R_o)\big], \tag{35}$$

and we get

$$L_r(\zeta^o) \le T(\theta, R_o) r^{1+n/2}\bigg(\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz\bigg)^{1/2} + Cr^{n+3}. \tag{36}$$

From Corollary 2 and (36) it follows that for u there is an $\mathcal{A}$-caloric h in $V_r(\zeta^o)$ such that

$$\begin{aligned} \fint_{Q_r(\zeta^o)} |u-h|^2\,dz &\le \varepsilon \fint_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\,dz \\ &\quad + C(\varepsilon)\bigg(T^2(\theta, R_o) r^2 \fint_{Q_r(\zeta^o)} |\nabla u|^2\,dz + r^4\bigg). \end{aligned} \tag{37}$$

Now we use Campanato's inequality for h and $\rho \le r < 1$, (37), (21) and Proposition 3 to get

$$\begin{aligned} &\Phi(\rho, \zeta^o) = \Phi(\rho) = \int_{Q_\rho(\zeta^o)} |u-(u)_\rho|^2\,dz \\ &\le 2\int_{Q_\rho(\zeta^o)} |h-(h)_\rho|^2\,dz + 2\int_{Q_\rho(\zeta^o)} |u-h-((u)_\rho-(h)_\rho)\,|^2\,dz \\ &\le 2\int_{Q_\rho(\zeta^o)} |h-(h)_\rho|^2\,dz + 2\int_{Q_\rho(\zeta^o)} |u-h|^2\,dz \\ &\le C\left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r(\zeta^o)} |h-(h)_r|^2\,dz + 2\int_{Q_r(\zeta^o)} |u-h|^2\,dz \\ &\le C\left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r(\zeta^o)} |h-(h)_r|^2\,dz + 2\varepsilon \int_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\,dz \\ &\quad + 2C(\varepsilon) T^2(\theta, R_o) r^2 \int_{Q_r(\zeta^o)} |\nabla u|^2\,dz + Cr^{n+4}. \end{aligned}$$

After rearranging the formula we get

$$\begin{aligned} \Phi(\rho) &\le C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right) \int_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\,dz \\ &\quad + 2C(\varepsilon)\left[T^2(\theta, R_o) r^2 \int_{Q_r(\zeta^o)} |\nabla u|^2\,dz + Cr^{n+4}\right] \\ &\le C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right) \Phi(r) \\ &\quad + \left[C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right) + 2C(\varepsilon) T^2(\theta, R_o)\right] r^2 \int_{Q_r(\zeta^o)} |\nabla u|^2\,dz + Cr^{n+4} \end{aligned} \tag{38}$$

$$\le C\left(\left(\frac{\rho}{r}\right)^{n+4}+2\varepsilon\right)\Phi(r)$$
$$+\left[C\left(\left(\frac{\rho}{r}\right)^{n+4}+2\varepsilon\right)+2C(\varepsilon)T^2(\theta,R_o)\right]\left(\Phi(lr)+r^{n+4}\right)+Cr^{n+4}.$$

As (38) is evidently true for $\rho\in(r,lr]$ and

$$\Phi(r)=\int_{Q_r(\zeta^o)}|u-(u)_r|^2\,dz\le\int_{Q_r(\zeta^o)}|u-(u)_{lr}|^2\,dz$$
$$\le\int_{Q_{lr}(\zeta^o)}|u-(u)_{lr}|^2\,dz=\Phi(lr),$$

we "rename" lr by r and simplify the constants to obtain

$$\Phi(\rho)\le C_9\left[\left(\frac{\rho}{r}\right)^{n+4}+\left\{\varepsilon+C(\varepsilon)T^2(\theta,R_o)\right\}\right]\Phi(r)+C(\varepsilon)r^{n+4},\tag{39}$$

for all $\zeta^o\in Q_{R_3}(z^o)$ and $0<\rho\le r\le R_3$. By Campanato's algebraic lemma, for any $\alpha\in(0,1)$ there exists an $\epsilon_0=\epsilon_0(C_9,n,\alpha)>0$ so that

$$\Phi(\rho,\zeta^o)\le C_{10}\rho^{n+2+2\alpha}\left[\frac{\Phi(r,\zeta^o)}{r^{n+2+2\alpha}}+1\right]\tag{40}$$

provided that

$$C_9\{\varepsilon+C(\varepsilon)T^2(\theta,R_o)\}<\epsilon_0.\tag{41}$$

Now, we can fix $\varepsilon\le\frac{\epsilon_0}{2C_9}$ and then fix $\theta\le\theta_1$ and R_o (we reduce R_o if needed) to supply the inequality

$$C_9C(\varepsilon)T^2(\theta,R_o)<\frac{\epsilon_0}{2}\,.\tag{42}$$

From (40) it follows that

$$[u]^2_{\mathcal{L}^{2,n+2+2\alpha}(Q_{R_3}(z^o))}\le C_{11}(\|\nabla u\|_{2,Q_R(z^o)},R^{-1},\alpha),$$
$$\langle u\rangle_{C^\alpha(Q_{R_3}(z^o))}\le C_{12}(\|\nabla u\|_{2,Q_R(z^o)},R^{-1},\alpha),\tag{43}$$

provided that u satisfies condition S_{θ,R_o} in $Q_R(z^o)$, $R\le R_o$ for the chosen θ and R_o. Note that the constants C_{11},C_{12} depend on α and do not depend on $\|\nabla u\|_{m,Q_R}$. □

Acknowledgments. This paper was partially prepared while the authors visited the Centro di Ricerca Matematica Ennio De Giorgi in Pisa. We wish to thank the administration of the center for the hospitality and especially we are grateful to Mariano Giaquinta who invited us and stimulated the work.

References

[1] Arkhipova A., *On the partial regularity up to the boundary of weak solutions to quasilinear parabolic systems with quadratic growth*, Zap. Nauchn. Sem. POMI, St-Petersburg **249** (1997), 3–23.

[2] Arkhipova A., *New a priori estimates for q-nonlinear elliptic systems with strong nonlinearities in the gradient,* $1 < q < 2$, Zap. Nauchn. Sem. POMI, St-Petersburg **310** (2003), 19–48.

[3] Arkhipova A., *Quasireverse Hölder inequalities in parabolic metric and their applications*, Amer. Math. Transl. (2) **220** (2007), 1–25.

[4] Arkhipova A., *New a priori estimates for nondiagonal strongly nonlinear parabolic systems*, Banach Center Publ. **81** (2008), 13–30.

[5] Arkhipova A., *Quasireverse Hölder inequalities and a priori estimates for quasilinear elliptic systems*, Rend. Mat. Accad. Lincei **14** (2003), 91–108.

[6] Arkhipova A., *Boundary a priori estimates for solutions of nondiagonal elliptic systems with strong nonlinearities*, Russian Acad. Izvestia Math. **68** (2004), 243–258.

[7] Bögelein V., *Partial regularity and singular sets of solutions of higher order parabolic systems*, Ann. Mat. Pura Appl. **188** (2009), no. 1, 61–122.

[8] Campanato S., *L^p-regularity for weak solutions of parabolic systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 65–85.

[9] Chen Y., Struwe M., *Existence and partial regularity results for the heat flow of harmonic maps*, Math. Z. **201** (1989), 83–103.

[10] Chiarenza P., Frasca M., Longo P., *$W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149–168.

[11] Daněček J., Viszus E., *$L^{2,\lambda}$-regularity for minima of variational integrals*, Boll. Unione Mat. Ital. **B 8** (2003), 39–48.

[12] De Giorgi E., *Frontiere orientate di misura minima*, Seminario di Mat. della Scuola Normale Superiore, Pisa, 1960–1961.

[13] Duzaar F., Grotowski J.F., *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximations*, Manuscripta Math. **103** (2000), 267–298.

[14] Duzaar F., Mingione G.: *Second order parabolic systems, optimal regularity and singular sets of solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 705–751.

[15] Giaquinta M., Giusti E., *Nonlinear elliptic systems with quadratic growth*, Manuscripta Math. **24** (1978), 323–349.

[16] Giaquinta M., Modica G., *Nonlinear systems of the type of the stationary Navier-Stokes system*, J. Reine Angew. Math. **330** (1982), 173–214.

[17] Giaquinta M., Struwe M., *On the partial regularity of weak solutions of nonlinear parabolic systems*, Math. Z. **179** (1982).

[18] Grotowski G.F., *Boundary regularity results for nonlinear elliptic systems in divergence form*, Habilitationsschrift, Math. Institut der Friedrich-Alexander-Univ., Erlangen-Nürenberg, 2000.

[19] Huang Q., *Estimates on the generalized Morrey spaces $L^{p,\lambda}_{\phi}$ and BMO for elliptic systems*, Indiana Univ. Math. J. **45** (1996), 397–439.

[20] Ivert P.-A., Naumann J., *Higher integrability by reverse Hölder inequality* (*the parabolic case*), Preprint, Dip. Matem., Univ. Catania, 1988.

[21] John O., Stará J., *On some regularity and nonregularity results for solutions to parabolic systems*, Matematiche (Catania) **55** (2000), suppl. 2, 145–163.

[22] Ladyzhenskaya O.A., Solonnikov V.A., Uraltseva N.N., *Linear and Quasilinear Equations of Parabolic Type*, Translations of Math. Monographs, 23, American Mathematical Society, Providence, RI, 1968.

[23] Marino M., Maugeri A., *Partial Hölder continuity of solutions of nonlinear parabolic systems of second order with quadratic growth*, Boll. Un. Mat. Ital. B (7) **3** (1989), 397–435, 437–451.

[24] Marino M., Maugeri A., *A remark on the note: Partial Hölder continuity of the spatial derivatives of the solutions to nonlinear parabolic systems with quadratic growth*, Rend. Sem. Mat. Univ. Padova **95** (1996), 23–28.

[25] Pošta M., *On partial regularity of solutions to the quasilinear parabolic system up to the boundary*, preprint no. MATH-KMA-2007/231, `http://www.karlin.mff.cuni.cz/kma-preprints/`

[26] Ragusa M.A., Tachikawa A., *Partial regularity of the minimizers of quadratic functionals with VMO coefficients*, J. London Math. Soc. **72** (2005), 609–620.

St. Petersburg State University, Department of Mathematics and Mechanics, Bibliotechnaya Pl. 2, 198904 St. Petersburg, Petrodvorets, Russia

Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

(*Received* March 10, 2010, *revised* May 20, 2010)