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ON  $a$ -KASCH SPACES

ALI AKBAR ESTAJI AND MELVIN HENRIKSEN

ABSTRACT. If  $X$  is a Tychonoff space,  $C(X)$  its ring of real-valued continuous functions. In this paper, we study non-essential ideals in  $C(X)$ . Let  $a$  be an infinite cardinal, then  $X$  is called  $a$ -Kasch (resp.  $\bar{a}$ -Kasch) space if given any ideal (resp.  $z$ -ideal)  $I$  with  $\text{gen}(I) < a$  then  $I$  is a non-essential ideal. We show that  $X$  is an  $\aleph_0$ -Kasch space if and only if  $X$  is an almost  $P$ -space and  $X$  is an  $\aleph_1$ -Kasch space if and only if  $X$  is a pseudocompact and almost  $P$ -space. Let  $C_F(X)$  denote the socle of  $C(X)$ . For a topological space  $X$  with only a finite number of isolated points, we show that  $X$  is an  $a$ -Kasch space if and only if  $\frac{C(X)}{C_F(X)}$  is an  $a$ -Kasch ring.

## 1. INTRODUCTION

Throughout this paper,  $C(X)$  will denote the ring of real valued continuous functions defined on a completely regular space  $X$  and  $C^*(X)$  will be the subring of bounded functions. Let us first recall some general notation from [8]. For  $f \in C(X)$ , let  $Z(f) = \{x \in X : f(x) = 0\}$  be the **zero set** of  $f$  and  $\text{Coz}(f) = X \setminus Z(f)$  be its **cozero set**. Whenever  $S$  is a subset in  $C(X)$ , then  $Z[S] = \{Z(f) : f \in S\}$  and  $\bigcap Z[S] = \bigcap_{f \in S} Z(f)$ . An ideal  $I$  of  $C(X)$  is called **fixed ideal** if  $\bigcap Z[I]$  is nonempty, otherwise  $I$  is called a **free ideal**. A subset  $Y$  of  $X$  is said to be  **$C$ -embedded** (resp.  **$C^*$ -embedded**) in  $X$  if the map that sends each element of  $C(X)$  (resp.  $C^*(X)$ ) to its restriction to  $Y$  is onto  $C(Y)$  ( $C^*(Y)$ ). An ideal  $I$  of  $C(X)$  is called a  **$z$ -ideal** (resp.  **$z^\circ$ -ideal**) if  $Z(f) = Z(g)$  (resp.  $\text{int } Z(f) = \text{int } Z(g)$ ) and  $g \in I$  imply that  $f \in I$ .  $X$  is called **extremally** (resp. **basically**) **disconnected** if each open (resp. cozero) set has an open closure and it is called  **$P$ -space** if each finitely generated ideal of  $C(X)$  is a direct summand, and so it is called an **almost  $P$ -space** if every non-empty  $G_\delta$  has a non-empty interior, see [12]. For undefined terms and notations, see [8], and so [7].

Throughout this paper,  $R$  is a commutative ring with unit. For each subset  $S$  of  $R$ , the annihilator of  $S$  denoted by  $\text{Ann}(S)$  is defined by  $\text{Ann}(S) = \{r \in R : rs = 0 \text{ for all } s \in S\}$ .

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An ideal  $I$  of  $R$  is said to be  **$a$ -generated**, where  $a$  is a cardinal number, if it admit a generating set  $A$  such that  $|A| \leq a$ . The least element in the set of cardinal numbers of all generating set of  $I$  is denoted by  $\text{gen}(I)$ .

A ring  $R$  is **Kasch** if every maximal ideal of  $R$  has a non zero annihilator; equivalently, every simple  $R$ -module embeds in  $R$ . In order to get more information, see [11].

An ideal  $I$  of  $R$  is called **essential** in  $R$  if  $I \cap J \neq (0)$  holds for every non-zero ideal  $J$  of  $R$ . Hence every reduce commutative ring is Kasch ring provided that every non-zero ideal is a non-essential ideal.

In [1], F. Azarpanah study essential ideal in  $C(X)$ , he proved that for completely regular space  $X$ , the following statements are equivalent:

- (1)  $X$  is finite.
- (2) Every ideal in  $C(X)$  is not essential ideal.

It is clear that  $C(X)$  is Kasch ring if and only if every ideal in  $C(X)$  is not essential ideal.

Now it is natural to ask, under which condition on  $X$ , every ideal  $I$  in  $C(X)$  with  $\text{gen}(I) < a$  is not essential ideal. By this question we introduce  $a$ -Kasch rings and study their properties.

## 2. PRELIMINARIES

In this section we review some propositions which will be used in the later sections. Also we prove that If every fixed ideal is a countable generated ideal then  $X$  is a countable discrete space.

It is not hard to prove that, an ideal  $B$  in a reduce commutative ring is an essential ideal if and only if  $\text{Ann}(B) = 0$ . We cite the following results which will be frequently referred to in the sequel. For the proof of the following results see [1] and [12].

**Proposition 2.1.** *If  $E$  is a nonzero ideal in  $C(X)$ , then the following statements are equivalent:*

- (1)  $E$  is an essential ideal in  $C(X)$ .
- (2)  $\text{Ann}(E) = 0$ .
- (3)  $\text{Int} \cap Z[E] = \emptyset$ .

**Proposition 2.2.**  *$X$  is an almost  $P$ -space if and only if every non-empty zero set has non-empty interior.*

Also, for the proof of the following results see [8].

**Proposition 2.3.** *For every  $f, g \in C(X)$ ,*

- (1) *If  $Z(f)$  is a neighborhood of  $Z(g)$ , then  $f$  is a multiple of  $g$ , that is,  $f = hg$  for some  $h \in C(X)$ .*
- (2) *If  $|f| \leq |g|^r$  for some real  $r > 1$ , then  $f$  is a multiple of  $g$ .*

**Proposition 2.4.** *Let  $X$  be a finite space.*

- (1)  $f$  is a multiple of  $g$  if and only if  $Z(g) \subseteq Z(f)$
- (2) Every ideal is a  $z$ -ideal.
- (3) Every ideal is principal, and in fact, is generated by an idempotent.

**Proposition 2.5.** *A point  $p$  of  $X$  is isolated if and only if the ideal  $M_p$  (resp.  $M_p^*$ ) is principal.*

In the theory of rings, many structure results were obtained with the help of minimal ideals, and the socle of a ring seems to be most efficient. The intersection of all essential ideals in any commutative ring  $R$ , or the sum of all minimal ideals of  $R$  is the **socle** of  $R$ , see [9]. Let  $C_F(X)$  denote the socle of  $C(X)$ . The socle  $C_F(X)$  of  $C(X)$  was first characterized via the following proposition in [10].

**Proposition 2.6.** *The socle  $C_F(X)$  of  $C(X)$  is a  $z$ -ideal, consisting of all functions that vanish everywhere except on a finite number of points of  $X$ .*

**Theorem 2.7.** *The following statements are equivalent:*

- (1)  $C(X)$  is a Kasch ring.
- (2) Every fixed ideal in  $C(X)$  is finitely generated ideal.
- (3)  $X$  is finite discrete space.
- (4) Every  $z$ -ideal in  $C(X)$  is non-essential ideal.

**Proof.** (1)  $\Rightarrow$  (2) By the hypothesis every maximal ideal of  $M$  in  $C(X)$  is non-essential ideal and since  $|\bigcap Z[M]| \leq 1$ , we can then conclude from of Proposition 2.1 that  $|\text{int} \bigcap Z[M]| = 1$ . It follows that there exists an isolated point  $p \in X$  such that  $M = M_p$ . So that  $X$  is a pseudocompact discrete space, see [8] Theorem 5.8(b), we infer that  $X$  is a finite discrete space, and by Proposition 2.4, this completes the proof.

(2)  $\Rightarrow$  (3) For every  $p \in X$ ,  $M_p$  is a finitely generated ideal and since  $M_p$  is a  $z$ -ideal, we conclude that  $M_p$  is principal ideal, see [13]. By Proposition 2.5,  $p$  is an isolated point, i.e.,  $X$  is a discrete space. Let  $x_0 \in X$  and for every  $x_0 \neq p \in X$  define  $f_p \in C(X)$  by  $f_p(p) = 1$  and  $f_p(X \setminus \{p\}) = 0$ . Now put  $I = \sum_{x_0 \neq p \in X} f_p C(X)$  then clearly  $I$  is a fixed ideal and in view of our hypothesis it is finitely generated ideal, and since  $X$  is discrete, we conclude that there exists idempotent  $e \in C(X)$  such that  $I = eC(X)$ , i.e.,  $Z(e) = \bigcap Z[I] = \{x_0\}$ . We also note that  $e \in I$ , so that there exists  $g_1, \dots, g_n \in C(X)$  and  $f_{p_1}, \dots, f_{p_n} \in I$  such that  $e = g_1 f_{p_1} + \dots + g_n f_{p_n}$ . This implies that  $X \setminus \{p_1, \dots, p_n\} = \bigcap_{i=1}^n Z(f_{p_i}) \subset Z(e) = \{x_0\}$ . This means that  $X = \{p_1, \dots, p_n, x_0\}$  is a finite discrete space.

(3)  $\Rightarrow$  (4) It is clear.

(4)  $\Rightarrow$  (1) If  $I$  is an ideal of  $C(X)$ , then there exists a maximal ideal of  $M$  in  $C(X)$  such that  $I \subseteq M$ , and in view of our hypothesis  $M$  is a non-essential ideal, therefore  $\emptyset \neq \text{int} \bigcap Z[M] \subseteq \text{int} \bigcap Z[I]$ , i.e.,  $I$  is a non-essential ideal of  $C(X)$  (by Proposition 2.1).  $\square$

In view of statement (2) of Theorem 2.7, it is natural to ask that what is the space  $X$  if every fixed ideal in  $C(X)$  is a countable generated ideal. We prove that  $X$  is a countable discrete space in this case.

**Lemma 2.8.** *For every cardinal number  $a \geq \aleph_0$ , if  $X$  is a discrete space and every fixed ideal in  $C(X)$  is  $\text{gen}(I) \leq a$ , then  $|X| \leq a$ .*

**Proof.** Let us, in contrary, assume that  $|X| > a$ . Let  $x_0 \in X$ . For every  $x_0 \neq p \in X$  define  $f_p \in C(X)$  by  $f_p(p) = 1$  and  $f_p(X \setminus \{p\}) = 0$ . Now we put  $I = \sum_{x_0 \neq p \in X} f_p C(X)$  then clearly  $I$  is a fixed ideal and in view of our hypothesis there exists generating set of  $S$  for  $I$  such that  $|S| \leq a$ . We also note that  $\bigcup_{f \in S} \text{Coz}(f) = X \setminus \{x_0\}$  and since  $|X \setminus \{x_0\}| = |X| > a \geq \aleph_0$ , we conclude that there exist  $f \in S$  such that  $|\text{Coz}(f)| = |X|$ , so that there exist  $g_1, \dots, g_n \in C(X)$  and  $f_{p_1}, \dots, f_{p_n} \in I$  such that  $f = g_1 f_{p_1} + \dots + g_n f_{p_n}$ . This implies that  $X \setminus \{p_1, \dots, p_n\} = \bigcap_{i=1}^n Z(f_{p_i}) \subseteq Z(f)$ . This means that  $\text{Coz}(f) \subseteq \{p_1, \dots, p_n\}$  is a finite set, which is a contradiction.  $\square$

**Proposition 2.9.** *If every fixed ideal is a countable generated ideal then  $X$  is a countable discrete space.*

**Proof.** By Lemma 2.8, it suffices to show that  $X$  is a discrete space. For every  $p \in X$ ,  $M_p$  is a fixed maximal ideal. By our hypothesis there exist  $f_1, f_2, \dots \in M$  such that  $M_p = (f_1, f_2, \dots)$ . Define

$$g = \sum_{i=1}^{\infty} \frac{f_i^{2/3}}{2^i(1 + f_i^{2/3})}.$$

Because of uniform convergence,  $g \in C(X)$ . Clearly  $g \in M_p$  and since  $|f_i| \leq (2^i(1 + f_i^{2/3})g)^{2/3}$ , we can then conclude from of Proposition 2.3 that  $f_i$  is a multiple  $g$ , for all  $i$ . So that  $M_p = gC(X)$  and in view of Proposition 2.5,  $p$  is an isolated point, i.e.,  $X$  is a discrete space.  $\square$

### 3. WHEN IS $X$ AN $a$ -KASCH SPACE

In this section the concept of an  $a$ -Kasch space is introduced and some fundamental properties are considered.

**Definition 3.1.** Let  $a$  be a infinite cardinal, then a ring  $R$  is said to be an  **$a$ -Kasch ring** if given any proper ideal  $I$  with  $\text{gen}(I) < a$ , then  $I$  is a non-essential ideal.  $X$  is called  **$a$ -Kasch space** if  $C(X)$  is  $a$ -Kasch ring and it is called  **$\bar{a}$ -Kasch space** if given any proper  $z$ -ideal  $I$  with  $\text{gen}(I) < a$ , then  $I$  is a non-essential ideal.

**Theorem 3.2.**  *$X$  is an almost  $P$ -space if and only if  $X$  is an  $\aleph_0$ -Kasch space.*

**Proof. Necessity.** Let  $I = (f_1, \dots, f_n)$  be a proper ideal of  $C(X)$ . Hence  $\text{int} \cap Z[I] = \text{int} Z(f_1^2 + \dots + f_n^2) \neq \emptyset$  and now by Proposition 2.1, we infer that  $I$  is a non-essential ideal.

**Sufficiently.** In view of Proposition 2.2, it suffices to show that for every  $f \in C(X)$ , if  $Z(f) \neq \emptyset$  then  $\text{int} Z(f) \neq \emptyset$ , and since  $I = fC(X)$  is non-essential ideal, by Proposition 2.1, we infer that  $\text{int} Z(f) \neq \emptyset$ .  $\square$

**Theorem 3.3.**  *$X$  is an  $\aleph_1$ -Kasch space if and only if  $X$  is a pseudocompact and almost  $P$ -space.*

**Proof. Necessity.** By Theorem 3.2,  $X$  is an almost  $P$ -space, then by Theorem 5.8(b) and 5.14 in [8], it suffices to show that if  $M$  is maximal ideal of  $C(X)$ , then  $Z[M]$  has the countable intersection property. Let  $f_1, f_2, \dots$  are belong to  $M$  and put  $I = (f_1, f_2, \dots)$  then  $\text{gen}(I) < \aleph_1$  and  $I$  is a proper ideal of  $C(X)$ . Now by our hypothesis we infer that  $\emptyset \neq \text{int} \bigcap_{i=1}^{\infty} Z(f_i) \subseteq \bigcap_{i=1}^{\infty} Z(f_i)$ .

**Sufficiently.** By our hypothesis and 6.6(b) in [8],  $C(X) = C^*(X) \cong C(\beta X)$  and  $\beta X$  is an almost  $P$ -space. So if  $I = (f_1, f_2, \dots)$  is an ideal of  $C(\beta X)$ , then there exists  $g \in C(\beta X)$  such that  $\bigcap Z[I] = \bigcap_{i=1}^{\infty} Z(f_i) = Z(g) \supseteq \text{int} Z(g) \neq \emptyset$ , see 1.14(a) and Theorem 4.11 in [8], i.e.,  $I$  is a non-essential ideal of  $C(\beta X)$ . Thus  $\beta X$  is an  $\aleph_1$ -Kasch space and since  $C(X) \cong C(\beta X)$ , we conclude that  $X$  is an  $\aleph_1$ -Kasch space.  $\square$

**Corollary 3.4.**  $\beta X$  is an almost  $P$ -space if and only if  $X$  is a pseudocompact almost  $P$ -space.

The fact that  $C(X) \cong C(vX)$  (see 8.8(a) in [8]) implies that  $X$  is  $a$ -Kasch space if and only if  $vX$  is  $a$ -Kasch space. Clearly  $\mathbb{N}$  is  $\aleph_0$ -Kasch, but  $\beta\mathbb{N}$  is not  $\aleph_0$ -Kasch, see [12]. However, we have the following.

**Corollary 3.5.** For cardinal number  $a \geq \aleph_1$ ,  $X$  is an  $a$ -Kasch space if and only if  $\beta X$  is an  $a$ -Kasch space.

**Proposition 3.6.** The following statements are equivalent:

- (1)  $X$  is  $a$ -Kasch (resp.  $\bar{a}$ -Kasch) space.
- (2) Every ideal (resp.  $z$ -ideal) of  $I$  in  $C(X)$  with  $\text{gen}(I) < a$  is in a non-essential principal ideal.

**Proof.** (1)  $\Rightarrow$  (2) Let  $I$  be an ideal (a  $z$ -ideal) of  $C(X)$  with  $\text{gen}(I) < a$ . Hence  $I$  is a non-essential ideal of  $C(X)$  and now by Proposition 2.1,  $\text{int} \bigcap Z[I] \neq \emptyset$ . Then by 3.2(b) in [8], there exists  $g \in C(X)$  such that  $\emptyset \neq \text{int} Z(g) \subseteq Z(g) \subseteq \text{int} \bigcap Z[I] \subseteq Z(f)$ , for all  $f \in I$ . Hence by Propositions 2.1 and 2.3,  $gC(X)$  is a non-essential ideal which containing of  $I$ .

(2)  $\Rightarrow$  (1) Let  $I$  be an ideal (resp. a  $z$ -ideal) of  $C(X)$  with  $\text{gen}(I) < a$ . Thus there exists  $g \in C(X)$  such that  $I \subseteq gC(X)$  and  $gC(X)$  is a non-essential ideal, i.e.,  $\text{int} Z(g) \neq \emptyset$ , and since  $\text{int} Z(g) \subseteq \bigcap Z[I]$ , we can then conclude from of Proposition 2.1 that  $I$  is a non-essential ideal of  $C(X)$ .  $\square$

**Lemma 3.7.** For every non-empty  $G_\delta$  set of  $G$  in  $X$ , there exists a countable generated  $z^\circ$ -ideal of  $I$  in  $C(X)$  such that  $\bigcap Z[I] \subseteq G$ .

**Proof.** Let  $x \in G = \bigcap_{i=1}^{\infty} G_i$ , where each  $G_i$  is an open subset of  $X$ . In view of 3.2(b) in [8], we define by induction a sequence of  $f_n \in C(X)$  as follows, there is a  $f_1 \in C(X)$  such that  $x \in \text{int} Z(f_1) \subseteq Z(f_1) \subseteq G_1$ . Now suppose that there exists  $f_{n-1} \in C(X)$  such that  $x \in \text{int} Z(f_{n-1}) \subseteq Z(f_{n-1}) \subseteq \bigcap_{i=1}^{n-1} G_i \cap \text{int} Z(f_{n-2})$ . Since  $x \in \bigcap_{i=1}^n G_i \cap \text{int} Z(f_{n-1})$ , we conclude that there is a  $f_n \in C(X)$  such that  $x \in \text{int} Z(f_n) \subseteq Z(f_n) \subseteq \bigcap_{i=1}^n G_i \cap \text{int} Z(f_{n-1})$ . So that for every  $n \in \mathbb{N}$ ,  $\text{int} Z(f_n) \subseteq Z(f_n) \subseteq Z(f_{n-1})$ . Hence  $I = (f_1, f_2, \dots)$  is a  $z^\circ$ -ideal such that  $\bigcap Z[I] \subseteq G$ .  $\square$

Clearly, every completely regular space is  $\overline{\aleph_0}$ -Kasch, because every finitely generated semiprime ideal is generated by an idempotent. Hence every finite generated  $z$ -ideal is a non-essential ideal, see [8, Theorem 2.8], and also we have the following.

**Theorem 3.8.** *The following statements are equivalent:*

- (1)  $X$  is an  $\overline{\aleph_1}$ -Kasch space.
- (2)  $X$  is an almost  $P$ -space and every countable generated  $z$ -ideal is fixed.
- (3) Every countable generated  $z^\circ$ -ideal is a non-essential ideal.

**Proof.** (1)  $\Rightarrow$  (2) It suffices to show that every non-empty  $G_\delta$  set has non-empty interior. Let  $G$  be a non-empty  $G_\delta$  set in  $X$ , in view of Lemma 3.7, there exists a countable generated  $z^\circ$ -ideal of  $I$  in  $C(X)$  such that  $\bigcap Z[I] \subseteq G$ , and since every  $z^\circ$ -ideal is  $z$ -ideal, we can then conclude from our hypothesis that  $\emptyset \neq \text{int} \bigcap Z[I] \subseteq \text{int} G$ .

(2)  $\Rightarrow$  (3) It is clear.

(3)  $\Rightarrow$  (1) In view of Lemma 3.7 and by our hypothesis is an almost  $P$ -space. So that every  $z$ -ideal is a  $z^\circ$ -ideal, see [4], and the proof is complete.  $\square$

#### 4. PROPERTIES OF $a$ -KASCH SPACES

Throughout this section all spaces are assumed to be infinity completely regular. A  $\sigma$ -compact space (i.e., it is the union of at most countably many compact) and locally compact space is  $\aleph_1$ -Kasch, see 8.2 and Theorem 14.17 in [8] and so [12].

In view of Theorem 3.3, the one-point compactification of an uncountable discrete space is an  $\aleph_1$ -Kasch, see [12]. Now it becomes of interest to ask: Under what condition a  $\aleph_1$ -Kasch space is a one-point compactification of a discrete space. We give a solution to this question, see Proposition 4.3. Before we proceed, we recall some definitions. The set of all cardinal numbers of the form  $|\mathcal{B}|$ , where  $\mathcal{B}$  is a basis for  $X$ , has the smallest element, this cardinal number is called the **weight** of  $X$  and is denoted by  $\mathcal{W}(X)$ . The **character of a point**  $x$  in  $X$  is defined as the smallest cardinal number of the form  $|\mathcal{B}(x)|$ , where  $\mathcal{B}(x)$  is a basis for  $X$  at the point  $x$ , this cardinal number is denoted by  $\chi(x, X)$ , see [7]. It is not hard to prove that  $\chi(x, X) = a$  if and only if  $\text{gen}(O_x) = a$ . So that we have the following.

**Proposition 4.1.** *For every  $a$ -Kasch space  $X$  and  $x \in X$ ,  $\chi(x, X) < a$  if and only if  $x$  is an isolated point in  $X$ .*

**Corollary 4.2.** *Suppose that  $X$  is an  $\aleph_0$ -Kasch space,  $X$  is a first countable if and only if it is a discrete space.*

By previous results, if  $X$  is an  $a$ -Kasch space and  $a \geq \aleph_1$ , then  $X$  is not a second countable space and also it is not a first countable space.

Suppose that  $b$  is a infinite cardinal number and it is a immediate successor of  $a$ . If  $X$  is a completely regular space and  $|X| = a$ , then  $X$  is not  $b$ -Kasch space. Otherwise we get a contradiction. Let  $x_0 \in X$ , by completely regularity of  $X$ , for every  $y \in X \setminus \{x_0\}$  there exists  $f_y \in M_{x_0}$  such that  $f_y(y) \neq 0$ . If  $I = \sum_{y \in X \setminus \{x_0\}} f_y C(X)$ , then  $\text{gen}(I) \leq a < b$ , thus  $I$  is a non-essential ideal in

$C(X)$  and by Proposition 2.1,  $\emptyset \neq \text{int} \bigcap Z[I] = \text{int} \{x_0\}$ , i.e.,  $x_0$  is an isolated point. It follows that  $X$  is a discrete space and since  $b \geq \aleph_1$ , we can then conclude from of Theorem 3.3 that  $X$  is a finite space, which is impossible.

**Proposition 4.3.** *For an  $\aleph_1$ -Kasch space  $X$ , the following statements are equivalent:*

- (1)  $X$  is a one-point compactification of a discrete space.
- (2) There exists the unique  $x \in X$  such that  $\chi(x, X) \geq \aleph_1$  and every neighborhood  $x$  has a compact complement.

**Proof.** (1)  $\Rightarrow$  (2) It is clear.

(2)  $\Rightarrow$  (1) For every  $y \in X \setminus \{x\} = Y$ , by Proposition 4.1,  $y$  is an isolated point in  $X$ . So that  $Y$  is a discrete space. It is clear that  $X$  is a one-point compactification of  $Y$ . □

By Proposition 3.2 a dense subset or an open subset of an  $\aleph_0$ -Kasch space is an  $\aleph_0$ -Kasch space (see [12]) and also we have the following.

**Proposition 4.4.** *Let  $X$  be a topological space.*

- (1) Every dense subset  $C$ -embedded of  $X$  is a-Kasch space if and only if  $X$  is a-Kasch space.
- (2) The free union  $\bigcup_{s \in S} X_s$  (see [7]) is  $\aleph_0$ -Kasch if and only if  $X_s$  is  $\aleph_0$ -Kasch space for all  $s \in S$ .
- (3) If  $f$  is a open continuous function from a-Kasch space  $X$  onto  $Y$ , then  $Y$  is a-Kasch space.
- (4) A  $C$ -embedded open subset in an  $\aleph_1$ -Kasch space is an  $\aleph_1$ -Kasch space.

**Proof.** (1) Let  $A$  be a dense subset  $C$ -embedded of  $X$ . Then every  $f \in C(A)$  has the unique extension  $\bar{f} \in C(X)$ . So that the mapping  $f \rightarrow \bar{f}$  is an isomorphism of  $C(X)$  onto  $C(A)$  and the proof is complete.

(2) Trivial.

(3) Let  $I$  be an ideal in  $C(Y)$  with  $\text{gen}(I) < a$ . There exists a generating set  $S$  for  $I$  such that  $|S| < a$ . Now we put  $T = \{g \circ f : g \in S\}$  and  $J = TC(X)$ , then  $J$  is a non-essential ideal in  $C(X)$ , i.e.,  $\emptyset \neq \text{int}_X \bigcap Z[J] = \text{int}_X \bigcap Z[T] = A$  (by Proposition 2.1). Since  $f$  is a open continuous function, we conclude that  $f(A)$  is a open subset in  $Y$  and it is clear that  $\emptyset \neq f(A) \subseteq \text{int}_Y \bigcap Z[I]$ , so that by Proposition 2.1,  $I$  is a non-essential ideal in  $C(Y)$ .

(4) It is clear. □

The **density** of a space  $X$  is defined as the smallest cardinal number of the form  $|A|$ , where  $A$  is a dense subset of  $X$ , this cardinal number is denoted by  $d(X)$ .

**Proposition 4.5.** *If  $X$  is an  $a$ -Kasch space then*

- (1)  $\mathcal{W}(X) \geq a$ .
- (2)  $d(X) \geq a$ .
- (3) If  $A$  is nowhere dense closed in  $X$ , then  $|X \setminus A| \geq a$ .



- (4) If  $C_K(X) = \{f \in C(X) : \overline{X \setminus Z(f)} \text{ is a compact space}\} = (0)$  and  $A$  is compact in  $X$ , then  $|X \setminus A| \geq a$ .
- (5) Every prime ideal of  $P$  in  $C(X)$  is an isolated maximal ideal or  $\text{gen}(I) \geq a$  and  $\text{Ann}(I) = (0)$ .
- (6) For every  $x \in \beta X$ ,  $O^x$  is an isolated maximal ideal or  $\text{gen}(O^x) \geq a$  and  $\text{Ann}(O^x) = (0)$ .
- (7) Every ideal of  $I$  with  $\text{gen}(I) < a$ , which  $O^x \subset I$  for some  $x \in \beta X$  is an isolated maximal ideal and  $O^x = I$ .

**Proof.** (1) Let  $\mathcal{W}(X) < a$ . Let  $p \in X$ , there exists  $S \subseteq C(X)$  with  $|S| < a$  such that  $\{p\} = \bigcap Z[S]$ , it follows that  $\emptyset \neq \text{int}\{p\} = \text{int}\bigcap Z[SC(X)]$ . Thus  $p$  is an isolated point, i.e.,  $X$  is a discrete space. So that  $\mathcal{W}(X) = |X| < a$ . If  $a = \aleph_0$  then  $X$  is finite space and if  $a \geq \aleph_1$ , in view of Theorem 3.3,  $X$  is pseudocompact, it follows that  $X$  is finite space, which is a contradiction.

(2) Let  $X$  is a discrete space and  $d(X) < a$ . It is clear that  $\text{gen}(C_F(X)) = |X| = d(X) < a$ , thus  $C_F(X)$  is a non-essential ideal in  $C(X)$ , it follows that  $C_F(X)$  is a fixed ideal. But by Corollary 3.6 in [10],  $C_F(X)$  is a free ideal, which is a contradiction.

If  $X$  is not a discrete space, there exists a dense proper subset of  $A$  in  $X$ . Let  $x_0 \in X \setminus A$ , for every  $p \in A$  there exists  $f_p \in C(X)$  such that  $f_p(p) = 1$  and  $f_p(x_0) = 0$ . So that  $I = \sum_{p \in A} f_p C(X) \subseteq M_{x_0}$ . We claim that  $F = \text{int}\bigcap Z[I] = \emptyset$ , for otherwise there exists  $p \in F \cap A$ . This means that  $f_p(p) = 0$  and  $f_p(p) = 1$ , which is the desired contradiction. So that  $I$  is a essential ideal with  $\text{gen}(I) \leq d(X)$ , it follows that  $a < \text{gen}(I) \leq d(X)$ , and the proof is complete.

(3) For every  $x \in X \setminus A$  there exists  $f_x \in C(X)$  such that  $f_x(A) = 0$  and  $f_x(x) = 1$ , so that  $I = \sum_{x \in X \setminus A} f_x C(X)$  is an ideal with  $\text{gen}(I) \leq |X \setminus A|$ . Since  $\text{int}\bigcap Z[I] = \text{int} A = \emptyset$ , we can then conclude from of Proposition 2.1 that  $I$  is an essential ideal. It follows that  $|X \setminus A| \geq \text{gen}(I) \geq a$ .

(4) By previous part it suffices to show that  $\text{int} A = \emptyset$ . Let us assume that  $x \in \text{int} A$  and seek a contradiction. By completely regularity  $X$  there exists  $g \in C(X)$  such that  $g(x) = 1$  and  $g(X \setminus \text{int} A) = 0$ , thus  $\overline{X \setminus Z(g)} \subseteq \text{int} A \subseteq A$ , and since  $A$  is a compact space we conclude that  $\overline{X \setminus Z(g)}$  is compact, i.e.,  $0 \neq g \in C_K(X)$ , which is the desired contradiction.

(5), (6) and (7) It is clear. □

In view of proposition next if  $X$  is an  $a$ -Kasch space, where  $a \geq \aleph_1$ , then  $X$  is not a  $P$ -space and also it is not a basically disconnected space.

**Proposition 4.6.** *Let  $X$  is an  $\aleph_0$ -Kasch space, then the following statements are equivalent:*

- (1)  $X$  is a  $P$ -space.
- (2) For every non-unit element of  $f$  in  $C(X)$ ,  $\text{Ann}(f)$  is a principal ideal.
- (3)  $X$  is a basically disconnected space.

**Proof.** (1)  $\Rightarrow$  (2) It is clear.

(2)  $\Rightarrow$  (3) Let  $f \in C(X)$ , it suffices to show that  $\overline{\text{Coz}(f)}$  is an open set. If  $Z(f) = \emptyset$  or  $Z(f) = X$  then  $\overline{\text{Coz}(f)}$  is an open set. Now let us assume that  $X \neq Z(f) \neq \emptyset$ , by our hypothesis  $\text{Ann}(f)$  is a nonzero principal  $z$ -ideal, so that  $\text{Ann}(f) = eC(X)$  for some an idempotent of  $e$  in  $C(X)$ , see [13]. Clearly,  $\overline{\text{Coz}(f)} \subseteq Z(e)$  therefore if  $x \in Z(e) \setminus \overline{\text{Coz}(f)}$  we get a contradiction. By completely regularity  $X$  there exists  $g \in C(X)$  such that  $g(x) = 1$  and  $g(\overline{\text{Coz}(f)}) = 0$ , which means that  $g \in \text{Ann}(f) = eC(X)$ , and other hand  $Z(e) \subseteq Z(g)$  then  $g(x) = 0$ , which is impossible.

(3)  $\Rightarrow$  (1) See Proposition 2.8 in [3]. □

**Proposition 4.7.** *If  $\frac{C(X)}{C_F(X)}$  is an  $\aleph_0$ -Kasch ring then  $X$  is an  $\aleph_0$ -Kasch space.*

**Proof.** Let  $I = (f_1, \dots, f_n)$  is a ideal in  $C(X)$ . If  $f_1, \dots, f_n \in C_F(X)$ , in view of Proposition 2.6,  $A = \bigcup_{i=1}^n \text{Coz}(f_i)$  is finite clopen set in  $X$ . Now we define  $g \in C(X)$  by  $g(x) = 0$  for all  $x \in A$  and  $g(x) = 1$  if  $x \notin A$ , see 1A in [8]. Hence  $g \in \text{Ann}(I)$  and  $g \neq 0$ . Now let us assume that  $f_i \notin C_F(X)$  for some  $f_i$  and put  $\bar{I} = \sum_{i=1}^n \bar{f}_i \frac{C(X)}{C_F(X)}$ , where  $\bar{f}_i = f_i + C_F(X)$ , it is clear that  $\bar{I} \neq (0)$ , and  $\text{gen}(\bar{I}) < \aleph_0$ .

If  $\bar{I} \neq \frac{C(X)}{C_F(X)}$ , there exists  $g \in C(X) \setminus C_F(X)$  such that  $g\bar{I} = (0)$ . BY Proposition 2.6 we get that  $B = \text{Coz}(g) \cap (\bigcup_{i=1}^n \text{Coz}(f_i))$  is finite clopen set in  $X$ . Thus if we define  $g_0(x) = 0$  for all  $x \in B$  and  $g_0(x) = g(x)$  if  $x \notin B$ , then  $0 \neq g_0 \in C(X)$  and  $g_0 \in \text{Ann}(I)$ .

If  $\bar{I} = \frac{C(X)}{C_F(X)}$ , then there exists  $h \in I$  such that  $\bar{h} = \bar{I}$ . Thus in view of Proposition 2.6,  $\text{Coz}(h - 1)$  is a finite clopen set in  $X$ . Since  $\bigcap Z[I] \subseteq Z(h)$  and  $Z(h) \subseteq \text{Coz}(h - 1)$ , we conclude that  $\bigcap Z[I]$  is a finite clopen set. If  $\bigcap Z[I] \neq \emptyset$  then by Proposition 2.1,  $I$  is a non-essential ideal in  $C(X)$ . If  $\bigcap Z[I] = \emptyset$  we get a contradiction. Let  $Z(h) = \{x_1, \dots, x_n\}$ , thus for every  $x_i \in Z(h)$ ,  $\exists h_i \in I$  such that  $h_i(x_i) \neq 0$ . So that  $Z(h_1^2 + \dots + h_n^2 + h^2) = \emptyset$  and  $h_1^2 + \dots + h_n^2 + h^2 \in I$ , which is the desired contradiction. □

**Lemma 4.8.** *If  $\text{gen}(C_F(X)) < a$  and  $X$  is an  $a$ -Kasch space then  $\frac{C(X)}{C_F(X)}$  is an  $a$ -Kasch ring.*

**Proof.** let  $\bar{I} = \frac{I}{C_F(X)}$  such that  $\text{gen}(\bar{I}) < a$ , where  $I$  is an ideal in  $C(X)$  which is contains  $C_F(X)$ . By our hypothesis  $\text{gen}(I) \leq \text{gen}(\bar{I}) + \text{gen}(C_F(X)) < a$ , thus  $I$  is a non-essential ideal in  $C(X)$ , i.e.,  $A = \text{int} \bigcap Z[I] \neq \emptyset$  (by Proposition 2.1). By completely regularity  $X$ , if  $x \in A$ , there exists  $f \in C(X)$  such that  $f(x) = 1$  and  $f(X \setminus A) = (0)$ , it follows that  $f \in I$  and  $J = \bar{f} \frac{C(X)}{C_F(X)} \neq (0)$ . Now let  $\frac{I}{C_F(X)}$  be an essential ideal in  $\frac{C(X)}{C_F(X)}$  and seek a contradiction. Thus  $J \cap \frac{I}{C_F(X)} \neq (0)$  and there exists  $g \in I$  such that  $g\bar{f} \neq 0$ . But  $X = A \cup (X \setminus A) \subseteq Z(g) \cup Z(f)$ , i.e.,  $g\bar{f} = 0$ , which is impossible. □

**Proposition 4.9.** *For a topological space  $X$  with only a finite number of isolated points, the following statements are equivalent:*

- (1)  $X$  is an  $a$ -Kasch space.

(2)  $\frac{C(X)}{C_F(X)}$  is an  $a$ -Kasch ring.

**Proof.** 1  $\Rightarrow$  2) By Lemma 3.1 and our hypothesis, it is clear.

2  $\Rightarrow$  1) Let  $I$  is an ideal in  $C(X)$  with  $\text{gen}(I) < a$ . By our hypothesis  $\text{gen}\left(\frac{C_F(X)+I}{C_F(X)}\right) < a$ .

If  $\frac{C_F(X)+I}{C_F(X)} \neq \frac{C(X)}{C_F(X)}$ , there exists  $0 \neq \bar{g} \in \frac{C(X)}{C_F(X)}$  such that  $g(C_F(X) + I) \subseteq C_F(X)$ . By Proposition 2.6, we infer that  $\text{Coz}(gf) \subseteq H$ , for all  $f \in I$ , where  $H$  is set of all isolated points in  $X$ . It follows that  $A = \bigcup_{f \in I} \text{Coz}(gf) \subseteq H$  is a finite clopen set in  $X$  (by hypothesis). By 1A in [8], we can define  $g_0 \in C(X)$  by  $g_0(x) = 0$  if  $x \in A$  and  $g_0(x) = g(x)$  if  $x \notin A$ . Since  $\bar{g} \neq 0$ ,  $\text{Coz}(g)$  is a infinity set, we conclude that  $0 \neq g_0 \in \text{Ann}(I)$ , i.e.,  $I$  is a non-essential ideal in  $C(X)$ .

If  $\frac{C_F(X)+I}{C_F(X)} = \frac{C(X)}{C_F(X)}$ , there exists  $g \in C_F(X)$  and  $f \in I$  such that  $g + f = 1$  then  $1 - f = g \in C_F(X)$ . By Proposition 2.6,  $\bigcap Z[I] \subseteq Z(f) \subseteq \text{Coz}(1 - f) \subseteq H$ . It follows that  $Z(f)$  and  $\bigcap Z[I]$  are finite clopen set in  $X$ . If  $\bigcap Z[I] \neq \emptyset$  then by Proposition 2.1,  $I$  is a non-essential ideal in  $C(X)$ , and if  $\bigcap Z[I] = \emptyset$ , as in the proof of the previous proposition we get a contradiction.

A subset  $S$  of  $R$  is said to be orthogonal provided  $xy = 0$  for all  $x, y \in S, x \neq y$ . If  $X$  is a connected space and  $S$  is a orthogonal subset in  $C(X)$  with more than two element, then  $I = \sum_{f \in S} fC(X) \neq C(X)$ , otherwise there exists  $f_1, \dots, f_n \in S$  and  $g_1, \dots, g_n \in C(X)$  such that  $g_1f_1 + \dots + g_nf_n = 1$  and  $g_1f_1 \neq 0$ . Clearly  $g_1f_1 = (g_1f_1)^2$  and by 1A in [8],  $g_1f_1 = 1$ , i.e.,  $f_1$  is a unit element in  $C(X)$ , it follows that  $|S| \leq 2$ , which is the desired contradiction.  $\square$

**Proposition 4.10.** *Let  $X$  is an  $a$ -Kasch space and  $S$  is a maximal orthogonal subset in  $C(X)$ .*

(1)  $S$  is a finite set if and only if  $C(X) = \sum_{f \in S} fC(X)$ .

(2) If  $S$  is a infinity subset in  $C(X)$  then  $|S| \geq a$  and  $I = \sum_{f \in S} fC(X)$  is an essential ideal in  $C(X)$ .

**Proof.** (1) **Necessity.** If  $I = \sum_{f \in S} fC(X) \neq C(X)$ , since  $\text{gen}(I) < \aleph_0$ , we conclude that there exists  $0 \neq g \in \text{Ann}(I) = \text{Ann}(S)$ . Thus  $A = S \cup \{g\}$  is a orthogonal subset in  $C(X)$ . So that  $A$  properly contains  $S$ , which violates the maximality of  $S$ .

**Sufficiency.** Let  $S$  is a infinity subset in  $C(X)$  we get a contradiction. Clearly, there exists  $f_1, \dots, f_n \in S$  and  $g_1, \dots, g_n \in C(X)$  such that  $g_1f_1 + \dots + g_nf_n = 1$ . Since  $S$  is a infinity set, we conclude that there exists  $0 \neq h \in S \setminus \{f_1, \dots, f_n\}$ . But  $h = g_1f_1h + \dots + g_nf_nh = 0$  which is impossible.

(2) In view of the proof of previous case,  $\text{Ann}(S) = (0)$  and  $I = \sum_{f \in S} fC(X)$  is a proper ideal in  $C(X)$ . So that  $I$  is an essential ideal in  $C(X)$  and  $|S| = \text{gen}(I) \geq a$ .  $\square$

If  $X$  is a pseudocmpact then  $\beta: C(X) \rightarrow C(\beta X)$  is a isomorphism onto, 6.6(b) in see [8]. It is not hard to prove that for every  $A \subseteq X, \bigcap Z[O_A] = cl_X A$  and also see [5, Lemma 6.1], so that we have the following.

**Proposition 4.11.** *Suppose that  $X$  is an  $a$ -Kasch space, if  $\beta X(\nu X)$  has a nowhere dense zero set then  $a = \aleph_0$ .*

**Proof.** Let  $a \geq \aleph_1$ . By Theorem 3.3 and 8A in [8],  $\nu X = \beta X$ . Let  $A$  be a nowhere dense zero set in  $\beta X$ . Now put  $O_\beta^A = \{f^\beta : A \subseteq \text{int}_{\beta X} Z(f^\beta) \text{ and } f \in C(X)\}$ , thus  $\text{int} \bigcap Z[O_\beta^A] = \text{int}_{\beta X} \text{cl}_{\beta X} A = \emptyset$ , by Proposition 2.1 we infer that  $O_\beta^A$  is an essential ideal in  $C(\beta X)$ . In view of Lemma 2.1 in [13],  $O^A = \{f \in C(X) : A \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)\}$  is a countable generated ideal in  $C(X)$ . Hence  $\text{gen}(O_\beta^A) = \text{gen}(\beta(O^A)) = \text{gen}(O^A) < \aleph_1$  and in view of Corollary 3.3,  $O_\beta^A$  is a non-essential ideal in  $C(\beta X)$ , which is a contradiction.  $\square$

**Proposition 4.12.** *If  $X$  is a pseudocompact space then every countable  $z$ -ideal in  $C(X)$  is of the form  $O^A$ , where  $A \in Z[\beta X]$ , and hence it is a  $z^\circ$ -ideal.*

**Proof.** Let  $I = (f_1, f_2, \dots)$  be a  $z$ -ideal in  $C(X)$ . Clearly  $\beta(I) = (f_1^\beta, f_2^\beta, \dots)$  is a  $z$ -ideal in  $C(\beta X)$ . By Corollary in [13] there exists  $A \in Z[\beta X]$  such that  $\beta(I) = O_\beta^A = \{f^\beta : A \subseteq \text{int}_{\beta X} Z(f^\beta) \text{ and } f \in C(X)\}$ . It follows that  $I = O^A = \{f \in C(X) : A \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)\}$  is a  $z^\circ$ -ideal, see [4].  $\square$

**Lemma 4.13.** *If  $I$  is a countable generated  $z$ -ideal and  $Z[I]$  is closed under countable intersection then  $I$  is generated by an idempotent.*

**Proof.** Let  $I = (f_1, f_2, \dots)$  and put  $g = \sum_{i=1}^\infty \frac{f_i^{\frac{2}{3}}}{2^i(1+f_i^{\frac{2}{3}})}$ . Then by uniform convergence,  $g \in C(X)$  and  $Z(g) = \bigcap_{i=1}^\infty Z(f_i)$ . But  $Z[I]$  is closed under countable intersection, hence  $Z(g) \in Z[I]$  implies that  $g \in I$ , for  $I$  is a  $z$ -ideal. Since  $|f_n| \leq (2^n(1+f_n^{\frac{2}{3}})g)^{\frac{3}{2}}$ , we can then conclude from Proposition 2.3 that  $f_n$  is a multiple of  $g$ , for all  $n \in N$ . We infer that  $I = gC(X)$ , hence  $I$  is generated by an idempotent, see [13].  $\square$

If for every  $j \in \Lambda, I_j$  is a  $z$ -ideal and  $Z[I_j]$  is closed under countable intersection then  $\bigcap_{j \in \Lambda} I_j$  is a  $z$ -ideal and  $Z[\bigcap_{j \in \Lambda} I_j]$  is closed under countable intersection. By Lemma 3.2 and Theorem 5.14, 5.8 in [8] if  $X$  is a pseudocompact space then every countably generated intersection of maximal ideals is generated by an idempotent, see [12].

**Proposition 4.14.** *If  $X$  is a pseudocompact space, then every countably generated ideal which contains finite intersection maximal ideals is generated by an idempotent.*

REFERENCES

[1] Azarpanah, F., *Essential ideals in  $C(X)$* , Period. Math. Hungar. **3** (12) (1995), 105–112.  
 [2] Azarpanah, F., *Intersection of essential ideals in  $C(X)$* , Proc. Amer. Math. Soc. **125** (1997), 2149–2154.  
 [3] Azarpanah, F., *On almost  $P$ -space*, Far East J. Math. Sci. Special volume (2000), 121–132.  
 [4] Azarpanah, F., Karamzadeh, O. A. S., Aliabad, A. R., *On  $z^\circ$ -ideals in  $C(X)$* , Fund. Math. **160** (1999), 15–25.

- [5] Dietrich, W. E., Jr., *On the ideal structure of  $C(X)$* , Trans. Amer. Math. Soc. **152** (1970), 61–77.
- [6] Donne, A. Le, *On a question concerning countably generated  $z$ -ideal of  $C(X)$* , Proc. Amer. Math. Soc. **80** (1980), 505–510.
- [7] Engelking, R., *General topology*, mathematical monographs, vol. 60 ed., PWN Polish Scientific publishers, 1977.
- [8] Gillman, L., Jerison, M., *Rings of continuous functions*, Springer-Verlag, 1979.
- [9] Goodearl, K. R., *Von Neumann regular rings*, Pitman, San Francisco, 1979.
- [10] Karamzadeh, O. A. S., Rostami, M., *On the intrinsic topology and some related ideals of  $C(X)$* , Proc. Amer. Math. Soc. **93** (1985), 179–184.
- [11] Lam, Tsit-Yuen, *Lectures on Modules and Rings*, Springer, 1999.
- [12] Levy, R., *Almost  $P$ -spaces*, Can. J. Math. **29** (1977), 284–288.
- [13] Marco, G. De, *On the countably generated  $z$ -ideal of  $C(X)$* , Proc. Amer. Math. Soc. **31** (1972), 574–576.
- [14] Nunzetta, P., Plank, D., *Closed ideal in  $C(X)$* , Proc. Amer. Math. Soc. **35** (2) (1972), 601–606.

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