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INVARIANT APPROXIMATION FOR FUZZY NONEXPANSIVE MAPPINGS

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Abstract. We establish results on invariant approximation for fuzzy nonexpansive mappings defined on fuzzy metric spaces. As an application a result on the best approximation as a fixed point in a fuzzy normed space is obtained. We also define the strictly convex fuzzy normed space and obtain a necessary condition for the set of all t-best approximations to contain a fixed point of arbitrary mappings. A result regarding the existence of an invariant point for a pair of commuting mappings on a fuzzy metric space is proved. Our results extend, generalize and unify various known results in the existing literature.

Keywords: fuzzy normed space, strictly convex fuzzy normed space, fixed point, fuzzy nonexpansive mapping, fuzzy best approximation, fuzzy Banach mapping

MSC 2010: 47H10, 47H09, 41A50, 46S40, 54H25, 41A65

1. Introduction and preliminaries

The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [17] in 1965, as a new way to represent vagueness in everyday life. The concept of a fuzzy metric space has been introduced and generalized in many ways ([5], [11]). Moreover, George and Veeramani ([8], [9]) modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [10]. They obtained a Hausdorff topology for this kind of fuzzy metric space which has applications in quantum particle physics, particularly in connection with both string and ε^{∞} theory (see [7] and references mentioned therein). Fixed point theorems in fuzzy metric spaces have applications to control theory, system theory and optimization problems. Study of fixed points in fuzzy normed spaces is a very recent development

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([1], [2], [3]). Recently, Veeramani [16] introduced the notion of t-best approximation in fuzzy metric spaces. Vaezpour and Karimi [14] studied the properties of the set of all t-best approximations on fuzzy normed spaces. In this paper we establish results on invariant approximation for fuzzy nonexpansive mappings defined on fuzzy metric spaces. As an application a result on the best approximation as a fixed point in a fuzzy normed space is obtained. We define the strictly convex fuzzy normed space and obtain a necessary condition for the set of all t-best approximations to contain a fixed point of arbitrary mappings. A result regarding the existence of an invariant point for a pair of commuting mappings on a fuzzy metric space is also proved. Our results extend, generalize and unify various known results in the existing literature.

For the sake of convenience, we first give some definitions and known results.

Definition 1.1 [17]**.** Let X be any set. A *fuzzy set* A in X is a function with domain X and values in $[0,1]$.

Definition 1.2. Let X be an arbitrary set. A fuzzy set M in $X \times X \times \mathbb{R}$ is called a *fuzzy metric* on X if and only if for each $x, y, z \in X$ the following conditions are satisfied:

(a) For all $t \in \mathbb{R}$ with $t \leqslant 0$, $M(x,y,t) = 0$;
(b) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;
(c) $M(x,y,t) = M(y,x,t)$ for all $t \in \mathbb{R}$;
(d) $\min\{M(x,y,t), M(y,z,s)\} \leqslant M(x,z,t+s)$ for all $t, s \in \mathbb{R}$;
(e) $M(x,y,.)\colon (0,\infty) \to [0,1]$ is continuous, and $\lim_{t\to\infty} M(x,y,t) = 1$.

Note that $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t. It is known that $M(x,y,.)$ is nondecreasing for all $x, y \in X$.

Definition 1.3 [4]**.** Let U be a linear space over the real $\mathbb{R}$. A fuzzy set N in $U \times \mathbb{R}$ is called a *fuzzy norm* on U if and only if for all $x, y \in U$ and $\mathbb{R}$, the following conditions are satisfied: (N1) For all $t \in \mathbb{R}$ with $t \leqslant 0$, $N(x,t) = 0$;
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x,t) = 1$ if and only if $x = 0$;
(N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;
(N4) for all $t, s \in \mathbb{R}$, $N(x+y, t+s) \geqslant \min\{N(x,t), N(y,s)\}$;
(N5) $N(x,.)\colon (0,\infty) \longrightarrow [0,1]$ is continuous, and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U,N) is called a *fuzzy normed space.* Let (U,N) be a fuzzy normed space. If we define

$$M(x,y,t) = N(x-y,t)$$

then M is a fuzzy metric on X which is called the fuzzy metric induced by the fuzzy norm N.

Definition 1.4. Let (X, M) be a fuzzy metric space. We define an *open ball* $B(x, r, t)$ and a *closed ball* $B[x, r, t]$ with a center $x \in X$ and a radius $0 < r < 1$, $t > 0$ as follows:

$$B(x, r, t) = \{y \in X\colon\ M(x, y, t) > 1 - r\}, \text{ and}$$
$$B[x, r, t] = \{y \in X\colon\ M(x, y, t) \geqslant 1 - r\}.$$

Definition 1.5. Let (X, M) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *Cauchy* if $\lim\limits_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p = 1, 2, 3, \ldots$

Definition 1.6. Let (X, M) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x \in X$ such that $\lim\limits_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. In that case x is called the *limit of the sequence* $\{x_n\}$.

A fuzzy metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

Definition 1.7. Let (X, M) be a fuzzy metric space and $f\colon X \to X$. A mapping $T\colon X \to X$ is called *fuzzy f-nonexpansive* if for all $x, y \in X$,

$$M(Tx, Ty, t) \geqslant M(fx, fy, t)$$

for all $t \in \mathbb{R}$.

If we put $f = I$ (identity map) in Definition 1.7, we obtain the definition of the *fuzzy nonexpansive* mapping [1].

Definition 1.8. Let (X, M) be a fuzzy metric space. Then $T\colon X \to X$ is called the *fuzzy Banach* mapping if there exists a $k \in (0, 1)$ such that

$$M(Tx, T^2x, kt) \geqslant M(x, Tx, t)$$

for all $x \in X$ and $t \in \mathbb{R}$.

Definition 1.9. Let (X, M) be a fuzzy metric space and let T, S be self mappings on X. A point $x \in X$ is called:

(1) a *fixed point of* T if $T(x) = x$;
(2) a *coincidence point of the pair* $\{T, S\}$ if $Tx = Sx$;
(3) a *common fixed point of the pair* $\{T, S\}$ if $x = Tx = Sx$.

$F(T)$, $C(T, S)$ and $F(T, S)$ denote the set of all fixed points of T, the set of all coincidence points of the pair $\{T, S\}$, and the set of all common fixed points of the pair $\{T, S\}$, respectively.

Our next definition is a fuzzy analogue of Dotson's notion [6] of the contractive jointly continuous family.

Definition 1.10. Let K be a subset of a fuzzy metric space (X, M) and let $\{f_\alpha\colon\ \alpha \in K\}$ be a family of maps from $[0,1]$ into K such that $f_\alpha(1) = \alpha$. Such a family is said to be:

(a) *fuzzy contractive* provided there exists a function $\varphi\colon\ (0,1) \to (0,1)$ such that for all $\varphi, \psi \in K$ and $s \in (0,1)$, the inequality

$$M\bigl(f_\varphi(s), f_\psi(s), \varphi(s)t\bigr) \geqslant M(\varphi, \psi, t)$$

holds for all $t \in \mathbb{R}$;

(b) *jointly continuous* if $s \to s_0$ in $(0,1)$ and $\varphi \to \varphi_0$ in K imply $\{f_\varphi(s)\}$ is fuzzy convergent to $f_{\varphi_0}(s_0)$.

Definition 1.11. Let K be a nonempty subset of a fuzzy metric space (X, M). For $x \in X$, $t > 0$, let

$$M(K, x, t) = \sup\{M(y, x, t)\colon\ y \in K\}.$$

An element $y_0 \in K$ is said to be the *t-best approximation* of x from K if

$$M(y_0, x, t) = M(K, x, t).$$

For $x \in X$, $t > 0$, $P_K^t(x)$ denotes the set of all t-best approximations of x from K.

Definition 1.12. A fuzzy normed space (U, N) is said to be strictly convex if for $x, y \in U$ and $t \in \mathbb{R}$ we have

$$N\bigl(u - \tfrac{1}{2}(x+y), t\bigr) = \min\{N(u-x, t), N(u-y, t)\}$$

then $x = y$.

Definition 1.13 [15]**.** Let U be a linear space over the real $\mathbb{R}$ and let K be a subset of X. K is said to be T*-regular* if and only if $T\colon\ K \to K$ and $\frac{1}{2}(x + Tx) \in K$ for each x in K.

Definition 1.14. A subset K of a fuzzy normed space (U, N) is called *q-starshaped* or *starshaped with respect to q* (called the star center of K) if $\lambda x + (1-\lambda)q \in F$ for all $x \in K$ and $\lambda \in [0,1]$.

2. Best approximation

Theorem 2.1. *Let T be a self mapping on a fuzzy metric space (X, M), let K be a nonempty T-invariant subset of X. If for $x \in F(T)$, $t > 0$, $P_K^t(x)$ is a nonempty compact set for which there exists a fuzzy contractive jointly continuous family of mappings and T is fuzzy nonexpansive on $P_K^t(x) \cup \{x\}$, then $P_K^t(x) \cap F(T) \cap K$ is nonempty.*

Proof. Let $x_0 \in F(T)$ and $t > 0$. If $x \in P_K^t(x_0)$, then

$$\begin{aligned} M(K, x_0, t) &\geqslant M(Tx, x_0, t) = M(Tx, Tx_0, t) \\ &\geqslant M(x, x_0, t) = M(K, x_0, t). \end{aligned}$$

Therefore $Tx \in P_K^t(x_0)$, that is, $T(P_K^t(x_0)) \subseteq P_K^t(x_0)$. For each $n \in \mathbb{N}$, let $k_n = \frac{n}{n+1} \in (0, 1)$. Let $\{f_\alpha \colon \alpha \in K\}$ be a family of fuzzy contractive jointly continuous mappings from $[0, 1]$ to K. Define mappings T_n on $P_K^t(x_0)$ by $T_n(x) = f_{Tx}(k_n)$. Now, for each $n \in \mathbb{N}$ and $x, y \in P_K^t(x_0)$, we have

$$\begin{aligned} M\big(T_n x, T_n y, \varphi(k_n)t\big) &= M\big(f_{Tx}(k_n), f_{Ty}(k_n), \varphi(k_n)t\big) \\ &\geqslant M(Tx, Ty, t) \geqslant M(x, y, t), \end{aligned}$$

where φ is a function on $(0, 1)$ corresponding to the fuzzy contractive family $\{f_\alpha \colon \alpha \in K\}$. This implies that each T_n is a fuzzy contraction mapping on $P_K^t(x_0)$. We obtain a sequence $\{x_n\}$ in $P_K^t(x_0)$ such that $T_n(x_n) = x_n = f_{Tx_n}(k_n)$. Since $P_K^t(x_0)$ is compact therefore the sequence $\{x_n\}$ has a fuzzy convergent subsequence $\{x_{n_j}\}$ in $P_K^t(x_0)$ which converges to $x^* \in P_K^t(x_0)$. Thus, $T_{n_j}(x_{n_j}) = x_{n_j} = f_{Tx_{n_j}}(k_{n_j})$. We have $T_{n_j}(x_{n_j}) \to x^*$. Also,

$$\begin{aligned} &T(x_{n_j}) \to Tx^*, \text{ and} \\ &T_{n_j}(x_{n_j}) = x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx^*}(1) = Tx^*. \end{aligned}$$

This yields that x^* is a fixed point of T which is a t-best approximation of x_0 from K. $\square$

Example 2.2. Let $X = \mathbb{R}^2$ and let $*$ be a minimum norm. Let M be the fuzzy metric defined by

$$M\big((x_1, x_2), (y_1, y_2), t\big) = \left[\exp\left(\frac{\sqrt{(x_1 - y_1)^2 - (x_2 - y_2)^2}}{t}\right)\right]^{-1}$$

for all $(x_1, x_2), (y_1, y_2) \in X$, $t > 0$. Take

$$K = \{(x, y)\colon\ 0 < x \leqslant y \leqslant 1\}.$$

Now define a map $T\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$T(x,y) = \begin{cases} (x,0) & \text{when } (x,y) \in \mathbb{R}^2 - K, \\ (1,1) & \text{when } (x,y) \in K \text{ and } y = 1, \\ (\frac{1}{2}x, \frac{1}{2}y) & \text{when } (x,y) \in K \text{ and } y < 1. \end{cases}$$

Now K is a T-invariant subset of $\mathbb{R}^2$. Take $(0,0) \in F(T)$, then for $t > 0$

$$M(K,(0,0),t) = \sup_{(x,y)\in K} \left[\exp \frac{\sqrt{x^2+y^2}}{t}\right]^{-1} = \left[\exp \frac{\sqrt{2}}{t}\right]^{-1}$$

and

$$M((1,1),(0,0),t) = \left[\exp \frac{\sqrt{2}}{t}\right]^{-1}$$

give that $(1,1)$ is the t-best approximation of $(0,0)$ from K. Note that T is fuzzy nonexpansive on $P_K^t((0,0)) \cup \{(0,0)\}$. Define $f_\alpha\colon\ [0,1] \to P_K^t((0,0))$ as $f_\alpha(x) = (1,1)$ for $\alpha \in P_K^t((0,0))$. All conditions of Theorem 2.1 are satisfied and $(1,1) \in P_K^t((0,0)) \cap F(T) \cap K$.

Theorem 2.3. *Let T be a self mapping on a fuzzy normed space (X, N) and let K be a nonempty T-invariant subset of X. If for $x \in F(T)$, $t > 0$, $P_K^t(x)$ is nonempty compact and starshaped and T is fuzzy nonexpansive on $P_K^t(x) \cup \{x\}$, then $P_K^t(x) \cap F(T) \cap K$ is nonempty.*

Proof. Assume that p is the star center of $P_K^t(x)$. Define a family of maps $\{f_\alpha\colon\ \alpha \in K\}$ from $[0,1]$ into K as

$$f_\alpha(\lambda) = (1-\lambda)p + \alpha\lambda, \quad \lambda \in [0,1].$$

For $s \in (0,1)$, take $\varphi(s) = s$. Let $\alpha, \beta \in K$ and $t \in \mathbb{R}$; then

$$N(f_\alpha(s) - f_\beta(s), \varphi(s)t) = N(\alpha s - \beta s, st) = N(\alpha - \beta, t),$$

which shows that $\{f_\alpha\colon\ \alpha \in K\}$ is fuzzy contractive. Also it is jointly continuous. Now the result follows from Theorem 2.1. $\square$

The following theorem is a direct application of Theorem 2.1.

Theorem 2.4. *Let S, T be selfmappings on a fuzzy metric space (X, M). Assume that (T, S) is a pair of commuting mappings on X such that T is S-nonexpansive, and $S^2 = I$. Let K be a TS-invariant subset of X. If for $x \in F(T, S)$, $t > 0$, $P_K^t(x)$ is a singleton, then $P_K^t(x) \cap F(T) \cap F(S)$ is a singleton.*

Proof. For any $x, y \in X$ and $t \in \mathbb{R}$ we have

$$M(TSx, TSy, t) \geqslant M(S^2x, S^2y, t) = M(x, y, t).$$

Thus TS is fuzzy nonexpansive on X. By Theorem 2.1, $P_K^t(x)$ contains a TS-invariant point x_0 as $P_K^t(x)$ is a singleton. Consider

$$TS(Tx_0) = T(TSx_0) = Tx_0.$$

By the uniqueness of x_0, we get $Tx_0 = x_0$. Also,

$$S(x_0) = S(Tx_0) = T(Sx_0) = x_0.$$

□

Next we prove the following lemma needed in the sequel.

Lemma 2.5. *Let K be a nonempty fuzzy closed subset of a fuzzy metric space (X, M). If $T\colon K \to K$ is a fuzzy Banach continuous mapping, then T has a fixed point in K provided $\mathrm{cl}(T(K))$ is fuzzy compact.*

Proof. Since T is a fuzzy Banach mapping therefore

$$M(T^{n+1}x, T^{n+2}x, kt) \geqslant M(T^n x, T^{n+1}x, kt)$$

for $n = 1, 2, \ldots$ By [13, Lemma 2.2], $\{T^n x\}$ is a Cauchy sequence in $T(K)$. Now $\mathrm{cl}(T(K))$ being fuzzy compact is complete, therefore there exists an x_0 in K such that $\lim\limits_{n\to\infty} M(T^n x, x_0, t) = 1$ for all $t > 0$. By continuity of T, x_0 is a fixed point of T. □

Theorem 2.6. *Let K be a nonempty fuzzy closed subset of a fuzzy metric space (X, M). Suppose there exists a fuzzy contractive, jointly fuzzy continuous family of maps associated with K, and $\mathrm{cl}(T(K))$ is fuzzy compact. If $T\colon K \to K$ is a fuzzy nonexpansive mapping, then T has a fixed point in K.*

Proof. For each $n \in \mathbb{N}$, let $k_n = \frac{n}{n+1} \in (0, 1)$. Define mappings $T_n\colon K \to K$ by $T_n(x) = f_{Tx}(k_n)$. Now, for each $n \in \mathbb{N}$ and $x \in K$, we have

$$\begin{aligned} M(T_n x, T_n^2 x, \varphi(k_n)t) &= M\big(f_{Tx}(k_n), f_T(f_{Tx}(k_n), \varphi(k_n)t)\big) \\ &\geqslant M\big(Tx, T(f_{Tx}(k_n)), t\big) \geqslant M(x, T_n x, t). \end{aligned}$$

This implies that each T_n is a fuzzy Banach mapping and therefore using Lemma 2.4 we obtain a sequence $\{x_n\}$ in K such that $T_n(x_n) = x_n = f_{Tx_n}(k_n)$. Since $\mathrm{cl}(T(K))$ is fuzzy compact therefore the sequence $\{x_n\}$ has a fuzzy convergent subsequence $\{x_{n_j}\}$ in K which converges to x^*. Thus, $T_{n_j}(x_{n_j}) = x_{n_j} = f_{Tx_{n_j}}(k_{n_j})$. We have $T_{n_j}(x_{n_j}) \to x^*$. Also,

$$\begin{aligned} T(x_{n_j}) &\to Tx^*, \text{ and} \\ T_{n_j}(x_{n_j}) = x_{n_j} &= f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx^*}(1) = Tx^*. \end{aligned}$$

This yields that x^* is a fixed point of T. □

Corollary 2.7. *Let K be a nonempty fuzzy closed starshaped subset of a fuzzy normed space (U, N) and let $\mathrm{cl}(T(K))$ be fuzzy compact. If $T\colon K \to K$ is a fuzzy nonexpansive mapping, then T has a fixed point in K.*

Theorem 2.8. *Let K be a nonempty subset of a strictly convex fuzzy normed space (U, N). If for $u \in U$ and $t > 0$, $P_K^t(u)$ is a nonempty T-regular set, then each point of $P_K^t(u)$ is a fixed point of T.*

Proof. First we note that for $x, y \in P_K^t(u)$ with $x \neq y$, we have $\frac{1}{2}(x+y) \notin K$. If $\frac{1}{2}(x+y) \in K$, then $x, y \in P_K^t(u)$ gives

$$\min\{N(u-x,t), N(u-y,t)\} \geqslant N\Big(u - \frac{x+y}{2}, t\Big).$$

On the other hand,

$$\begin{aligned} N\Big(u - \frac{x+y}{2}, t\Big) &= N\Big(\frac{u}{2} - \frac{x}{2} + \frac{u}{2} - \frac{y}{2}, t\Big) \\ &\geqslant \min\Big\{N\Big(\frac{u}{2} - \frac{x}{2}, \frac{t}{2}\Big), N\Big(\frac{u}{2} - \frac{y}{2}, \frac{t}{2}\Big)\Big\} \\ &= \min\{N(u-x,t), N(u-y,t)\}. \end{aligned}$$

Thus

$$\min\{N(u-x,t), N(u-y,t)\} = N\Big(u - \frac{x+y}{2}, t\Big).$$

Since U is a strictly convex fuzzy normed space, we arrive at a contradiction. Hence $\frac{1}{2}(x+y) \notin K$. Now if for some x in $P_K^t(u)$ we have $x \neq Tx$ then $\frac{1}{2}(x+Tx) \notin K$ and $\frac{1}{2}(x+Tx) \notin P_K^t(u)$. Since $P_K^t(u)$ is a T-regular set, therefore $x = Tx$ must hold. Thus each fuzzy t-best approximation of u is a fixed point of T.

Remark 2.9. [12, Th. 1 and Th. 3] are deterministic analogues of our Theorems 2.1 and 2.6.

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