KYBERNETIKA — VOLUME *46* (2010), NUMBER 6, PAGES 1069–1077

DUCS COPULAS

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Based on a recent representation of copulas invariant under univariate conditioning, a new class of copulas linked to a distortion of the identity function is introduced and studied.

Keywords: copula, distortion, univariate conditioning, invariantness

Classification: 93E12

1. INTRODUCTION

Copulas [18] link univariate marginal distribution functions into a joint distribution function of the corresponding random vector. In this paper we will deal with bivariate copulas only. Recall that a function $C\colon [0,1]^2 \to [0,1]$ is a (bivariate) copula whenever it is grounded, $C(x,y) = 0$ whenever $0 \in \{x,y\}$, it has neutral element 1, $C(x,y) = x \wedge y$, whenever $1 \in \{x,y\}$ and it is 2-increasing, $C(x+\epsilon, y+\delta) - C(x, y+\delta) \geq C(x+\epsilon, y) - C(x,y)$ for all $x, y, \epsilon, \delta \in [0,1]$ such that $x+\epsilon, y+\delta \in [0,1]$. Three basic copulas Π, M, W given by $\Pi(x,y) = xy$, $M(x,y) = x \wedge y$, $W(x,y) = (x+y-1) \vee 0$, express the independence, total comonotone dependence ($Y = \varphi(X)$ for an increasing function φ) and total countermonotone dependence ($Y = \eta(X)$ for a decreasing function η) of the univariate random variables X and Y, respectively. For modelling purposes, the knowledge of a large class of copulas is required. Thus several parametric classes of copulas have been introduced. For an overview we recommend monographs [8, 16]. It seems so that the most prominent class of copulas is the class of Archimedean copulas together with their M-ordinal sums.

For more details we recommend [16, 19]. Note only that by C_f we denote an Archimedean copula $C_f\colon [0,1]^2 \to [0,1]$ given by

$$C_f(x,y) = f^{(-1)}(f(x) + f(y)), \tag{1}$$

where $f\colon [0,1] \to [0,\infty]$ is a continuous strictly decreasing convex function satisfying $f(1) = 0$ and $f^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of f, $f^{(-1)}(u) = f^{-1}(\min(u, f(0)))$. The function f is called a generator of copula C_f and we denote by $\mathcal{F}$ the set of all generators.

We introduce some well-known examples of parametric families of Archimedean copulas, compare [8, 16]:

i) For real $\lambda \neq 0$, define $f_\lambda\colon [0,1] \to [0,\infty]$ by $f_\lambda(x) = \frac{x^{-\lambda}-1}{\lambda}$. Then $f_\lambda \in \mathcal{F}$ whenever $\lambda \geq -1$. Adding $f_0 = f_\Pi$, $f_\Pi(x) = -\log x$, the Clayton family $(C_{f_\lambda})_{\lambda \geq -1}$ is recognized, with $C_{f_{-1}} = W, C_{f_0} = \Pi$ and $C_{f_1} = H$ (Ali-Mikhail-Haq copula) given by $H(x,y) = \frac{xy}{x+y-xy}$ whenever $xy \neq 0$.

ii) The Gumbel family can be seen as $(C_{f_\Pi^\lambda})_{\lambda \geq 1}$.

iii) The Yager family can be seen as $(C_{f_W^\lambda})_{\lambda \geq 1}$.

The idea of a modification of formula (1) by means of a dependence function known from the extreme value copulas (EV-copulas, in short), was proposed in [1] as Archimax copulas. We give some more details on Archimax copulas in Section 3. On the other side, a recent study of copulas invariant under univariate conditioning initiated in [14] and [7] was completed by Durante and Jaworski in [4], providing a complete description of copulas with the above mentioned property. In the next section, we give more details on this result. The main aim of this paper is a generalization of construction provided by the results of Durante and Jaworski. Motivated by Archimax copulas, we introduce a new class of DUCS (Distorted Univariate Conditioning Stable) copulas in Section 3. In Section 4, several examples and properties of DUCS copulas are discussed. Finally, some concluding remarks are added (see Section 5).

2. COPULAS INVARIANT UNDER UNIVARIATE CONDITIONING

As already mentioned, each random vector (X,Y) is characterized by a copula C. This copula C is unique whenever X and Y have continuous distribution functions. Conditional random vector $((X,Y)|X \leq t)$, where $F_X(t) > 0$, is then characterized by a threshold copula $C_{(F_X(t))}$, see [14]. A copula C is called invariant under left univariate truncation whenever $C_{(F_X(t))} = C$ for any $t \in \mathbb{R}$, $F_X(t) > 0$. Similarly, copulas invariant under right univariate truncation can be introduced.

It is not difficult to check that a copula C is invariant under right univariate truncation if and only if the copula D given by $D(x,y) = C(y,x)$ is invariant under left univariate trucnation. Therefore, we will consider copulas invariant under left univariate truncation only, and we will call them briefly copulas invariant under univariate conditioning. Similarly as M-ordinal sums of copulas play a key role in the representation and construction of associative copulas, copulas invariant under univariate conditioning are linked to g-ordinal sums based on the product copula and introduced in [14]. Recall that a g-ordinal sum copula $C = g - (\langle a_k, b_k, C_k\rangle \,|k \in \mathcal{K})$ is defined for any disjoint system $(]a_k, b_k[)_{k\in\mathcal{K}}$ of open subintervals of $]0,1[$ and any system $(C_k)_{k\in\mathcal{K}}$ of copulas by

$$C(x,y) = \begin{cases} a_k y + (b_k - a_k) C_k\left(\frac{x-a_k}{b_k-a_k}, y\right) & \text{if} \quad x \in]a_k, b_k[, \\ xy & \text{elsewhere.} \end{cases} \tag{2}$$

In [4] the next results were shown.

Proposition 2.1. Let $f \in \mathcal{F}$ and let $\bar{f}\colon [0,1] \to [0,\infty]$ be given by $\bar{f}(x) = f(1-x)$. Then the functions $C_{(f)}$, $C_{(\bar{f})}\colon [0,1]^2 \to [0,1]$ given by

$$C_{(f)}(x,y) = xf^{(-1)}\left(\frac{f(y)}{x}\right) \tag{3}$$

whenever $x \in]0,1]$, and

$$C_{(\bar{f})}(x,y) = x\left(1 - f^{(-1)}\left(\frac{\bar{f}(y)}{x}\right)\right) \tag{4}$$

whenever $x \in]0,1]$, are copulas, which are invariant under univariate conditioning.

Note that for any $f \in \mathcal{F}$ and $c > 0$, $C_{(cf)} = C_{(f)}$. Moreover, $C_{(f)} \leq \Pi$ and for each $x \in]0,1[$ there is $y \in]0,1[$ so that $C_{(f)}(x,y) < xy$. Similarly, $C_{(\bar{f})} \geq \Pi$ and for each $x \in]0,1[$ there is $y \in]0,1[$ such that $C_{(\bar{f})}(x,y) > xy$. Observe also that for each $(x,y) \in [0,1]^2$, $C_{(\bar{f})}(x,y) = x - C_{(f)}(x,1-y)$, i. e., $C_{(\bar{f})}$ is a flipping of the copula $C_{(f)}$, compare [3, 16].

Theorem 2.2. A copula $C\colon [0,1]^2 \to [0,1]$ is invariant under univariate conditioning if and only if C is a g-ordinal sum where each summand C_k, $k \in \mathcal{K}$, satisfies $C_k \in \{C_{(f_k)}, C_{(\bar{f}_k)}\}$ for some $f_k \in \mathcal{F}$.

The main aim of our paper is a generalization of copulas introduced in Proposition 2.1, and thus we give now some examples.

Example 2.3.

i) Let $f = f_W$. Then $C_{(f_W)} = C_{f_W} = W$, and $C_{(\bar{f}_W)} = M$.

ii) For $p \in]0,1]$, define $h_p\colon [0,1] \to [0,\infty]$ by $h_p(x) = (1-x^p)^{\frac{1}{p}}$. Then $h_p \in \mathcal{F}$ and $C_{(h_p)} = C_{f_{-p}}$ is a Clayton copula with parameter $-p \in [-1,0[$, (see Introduction item (i)). Observe that $C_{f_{-p}} \leq \Pi$.

iii) For $\lambda \in]0,\infty[$, define $g_\lambda\colon [0,1] \to [0,\infty]$ by $g_\lambda(x) = ((1-x)^{-\lambda} - 1)^{-\frac{1}{\lambda}}$. Then $g_\lambda \in \mathcal{F}$ and $C_{(\bar{g}_\lambda)} = C_{f_\lambda}$ is a Clayton copula with parameter $\lambda \in]0,\infty[$. Note that then $C_{f_\lambda} \geq \Pi$.

iv) Recall that $f_1\colon [0,1] \to [0,\infty]$ in Introduction item (i) is given by $f_1(x) = \frac{1}{x} - 1$. Then $C_{(f_1)}(x,y) = \frac{x^2 y}{1-y+xy}$, and $C_{(\bar{f}_1)}(x,y) = \frac{xy}{x+y-xy} = C_{f_1}(x,y)$.

3. DUCS COPULAS

Based on the description of EV-copulas (extreme value copulas), see [20] or overview chapter [6], Capéraà et al. [1] have introduced Archimax copulas as a common generalization of Archimedean copulas and EV-copulas. Recall that for $f \in \mathcal{F}$, an Archimax copula $C_{f,D}\colon [0,1]^2 \to [0,1]$ is given by

$$C_{f,D}(x,y) = f^{(-1)}\left((f(x)+f(y))D\left(\frac{f(x)}{f(x)+f(y)}\right)\right), \tag{5}$$

with convention $\frac{0}{0} = \frac{\infty}{\infty} = 1$. Here $D\colon [0,1] \to [0,1]$ is a dependence function which is convex and satisfies $x \vee (1-x) \leq D(x) \leq 1$ for all $x \in [0,1]$. Evidently, for the strongest dependence funtction $D^*\colon [0,1] \to [0,1], D^*(x) = 1$, the Archimax copula C_{f,D^*} is just the Archimedean copula $C_f, C_{f,D^*} = C_f$. On the other side, for the weakest dependence function $D_*\colon [0,1] \to [0,1], D_*(x) = x \vee (1-x)$, for any $f \in \mathcal{F}$ it holds $C_{f,D_*} = M$. For $D \neq D^*$, Archimax copulas $C_{f,D}$ can be seen as distorted Archimedean copulas. Inspired by this observation, we propose to consider distorted univariate conditioning stable copulas, briefly DUCS copula.

Proposition 3.1. Let $f \in \mathcal{F}$ and let $d\colon [0,1] \to [0,1]$ be a function. Define $C_{(f,d)}\colon [0,1]^2 \to [0,1]$ by

$$C_{(f,d)}(x,y) = xf^{(-1)}\left(\frac{f(y)}{d(x)}\right), \tag{6}$$

with convention $\frac{0}{0} = 0$. Then:

i) $C_{(f,d)}$ is grounded.

ii) 1 is neutral element of $C_{(f,d)}$ if and only if $d(1) = 1$.

iii) $C_{(f,d)}$ is a copula for any $f \in \mathcal{F}$ if and only if there is a function $\tilde{d}\colon [0,1] \to [0,1]$ so that $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$, and both d and $\tilde{d}$ are non-decreasing on $]0,1]$.

Proof.

i) This result is evident due to the fact that $\frac{f(0)}{d(x)} \geq f(0)$ for each $x \in [0,1]$.

ii) $C_{(f,d)}(x,1) = xf^{(-1)}(0) = x$ for any $x \in [0,1]$ due to the convention $\frac{0}{0} = 0$. Moreover, $C_{(f,d)}(1,y) = f^{(-1)}\left(\frac{f(y)}{d(1)}\right) = y$ for all $y \in [0,1]$ if and only if $d(1) = 1$.

iii) Suppose that $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$.
Then $C_{(f,d)}(x,y) = \tilde{d}(x)C_{(f)}(d(x),y) = \Pi(\tilde{d}(x),1)C_{(f)}(d(x),y)$.

Due to i) and ii), it is enough to show the 2-increasigness of $C_{(f,d)}$. Suppose that $x, x', y, y' \in [0,1], x \le x'$ and $y \le y'$. Then

$$\begin{aligned} C_{(f,d)}(x',y') &- C_{(f,d)}(x',y) - C_{(f,d)}(x,y') + C_{(f,d)}(x,y) = \tilde{d}(x')(C_{(f)}(d(x'),y') \\ &- C_{(f)}(d(x'),y)) - \tilde{d}(x)(C_{(f)}(d(x),y') - C_{(f)}(d(x),y)) \ge 0 \end{aligned}$$

whenever both $\tilde{d}$ and d are non-decreasing, due to the fact that the 2-increasigness of the copula $C_{(f)}$ ensures

$$C_{(f)}(d(x'),y') - C_{(f)}(d(x'),y) \ge C_{(f)}(d(x),y') - C_{(f)}(d(x),y)) \ge 0.$$

On the other side, considering $f_W \in \mathcal{F}$, we have $C_{(f_W,d)} = \max(0, x+(y-1)\tilde{d}(x))$ which is 2-increasing (supermodular) only if $\tilde{d}$ is non-decreasing.

Moreover, let $h\colon [0,1] \to [0,1]$ be given by $h(x) = 1 - d(x)$. Then $C_{(f,d)}(x,y) = 0$ if and only if $xy = 0$ or $y \le h(x)$. Suppose that d is not non-decreasing on $]0,1]$, i. e., there are $0 < x_1 < x_2 \le 1$ such that $h(x_1) < h(x_2)$. However, then $C_{(f,d)}(x_1, h(x_2)) > 0 = C_{(f,d)}(x_2, h(x_2))$, violating the non-decreasigness of $C_{(f,d)}$ in the first coordinate, i. e., $C_{(f,d)}$ cannot be then a copula. Thus if $C_{(f,d)}$ is a copula, d is necessarily non-decreasing on $]0,1]$. □

Remark 3.2.

An alternative proof is added.

Due to Liebscher [12], see also [9] and [11] if $\tilde{d}$ and d are non-decreasing then $C_{(f,d)}$ is a copula (recall that $C(x,y) = C_1(f_1(x), g_1(y))C_2(f_2(x), g_2(y))$ defines a copula C whenever C_1, C_2 are copulas and $f_1, f_2, g_1, g_2\colon [0,1] \to [0,1]$ are non-decreasing functions such that $f_1(x)f_2(x) = g_1(x)g_2(x) = x$ for all $x \in [0,1]$).

Note that given $d\colon [0,1] \to [0,1]$ such that there is a function $\tilde{d}\colon [0,1] \to [0,1]$ satisfying $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$, necessarily $d(x) \ge x$, $\tilde{d}(x) \ge x$ and both d and $\tilde{d}$ are continuous and positive on $]0,1]$. If $d(0) = 0$, the value $\tilde{d}(0)$ can be chosen arbitrarily. However, in order to have the uniqueness of the relation of d and $\tilde{d}$, we will consider continuous d and $\tilde{d}$ only. Evidently, then $\tilde{}$ can be seen as duality, $\tilde{\tilde{d}} = d$.

Denote by $\mathcal{D}$ the set of all continuous non-decreasing functions $d\colon [0,1] \to [0,1]$ such that there is a continuous non-decreasing function $\tilde{d}\colon [0,1] \to [0,1]$ for which $d(x)\tilde{d}(x) = x$ for all $x \in [0,1]$. Elements of $\mathcal{D}$ will be called distortions. Clearly $d \in \mathcal{D}$ if and only if $\tilde{d} \in \mathcal{D}$. Now we are ready to define DUCS copulas.

Definition 3.3. A copula $C\colon [0,1]^2 \to [0,1]$ is called a DUCS copula whenever there is a generator $f \in \mathcal{F}$ and a distortion $d \in \mathcal{D}$ so that $C = C_{(f,d)}$.

Example 3.4.

i) The strongest distortion $d^* \in \mathcal{D}$ is given by $d^*(x) = 1$. For any $f \in \mathcal{F}$, $C_{(f,d^*)} = \Pi$, i. e., the product copula is the strongest DUCS copula. Moreover, $d_* = (\tilde{d^*})$ the weakest distortion is given by $d_*(x) = x$, and $C_{(f,d_*)} = C_{(f)}$ for any generator $f \in \mathcal{F}$ (observe the striking similarity with the bounds of Archimax copulas).

ii) For any $d \in \mathcal{D}$, the copula $C_{(f_W,d)}\colon [0,1]^2 \to [0,1]$ is given by $C_{(f_W,d)}(x,y) = \max(0, x + (y-1)\tilde{d}(x))$. Take a parametric family $(d_{(\alpha)})_{\alpha\in[0,1]}$ of distortions, $d_{(\alpha)}(x) = \frac{x}{\alpha+(1-\alpha)x}$. Then $\tilde{d}_{(\alpha)}(x) = \alpha + (1-\alpha)x$, and $C_{(f_W,d_{(\alpha)})}(x,y) = \max(0, \alpha(x+y-1) + (1-\alpha)xy)$. Observe that the family $(C_{(f_W,d_{(\alpha)})})_{\alpha\in[0,1]}$ is a parametric family of Archimedean copulas continuous and decreasing in parameter α, with extremal elements $W = C_{(f_W,d_{(1)})}$ and $\Pi = C_{(f_W,d_{(0)})}$. Note that the generator $f_{(\alpha)}$ of $C_{(f_W,d_{(\alpha)})}$ for $\alpha < 1$ is given by $f_{(\alpha)}(x) = -\log(\alpha + (1-\alpha)x)$.

iii) For $\alpha \in]0,1[$, define $d_{\{\alpha\}}\colon [0,1] \to [0,1]$ by $d_{\{\alpha\}}(x) = \max(\alpha, x)$. Then $d_{\{\alpha\}} \in \mathcal{D}$, and the DUCS copula $C_{(f_W,d_{\{\alpha\}})}\colon [0,1]^2 \to [0,1]$ is given by

$$C_{(f_W,d_{\{\alpha\}})}(x,y) = \begin{cases} \max(0, \frac{x}{\alpha}(y-1+\alpha)) & \text{if} \quad x \in [0,\alpha], \\ W(x,y) & \text{elsewhere.} \end{cases} \tag{7}$$

Observe that this copula is a W-ordinal sum copula as introduced in [15], see also [2, 5], $C_{(f_W,d_{\{\alpha\}})} = W - (\langle 0, \alpha, \Pi\rangle)$.

4. PROPERTIES AND EXAMPLES OF DUCS COPULAS

DUCS copulas are based on generators from $\mathcal{F}$ and distortions from $\mathcal{D}$. The structure of $\mathcal{F}$, especially construction methods for generators, were deeply studied in [13]. Concerning the distortions set $\mathcal{D}$, we have the next important result.

Proposition 4.1. Let $A\colon [0,1]^n \to [0,\infty]$ be a continuous idempotent homogeneous aggregation function. Then for any $d_1, \cdots, d_n \in \mathcal{D}$, also the function $d\colon [0,1] \to [0,1]$ given by $d(x) = A(d_1(x), \cdots, d_n(x))$ is a distortion.

Proof. Evidently, d is a non-decreasing continuous function satisfying $d(x) \geq x$ for all $x \in [0,1]$. Moreover,

$$\tilde{d}(x) = \frac{x}{A(d_1(x), \cdots, d_n(x))} = \frac{x}{A\left(\frac{x}{\tilde{d}_1(x)}, \cdots, \frac{x}{\tilde{d}_n(x)}\right)} = \frac{1}{A\left(\frac{1}{\tilde{d}_1(x)}, \cdots, \frac{1}{\tilde{d}_n(x)}\right)}$$

for all $x \in]0,1]$ due to the homogenity of A (for more details on homogeneous aggregation function see [17]). The non-decreasigness of $\tilde{d}_1, \cdots, \tilde{d}_n$ ensures the non-decreasigness of $\tilde{d}$, and thus $d \in \mathcal{D}$. □

As a corollary of Proposition 4.1, $\mathcal{D}$ is a convex class which is also a lattice with top element d^* and bottom element d_*. Note that a similar conclusion holds for the class of DUCS copulas with a fixed generator f.

Corollary 4.2. Let $f \in \mathcal{F}$ and $d_1, d_2 \in \mathcal{D}$. For DUCS copulas $C_{(f,d_1)}$ and $C_{(f,d_2)}$, denote $C = C_{(f,d_1)} \vee C_{(f,d_2)}$ and $D = C_{(f,d_1)} \wedge C_{(f,d_2)}$. Then both C and D are DUCS copulas, $C = C_{(f,d_1\vee d_2)}$ and $D = C_{(f,d_1\wedge d_2)}$.

P r o o f. As already observed, $d_1 \wedge d_2$ and $d_1 \vee d_2$ are distortion. Then

$$\begin{aligned} C_{(f,d_1\vee d_2)}(x,y) &= xf^{(-1)}\left(\frac{f(y)}{d_1(x)\vee d_2(x)}\right) = xf^{(-1)}\left(\frac{f(y)}{d_1(x)}\wedge\frac{f(y)}{d_2(x)}\right) \\ &= x\left(f^{(-1)}\left(\frac{f(y)}{d_1(x)}\right)\vee f^{(-1)}\left(\frac{f(y)}{d_2(x)}\right)\right) = C(x,y) \end{aligned} \tag{8}$$

□

(recall that $f^{(-1)}$ is non-increasing). Similarly, the result for D can be shown.

For special distortions we can get interesting DUCS copulas.

Proposition 4.3. Let $d^{(\lambda)} \in \mathcal{D}$ be given by $d^{(\lambda)}(x) = x^{\frac{1}{\lambda}}, \lambda \in [1,\infty[$. Then, for any $f \in \mathcal{F}$, $C_{(f,d^{(\lambda)})} = C_{(f^\lambda)}$.

P r o o f. For $f \in \mathcal{F}$ and $\lambda \in [1,\infty[$, denote $g = f^\lambda$. Then $g^{-1}(x) = f^{-1}(x^{\frac{1}{\lambda}})$ for all $x \in \mathrm{Ran}\ g$, and thus

$$C_{(f^\lambda)}(x,y) = xf^{(-1)}\left(\left(\frac{f^\lambda(y)}{x}\right)^{\frac{1}{\lambda}}\right) = xf^{(-1)}\left(\frac{f(y)}{x^{\frac{1}{\lambda}}}\right) = C_{(f,d^{(\lambda)})}(x,y). \tag{9}$$

□

Proposition 4.4. Let $d_{[\alpha]} \in \mathcal{D}$ be given by $d_{[\alpha]}(x) = \frac{x}{\alpha} \wedge 1, \alpha \in]0,1]$. Note that $d_{[\alpha]} = \tilde{d}_{\{\alpha\}}$, (see Example 3.4 iii). Then, for any $f \in \mathcal{F}$, $C_{(f,d_{[\alpha]})} = g - (\langle 0, \alpha, C_f\rangle)$, i. e., DUCS copula $C_{(f,d_{[\alpha]})}$ is a g-ordinal sum.

P r o o f. It is enough to express both $C_{(f,d_{[\alpha]})}(x,y)$ and $g - (\langle 0, \alpha, C_{(f)}\rangle)(x,y)$. □

Remark 4.5. For any distortion $d \in \mathcal{D}$ and constant $\alpha \in]0,1]$, the function $d_{\langle\alpha\rangle}\colon [0,1] \to [0,1]$ given by $d_{\langle\alpha\rangle}(x) = \frac{d(\alpha x)}{d(\alpha)}$ is also a distortion (formally, conditional distortion), and $(\tilde{d}_{\langle\alpha\rangle})(x) = \frac{\tilde{d}(\alpha x)}{\tilde{d}(\alpha)}$, i. e., $(\tilde{d}_{\langle\alpha\rangle}) = (\tilde{d})_{\langle\alpha\rangle}$. After some processing concerning the univariate conditioning, see [14], it can be shown that the left conditioning with threshold α of DUCS copula $C_{(f,d)}$ is just the DUCS copula $C_{(f,d_{\langle\alpha\rangle})}$, i. e., $(C_{(f,d)})_{(\alpha)} = C_{(f,d_{\langle\alpha\rangle})}$.

Moreover, $d_{\langle\alpha\rangle} = d$ for all $\alpha \in]0,1]$ yields the Cauchy equation $d(\alpha x) = d(\alpha)d(x)$, with solution $d(x) = x^{\frac{1}{\lambda}}$, $\lambda \in [1,\infty[$, i. e. $d = d^{(\lambda)}$, see Proposition 4.3. Thus the only univariate conditioning invariant DUCS copulas are copulas $C_{(f,d^{(\lambda)})} = C_{(f^\lambda)}$.

5. CONCLUDING REMARKS

Based on convex generators from the class $\mathcal{F}$ (these functions are just generators of Archimedean copulas) and distortions from the class $\mathcal{D}$, we have introduced and discussed a new class of DUCS copulas. Each DUCS copula $C_{(f,d)}$ can be seen as a distorted copula C, which is invariant under univariate conditioning. However, there is also another class of generated copulas invariant under univariate conditioning,

namely copulas $C_{(\bar{f})} = C_{(g)}$ related with $C_{(f)}$, by the flipping transformation. Recall that for $f \in \mathcal{F}$, $\bar{f}, g\colon [0,1] \to [0,\infty]$ is given by $\bar{f}(x) = g(x) = f(1-x)$, and then $C_{(g)}(x,y) = xg^{(-1)}\left(\frac{g(y)}{x}\right) = x - C_{(f)}(x,1-y)$, i. e., $C_{(g)} = \mathrm{flip}(C_{(f)})$. Consider, for any DUCS copula $C_{(f,d)}$, its flipped copula $\mathrm{flip}(C_{(f,d)})$. Then $\mathrm{flip}(C_{(f,d)})(x,y) = x - C_{(f,d)}(x,1-y) = x - xf^{(-1)}\left(\frac{f(1-y)}{d(x)}\right) = x(1 - f^{(-1)}\left(\frac{f(1-y)}{d(x)}\right) = xg^{(-1)}\left(\frac{g(y)}{d(x)}\right)$, i. e., $\mathrm{flip}(C_{(f,d)}) = C_{(g,d)}$.

Hence the concept of DUCS copulas can be straightforwardly extended considering arbitrary generator of a copula invariant under univariate conditioning, as characterized in [4]. Finally note that the concept of DUCS copulas, based on two univariate functions $f \in \mathcal{F}$ and $d \in \mathcal{D}$, can be seen as a particular case of distortion of general copulas. Namely, based on [9, 11, 12], see also alternative proof of iii) of Proposition 3.1, for any copula C and distortion d, the function $C_{\{d\}}\colon [0,1]^2 \to [0,1]$ given by

$$C_{\{d\}}(x,y) = \tilde{d}(x) \cdot C(d(x),y) = x \cdot \frac{C(d(x),y)}{d(x)} \tag{10}$$

is a copula, and $C_{\{d^*\}} = \Pi$, $C_{\{d_*\}} = C$, i. e., the values of the distorted copula $C_{\{d\}}$ are always between the values of the product copula Π, and of the original copula C.

ACKNOWLEDGEMENT

This work was partially supported by APVV LPP-0004-07, VEGA 1/0496/08, APVV-0443-07.

The authors are grateful to both anonymous referees whose comments have enabled us to improve the original version of this paper.

(Received June 9, 2010)

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