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Włodzimierz M. Mikulski
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ON SYMMETRIZATION OF JETS

W. M. Mikulski, Kraków

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Dedicated to Professor Ivan Kolář on the occasion of his 75th birthday

Abstract. Let $F = F^{(A,H,t)}$ and $F^1 = F^{(A^1,H^1,t^1)}$ be fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibred manifolds Y with m-dimensional bases and fibred maps covering local diffeomorphisms. We define a quasi-morphism $(A,H,t) \to (A^1,H^1,t^1)$ to be a $GL(m)$-invariant algebra homomorphism $\nu\colon A \to A^1$ with $t^1 = \nu \circ t$. The main result is that there exists an $\mathcal{FM}_m$-natural transformation $FY \to F^1Y$ depending on a classical linear connection on the base of Y if and only if there exists a quasi-morphism $(A,H,t) \to (A^1,H^1,t^1)$. As applications, we study existence problems of symmetrization (holonomization) of higher order jets and of holonomic prolongation of general connections.

Keywords: jets, higher order connections, Ehresmann prolongation, Weil functors, bundle functors, natural operators

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0. Introduction

The classical theory of higher order jets was established by C. Ehresmann [6]. For nonholonomic and semiholonomic jets we refer to the paper [16] by P. Libermann. It is well-known that higher order jets are a very powerful tool in differential geometry and in mathematical physics. For example, holonomic jets globalize the theory of differential systems and semiholonomic jets play an important role in the calculus of variations and in the theory of partial differential equations, [22], [24]. Further, the theory of jets and connections forms the geometrical background for field theories and theoretical physics [18], [15]. The theory of higher order jets is closely connected with the theory of natural operations in differential geometry, [13]. Holonomic, semiholonomic and nonholonomic prolongation functors $J^r, \overline{J}^r, \tilde{J}^r$ on the category $\mathcal{FM}_m$ of fibred manifolds with m-dimensional bases and fibred maps covering local diffeomorphisms are "classical" examples of fiber product preserving bundle functors (i.e. bundle functors F on $\mathcal{FM}_m$ in the sense of [13] such that $F(Y_1 \times_M Y_2) = FY_1 \times_M FY_2$ for any $\mathcal{FM}_m$-objects $Y_1 \to M$ and $Y_2 \to M$).

We recall that a nonholonomic r-th order connection on a fibred manifold $p\colon Y \to M$ is a section $\Gamma\colon Y \to \tilde{J}^r Y$ of the nonholonomic r-th jet prolongation $\tilde{J}^r Y \to Y$ of $p\colon Y \to M$. It is called a semiholonomic or a holonomic r-th order connection if it has values in $\overline{J}^r Y$ or in $J^r Y$. For $r = 1$ we obtain the concept of general connections $\Gamma\colon Y \to J^1 Y$ on $p\colon Y \to M$. Higher order connections were introduced first on groupoids by Ehresmann [7]. Then I. Kolář [12] extended the concept of higher order connections to fibred manifolds.

Given an r-th order nonholonomic connection $\Gamma_1\colon Y \to \tilde{J}^r Y$ and an s-th order nonholonomic connection $\Gamma_2\colon Y \to \tilde{J}^s Y$ we have an $(r+s)$-th order nonholonomic connection $\Gamma_1 * \Gamma_2 := \tilde{J}^s \Gamma_1 \circ \Gamma_2\colon Y \to \tilde{J}^s \tilde{J}^r Y = \tilde{J}^{r+s} Y$. Now, by iteration, given a general connection $\Gamma\colon Y \to J^1 Y$ one can define $\Gamma^{(r-1)}\colon Y \to \tilde{J}^r Y$ by $\Gamma^{(1)} = \Gamma * \Gamma$, $\Gamma^{(r-1)} = \Gamma^{(r-2)} * \Gamma$ which is called the Ehresmann prolongation of $\Gamma\colon Y \to J^1 Y$. Clearly, $\Gamma^{(r-1)}$ is a semiholonomic r-th order connection, see [9]. However, the most important role in differential geometry and its applications in mathematical physics is played by classical holonomic jets and holonomic connections. That is why it is useful to study holonomic prolongations of connections.

The following results on holonomic prolongations are known.

If $r = 2$, we have the well-known symmetrization (holonomization) $C\colon \overline{J}^2 Y \to J^2 Y$ of second order semiholonomic jets, [10]. Composing $\Gamma^{(1)}\colon Y \to \overline{J}^2 Y$ with C we obtain a second order holonomic connection $A^2(\Gamma) := C \circ \Gamma^{(1)}\colon Y \to J^2 Y$.

In [2], M. Doupovec and the author proved that for $r \geqslant 3$ and $m \geqslant 2$ it is impossible to construct an r-th order holonomic connection $D(\Gamma)\colon Y \to J^r Y$ from a general connection $\Gamma\colon Y \to J^1 Y$. In particular, for $r \geqslant 3$ and $m \geqslant 2$ there is no symmetrization (holonomization) $\overline{J}^r Y \to J^r Y$. Further, the authors constructed an r-th order holonomic connection $A^r(\Gamma, \widetilde{\nabla})\colon Y \to J^r Y$ from a general connection $\Gamma\colon Y \to J^1 Y$ by means of a projectable classical linear connection $\widetilde{\nabla}$ on Y.

In [19], we constructed (in a rather complicated way) an r-th order holonomic connection $B^r(\Gamma, \nabla)\colon Y \to J^r Y$ from a general connection $\Gamma\colon Y \to J^1 Y$ by means of a classical linear connection ∇ on the base M.

The above facts show that it is useful to investigate the existence problem of symmetrization of higher order jets of Y by means of classical linear connections ∇ on the base M of Y. But higher order jet prolongation functors are fiber product preserving. That is, why it is useful to study the existence problem of natural transformations $FY \to F^1 Y$ depending on ∇ on M for arbitrary fiber product preserving bundle functors F and F^1 on $\mathcal{FM}_m$ instead of the higher order jet prolongation ones. The complete description of fiber product preserving bundle functors F on

$\mathcal{FM}_m$ in terms of the so called admissible triples (A, H, t) was given by I. Kolář and the author in [14] (see also [11]).

In the present paper we investigate the existence problem of natural transformations $\eta_{(Y,\nabla)}\colon FY \to F^1Y$ depending on ∇ on the base M of Y for arbitrary fiber product preserving bundle functors $F = F^{(A,H,t)}$ and $F^1 = F^{(A^1,H^1,t^1)}$ on $\mathcal{FM}_m$. We define a quasi-morphism $\nu\colon (A, H, t) \to (A^1, H^1, t^1)$ to be a $GL(m)$-invariant algebra homomorphism $\nu\colon A \to A^1$ such that $t^1 = \nu \circ t$. The main result we prove is that there exists an $\mathcal{FM}_m$-natural transformation $FY \to F^1Y$ depending on a classical linear connection ∇ on the base of Y if and only if there exists a quasi-morphism $(A, H, t) \to (A^1, H^1, t^1)$. As applications we study existence problems of symmetrization of higher order jets and of holonomic prolongation of general connections.

All manifolds and maps are assumed to be of C^∞. Manifolds are assumed to be finite dimensional paracompact and without boundaries.

The main result

A finite dimensional real commutative associative algebra A with unity is called a Weil algebra of order r if it is of the form $A = \mathbb{R} \cdot 1 \oplus N$, where N is a nilpotent ideal with $N^{r+1} = \{0\}$.

In [23], A. Weil constructed the functor $T^A\colon \mathcal{M}f \to \mathcal{FM}$ of near A-points for any Weil algebra A. T^A is a product preserving bundle functor (ppb-functor). Any algebra homomorphism $\mu\colon A \to B$ of Weil algebras can be extended to the natural transformation $\mu\colon T^A \to T^B$ of ppb-functors.

It turned out that any ppb-functor F on manifolds is of the form $F = T^A$ for the Weil algebra $A = F\mathbb{R}$, and natural transformations $\mu_M\colon T^AM \to T^BM$ are in bijection with algebra homomorphisms $\mu = \mu_{\mathbb{R}}\colon A = T^A\mathbb{R} \to B = T^B\mathbb{R}$. This result was proved (independently) by Eck [5], Kainz and Michor [8], and Luciano [17].

An admissible triple of order r and dimension m is (by definition) a system (A, H, t), where A is a Weil algebra of order r, $H\colon G^r_m \to \mathrm{Aut}(A)$ is a Lie group homomorphism from the r-th order differential Lie group $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ into the Lie group $\mathrm{Aut}(A)$ of algebra automorphisms of A (i.e. H is an algebra action of G^r_m on A) and $t\colon \mathcal{D}^r_m \to A$ is a G^r_m-invariant algebra homomorphism from the Weil algebra $\mathcal{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ (with the pull-back action H^r_m of G^r_m on $\mathcal{D}^r_m$) into A (with the action H of G^r_m on A). A morphism $\nu\colon (A, H, t) \to (A^1, H^1, t^1)$ of admissible triples of order r and dimension m is a G^r_m-invariant algebra homomorphism $\nu\colon A \to A^1$ with $t^1 = \nu \circ t$.

In [14], I. Kolář and the author constructed a bundle functor $F^{(A,H,t)}\colon \mathcal{FM}_m \to \mathcal{FM}$ of order r for any admissible triple (A, H, t) of order r and dimension m as follows. Every $\mathcal{FM}_m$-object $p\colon Y \to M$ defines the bundle

$$F^{(A,H,t)}Y := \{\langle u, X\rangle \in P^rM[T^AY, H_Y]\colon\ t_M(u) = T^Ap(X)\}$$

over Y, where $P^rM \subset T^r_mM$ is the principal bundle of frames of M of order r with the standard group G^r_m, $t_M\colon T^r_mM = T^{\mathcal{D}^r_m}M \to T^AM$ is the extension of t, T^AY is the Weil bundle of near A-points and $P^rM[T^AY, H_Y]$ is the associated bundle with the standard fiber T^AY and the left action $H_Y\colon G^r_m \times T^AY \to T^AY$ by $H_Y(\xi, X) = H(\xi)_Y(X)$. Every $\mathcal{FM}_m$-map $f\colon Y_1 \to Y_2$ over $\underline{f}$ induces $F^{(A,H,t)}f\colon F^{(A,H,t)}Y_1 \to F^{(A,H,t)}Y_2$ by

$$F^{(A,H,t)}f(\langle u, X\rangle) := \left\langle P^r\underline{f}(u), T^Af(X)\right\rangle,$$

$\langle u, X\rangle \in F^{(A,H,t)}Y_1$, $u = j^r_0\varphi \in P^rM_1$, $X \in T^AY_1$, $P^r\underline{f}(j^r_0\varphi) = j^r_0(\underline{f}\circ\varphi) \in P^rM_2$. The bundle functor $F^{(A,H,t)}$ is fiber product preserving.

It turned out that any fiber product preserving bundle functor (fppb-functor) F of order r on $\mathcal{FM}_m$ is of the form $F = F^{(A,H,t)}$ for an admissible triple (A, H, t) of order r and dimension m, and the natural transformations $\nu\colon F^{(A,H,t)} \to F^{(A^1,H^1,t^1)}$ of fppb-functors are in bijection with morphisms $\nu\colon (A, H, t) \to (A^1, H^1, t^1)$ of admissible triples. This result was proved in [14] (see also [11]).

Let $F, F^1\colon \mathcal{FM}_m \to \mathcal{FM}$ be fppb-functors. An $\mathcal{FM}_m$-natural transformation $\eta_{(Y,\nabla)}\colon FY \to F^1Y$ depending on classical linear connections ∇ on bases M of $\mathcal{FM}_m$-objects $p\colon Y \to M$ is by definition an $\mathcal{FM}_m$-natural operator $\eta = \{\eta_Y\}\colon Q \circ \mathcal{B} \rightsquigarrow (F, F^1)$ in the sense of [13], i.e. η is a family of $\mathcal{FM}_m$-invariant regular operators $\eta_Y\colon QM \to C^\infty_Y(FY, F^1Y)$ for all $\mathcal{FM}_m$-objects $p\colon Y \to M$, where QM is the set of classical linear connections on M and $C^\infty_Y(FY, F^1Y)$ is the set of all fibred maps $FY \to F^1Y$ covering the identity map of Y. If we replace (in the above definition) $\mathcal{FM}_m$ by the category $\mathcal{FM}_{m,n}$ of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred local isomorphisms we obtain the concept of $\mathcal{FM}_{m,n}$-natural transformations $\eta_{(Y,\nabla)}\colon FY \to F^1Y$ depending on classical linear connections on bases of $\mathcal{FM}_{m,n}$-objects.

We define a quasi-morphism $\nu\colon (A, H, t) \to (A^1, H^1, t^1)$ of admissible triples of order r to be a $GL(m)$-invariant algebra homomorphism $\nu\colon A \to A^1$ such that $t^1 = \nu \circ t$, where $GL(m) \subset G^r_m$ is the group of linear automorphisms of $\mathbb{R}^m$. (Clearly, a quasi-morphism ν of admissible triples is a morphism of admissible triples if it is G^r_m-invariant.)

The following theorem is the main result of the present paper.

Theorem 1. *Let $F = F^{(A,H,t)}$ and $F^1 = F^{(A^1,H^1,t^1)}$ be fppb-functors of order r on $\mathcal{FM}_m$. Then the following conditions are equivalent:*

(a) *There is an $\mathcal{FM}_m$-natural transformation $FY \to F^1Y$ depending on classical linear connections on the base of Y.*

(b) *There is a quasi-morphism $(A,H,t) \to (A^1,H^1,t^1)$.*

(c) *For some $n \geqslant 1$, there is an $\mathcal{FM}_{m,n}$-natural transformation $FY \to F^1Y$ depending on classical linear connections on the base of Y.*

(d) *For any $n \geqslant 1$, there is an $\mathcal{FM}_{m,n}$-natural transformation $FY \to F^1Y$ depending on classical linear connections on the base of Y.*

Proof. The proof of Theorem 1 will occupy the rest of this section. □

Let M be an m-dimensional manifold and $u = j^r_0\varphi \in P^rM$ a frame of order r with $\varphi(0) = x \in M$. Let ∇ be a classical linear connection on M. Let $\underline{f}$ be a ∇-normal coordinate system on M with center x such that $j^1_0(\underline{f} \circ \varphi) = j^1_0(\mathrm{id})$. The germ of $\underline{f}$ at x is uniquely determined. Denote $\xi(u,\nabla) := j^r_0(\underline{f} \circ \varphi) \in G^r_m$. We prove the following lemma.

Lemma 1.

(i) *Let M be an m-manifold and ∇ a classical linear connection on M. Let $u = j^r_0\varphi \in P^rM$ and $u_1 = j^r_0\varphi \circ j^r_0\psi^{-1} \in P^rM$, $j^r_0\psi \in G^r_m$, $B = T_0\psi \in GL(m)$. Then $\xi(u_1,\nabla) = j^r_0B \circ \xi(u,\nabla) \circ j^r_0\psi^{-1}$.*

(ii) *Let M and M_1 be m-manifolds and let ∇ and ∇_1 be classical linear connections on M and M_1, respectively, and let $\underline{g}\colon M \to M_1$ be (∇,∇_1)-affine local diffeomorphism. Let $u = j^r_0\varphi \in P^rM$ and $u_1 = P^r\underline{g}(u) = j^r_0(\underline{g} \circ \varphi) \in P^rM_1$. Then $\xi(u_1,\nabla_1) = \xi(u,\nabla)$.*

(iii) *If $\{\nabla_t\}$ is a smoothly parametrized family of classical linear connections on M, then $\xi(u,\nabla_t)$ is smooth in $(t,u) \in \mathbb{R} \times P^rM$.*

Proof of Lemma 1. ad (i). Let $\underline{f}$ be a ∇-normal coordinate system on M with center $x = \varphi(0)$ such that $j^1_0(\underline{f} \circ \varphi) = j^1_0(\mathrm{id})$. Then $B \circ \underline{f}$ is a ∇-normal coordinate system on M with center x such that $j^1_0((B \circ \underline{f}) \circ (\varphi \circ \psi^{-1})) = j^1_0(\mathrm{id})$. Then $\xi(u_1,\nabla) = j^r_0((B\circ\underline{f})\circ(\varphi\circ\psi^{-1})) = j^r_0B\circ j^r_0(\underline{f}\circ\varphi)\circ j^r_0\psi^{-1} = j^r_0B\circ\xi(u,\nabla)\circ j^r_0\psi^{-1}$.

ad (ii) Let $\underline{f}$ be a ∇-normal coordinate system on M with center $x = \varphi(0)$ such that $j^1_0(\underline{f} \circ \varphi) = j^1_0(\mathrm{id})$. Then $\underline{f} \circ g^{-1}$ is a ∇_1-normal coordinate system on M_1 with center $x_1 = \underline{g} \circ \varphi(0)$ such that $j^1_0((\underline{f} \circ g^{-1}) \circ (\underline{g} \circ \varphi)) = j^1_0(\mathrm{id})$. Then $\xi(u_1,\nabla_1) = j^r_0((\underline{f} \circ \underline{g}^{-1}) \circ (\underline{g} \circ \varphi)) = j^r_0(\underline{f} \circ \varphi) = \xi(u,\nabla)$.

ad (iii) It follows from the easy to see fact that the map $\underline{f}$ in the definition of $\xi(u,\nabla)$ is $\underline{f} = (T_0\varphi)^{-1} \circ (Exp^{\nabla}_x)^{-1}$.

The proof of of Lemma 1 is complete. □

We continue the proof of Theorem 1. That (b) implies (a) is an immediate consequence of the following example.

Example 1. Let $\nu\colon (A,H,t)\to(A^1,H^1,t^1)$ be a quasi-morphism of admissible triples. For any $\mathcal{FM}_m$-object $p\colon Y\to M$ and any classical linear connection ∇ on M we define a map $\overline{\nu}_{(Y,\nabla)}\colon F^{(A,H,t)}Y\to P^rM[T^{A^1}Y,H^1_Y]$ by

$$\overline{\nu}_{(Y,\nabla)}(w):=\big\langle u,H^1_Y(\xi^{-1})\circ\nu_Y\circ H_Y(\xi)(X)\big\rangle,$$

$w=\langle u,X\rangle\in F^{(A,H,t)}Y$, $u=j^r_0\varphi\in P^rM$, $X\in T^A_yY$, $y\in Y$, where $\xi=\xi(u,\nabla)$ is as in Lemma 1, $\nu_Y\colon T^AY\to T^{A^1}Y$ is the extension of $\nu\colon A\to A^1$ and $H_Y(\xi)\colon T^AY\to T^AY$ and $H^1_Y(\xi)\colon T^{A^1}Y\to T^{A^1}Y$ are the extensions $H(\xi)_Y$ and $H^1(\xi)_Y$ of $H(\xi)\in\mathrm{Aut}(A)$ and $H^1(\xi)\in\mathrm{Aut}(A^1)$. If $w=\langle u_1,X_1\rangle$ is another representation of w, then $u_1=j^r_0\varphi\circ j^r_0\psi^{-1}$ and $X_1=H_Y(j^r_0\psi)(X)$ for some $j^r_0\psi\in G^r_m$. Denote $\xi_1=\xi(u_1,\nabla)$. By Lemma 1 (i), $\xi_1=j^r_0B\circ\xi\circ j^r_0\psi^{-1}$, where $B=T_0\psi\in GL(m)$. Then

$$H^1_Y(\xi_1^{-1})\circ\nu_Y\circ H_Y(\xi_1)(X_1)=H^1_Y(j^r_0\psi)\circ H^1_Y(\xi^{-1})\circ\nu_Y\circ H_Y(\xi)(X)$$

because of $H^1_Y(j^r_0B^{-1})\circ\nu_Y\circ H_Y(j^r_0B)=\nu_Y$ as ν is $GL(m)$-invariant. That is why $\overline{\nu}_{(Y,\nabla)}(w)$ is well-defined. We have $t^1=\nu\circ t$ and (since $w\in F^{(A,H,t)}Y$) $t_M(u)=T^Ap(X)$. Then

$$\begin{aligned}t^1_M(u)&=H^1_M(\xi^{-1})\circ t^1_M\circ(H^r_m)_M(\xi)(u)=H^1_M(\xi^{-1})\circ\nu_M\circ t_M\circ(H^r_m)_M(\xi)(u)\\&=H^1_M(\xi^{-1})\circ\nu_M\circ H_M(\xi)\circ t_M(u)=H^1_M(\xi^{-1})\circ\nu_M\circ H_M(\xi)\circ T^Ap(X)\\&=T^{A^1}p\circ H^1_Y(\xi^{-1})\circ\nu_Y\circ H^1_Y(\xi)(X).\end{aligned}$$

Moreover, $H^1_Y(\xi^{-1})\circ\nu_Y\circ H_Y(\xi)(X)\in T^{A^1}_yY$ (as natural transformations of ppb-functors on $\mathcal{M}f$ covering the identity map of Y). Then $\overline{\nu}_{(Y,\nabla)}(w)\in F^{(A^1,H^1,t^1)}_yY$. Hence $\overline{\nu}_{(Y,\nabla)}\colon F^{(A,H,t)}Y\to F^{(A^1,H^1,t^1)}Y$ is a fibred map covering the identity map of Y. It is smooth because of Lemma 1(iii). Even $\{\overline{\nu}_{(Y,\nabla_t)}\}$ is a smoothly parametrized family if $\{\nabla_t\}$ is. To prove $\mathcal{FM}_m$-invariance of $\overline{\nu}_{(Y,\nabla)}$, we consider $\mathcal{FM}_m$-objects $p\colon Y\to M$ and $p_1\colon Y_1\to M_1$, connections $\nabla\in QM$ and $\nabla_1\in QM_1$ and an $\mathcal{FM}_m$-map $g\colon Y\to Y_1$ covering (∇,∇_1)-affine local diffeomorphism $\underline{g}\colon M\to M_1$, and verify $F^{(A^1,H^1,t^1)}g\circ\overline{\nu}_{(Y,\nabla)}=\nu_{(Y_1,\nabla_1)}\circ F^{(A,H,t)}g$ as follows. Let $w=\langle u,X\rangle\in F^{(A,H,t)}Y$, $u=j^r_0\varphi\in P^rM$, $X\in T^AY$. Let $u_1=P^r\underline{g}(u)\in P^rM_1$. By Lemma 1(ii), $\xi(u_1,\nabla_1)=\xi(u,\nabla)$. Denote $\xi=\xi(u,\nabla)=\xi(u_1,\nabla_1)$. Then

$$\begin{aligned}F^{(A^1,H^1,t^1)}g\circ\overline{\nu}_{(Y,\nabla)}(w)&=F^{(A^1,H^1,t^1)}g(\big\langle u,H^1_Y(\xi^{-1})\circ\nu_Y\circ H_Y(\xi)(X)\big\rangle)\\&=\langle u_1,T^{A^1}g\circ H^1_Y(\xi^{-1})\circ\nu_Y\circ H_Y(\xi)(X)\rangle\\&=\big\langle u_1,H^1_{Y_1}(\xi)\circ\nu_{Y_1}\circ H_{Y_1}\circ T^Ag(X)\big\rangle\\&=\overline{\nu}_{(Y_1,\nabla_1)}(\big\langle u_1,T^Ag(X)\big\rangle)=\overline{\nu}_{(Y_1,\nabla_1)}\circ F^{(A,H,t)}g(w).\end{aligned}$$

So, $\overline{\nu}_{(Y,\nabla)}\colon F^{(A,H,t)}Y \to F^{(A^1,H^1,t^1)}Y$ is an $\mathcal{FM}_m$-natural transformation depending on classical linear connections ∇ on the base of Y.

We continue the proof of Theorem 1. That (a) implies (d) is clear. Similarly, that (d) implies (c) is clear, too.

The proof of the fact that (c) implies (b) is a direct modification of the corresponding part of the proof of Theorem 2 in [20]. More precisely, for an $\mathcal{FM}_{m,n}$-natural transformation $\eta\colon F \to F^1$ depending on ∇ we define $(\eta_1,\dots,\eta_n) := \eta_{(\mathbb{R}^m\times\mathbb{R}^n,\nabla^0)|F_0(\mathbb{R}^m\times\mathbb{R}^n)}\colon A^n \to (A^1)^n$, where ∇^0 is the usual flat classical linear connection on $\mathbb{R}^m$. Then, similarly to Steps 1–3 of the proof of Theorem 2 in [20], we can prove that $\sigma\colon A\to A^1$, $\sigma(a) := \eta_1(a,0,\dots,0)$, $a\in A$ is a quasi-morphism $(A,H,t)\to(A^1,H^1,t^1)$.

The proof of Theorem 1 is complete. □

2. On symmetrization of jets

Let $\overline{J}^r$ and J^r be respectively the semiholonomic and holonomic r-jet prolongation functors on $\mathcal{FM}_m$. Since $\overline{J}^r$ and J^r are fiber product preserving bundle functors on $\mathcal{FM}_m$, we can write $\overline{J}^r = F^{(\bar{A}^r,\overline{H}^r,\bar{t}^r)}$ and $J^r = F^{(A^r,H^r,t^r)}$. Vector $GL(m)$-spaces $\bar{A}^r$ and A^r (with respect to $\overline{H}^r_{|GL(m)}$ and $H^r_{|GL(m)}$) are of the form

$$\bar{A}^r = \bigoplus_{k=0}^{r} \otimes^k \mathbb{R}^{m*} \quad \text{and} \quad A^r = \bigoplus_{k=0}^{r} S^k\mathbb{R}^{m*}$$

with the standard tensor actions of $GL(m)$ (this is an easy observation, e.g. by the standard coordinate description). The algebra multiplications of $\bar{A}^r$ and A^r will be denoted by $\cdot$ (they are given by rather complicated formulas, which will not be used in the sequel). Clearly, the obvious inclusion $i\colon A^r\to\bar{A}^r$ is a morphism $i\colon (A^r,H^r,t^r)\to(\bar{A}^r,\overline{H}^r,\bar{t}^r)$ of admissible triples (the one corresponding to the inclusion $J^rY\subset\overline{J}^rY$).

Lemma 2. *Let $C_1,C_2\colon \bar{A}^r\times\bar{A}^r\to A^r$ be $GL(m)$-invariant maps such that $C_{1|A^r\times A^r} = C_{2|A^r\times A^r}$. Then $C_1=C_2$.*

Proof. We have to show that $\langle C_1(\bar{u}_1,\bar{u}_2),w\rangle = \langle C_2(\bar{u}_1,\bar{u}_2),w\rangle$ for any $\bar{u}_1,\bar{u}_2\in\bar{A}^r$ and any $w\in S^k\mathbb{R}^m$ for $k=0,\dots,r$. Because of the $GL(m)$-invariance of C_1 and C_2 we can assume that $w=\odot^k e_1\in S^k\mathbb{R}^m$, where $e_1 = (1,0,\dots,0)\in\mathbb{R}^m$, $k=0,\dots,r$. Using the invariance of C_1 and C_2 with respect to $a_t\colon \mathbb{R}^m\to\mathbb{R}^m$, $a_t(x_1,\dots,x_m) = (x_1,tx_2,\dots,tx_m)$ for $t>0$ we obtain

$\langle C_i(\bar u_1,\bar u_2),w\rangle = \langle C_i((a_t)^*\bar u_1,(a_t)^*\bar u_2),w\rangle$ for $i=1,2$ and any $t>0$ (as a_t preserves $w=\odot^k e_1$). Putting $t\to 0$ we get $\langle C_i(\bar u_1,\bar u_2),w\rangle = \big\langle C_i(u_1^0,u_2^0),w\big\rangle$ for $i=1,2$, where $u_i^0=\lim\limits_{t\to 0}((a_t)^*\bar u_i)\in A^r$. Then $C_1(u_1^0,u_2^0)=C_2(u_1^0,u_2^0)$ completes the proof. □

Using Lemma 2, we prove the following proposition.

Proposition 1. *Let $s\colon \bar A^r\to A^r$ be the usual symmetrization. Then $s\colon (\bar A^r,\overline{H}^r,\bar t^r)\to(A^r,H^r,t^r)$ is a quasi-morphism of admissible triples.*

Proof. Of course, s is $GL(m)$-invariant $\mathbb{R}$-linear and $s(1)=1$. Moreover, since $s\circ i=\mathrm{id}_{A^r}$ and $\bar t^r=i\circ t^r$ (as $i\colon (A^r,H^r,t^r)\to(\bar A^r,\overline{H}^r,\bar t^r)$ is a morphism of admissible triples corresponding to the inclusion $J^rY\subset \overline{J}^rY$), we have $s\circ\bar t^r=t^r$. To prove that s is multiplicative, we define two maps $C_1,C_2\colon \bar A^r\times\bar A^r\to A^r$ by

$$C_1(\bar u_1,\bar u_2):=s(\bar u_1)\cdot s(\bar u_2)\qquad\text{and}\qquad C_2(\bar u_1,\bar u_2):=s(\bar u_1\cdot\bar u_2)\ .$$

They are $GL(m)$-invariant because of $\overline{H}^r\colon G^r_m\to\mathrm{Aut}(\bar A^r)$ and $H^r\colon G^r_m\to\mathrm{Aut}(A^r)$ and s is $GL(m)$-invariant. Moreover, $C_{1|A^r\times A^r}=C_{2|A^r\times A^r}$ because of $s\circ i=\mathrm{id}_{A^r}$. Then $C_1=C_2$ because of Lemma 2. But this means that s is multiplicative. Summing up, we see that $s\colon \bar A^r\to A^r$ is a $GL(m)$-invariant algebra homomorphism such that $s\circ\bar t^r=t^r$. In other words, s is a quasi-morphism of admissible triples in question. □

So, one can symmetrize semiholonomic jets by means of classical linear connections on the base. Namely, we have the following example.

Example 2. Given an $\mathcal{FM}_m$-object $p\colon Y\to M$ and a classical linear connection ∇ on M we have the fibred map $\bar s_{(Y,\nabla)}\colon \overline{J}^rY\to J^rY$ over the identity of Y corresponding to the symmetrization $s\colon \bar A^r\to A^r$ (see Example 1 for $\nu=s$). One can easily see that for $r=2$, $\bar s_{(Y,\nabla)}\colon \overline{J}^2Y\to J^2Y$ is independent of ∇ and equal to the classical symmetrization $\overline{J}^2Y\to J^2Y$ as in [10].

For non-holonomic jets, we have the following non-existence result.

Proposition 2. *Let $(\tilde A^r,\tilde H^r,\tilde t^r)$ and (A^r,H^r,t^r) be the admissibles triple corresponding to the non-holonomic and holonomic jet fppb-functors $\tilde J^r,J^r\colon \mathcal{FM}_m\to\mathcal{FM}$, respectively. For $r\geqslant 2$ and $m\geqslant 1$, there is no quasi-morphism*

$$\nu\colon (\tilde A^r,\tilde H^r,\tilde t^r)\to(A^r,H^r,t^r).$$

Consequently, for $r\geqslant 2$ we cannot symmetrize nonholonomic r-jets by means of classical linear connections on the base. More precisely, for $r\geqslant 2$ and $m\geqslant 1$ and

$n \geqslant 1$ *there is no* $\mathcal{FM}_{m,n}$*-natural transformation* $\eta_{(Y,\nabla)}\colon \tilde{J}^rY \to J^rY$ *depending on classical linear connections* ∇ *on the base of* Y.

S k e t c h o f t h e p r o o f. We have $\tilde{A}^r = \otimes^r A^1$ (as an algebra) and $\tilde{H}^r_{|GL(m)} = \otimes^r(H^1_{|GL(m)})$ (see, e.g., Subsection 4.5 in [11]). Then $\tilde{A}^r$ is generated by elements $a^{\langle i\rangle} = 1 \otimes \ldots \otimes a \otimes \ldots \otimes 1$ for $a \in A^1$ with $a^2 = 0$ and $i = 1, \ldots, r$, where a is in the i-th position. We have $(a^{\langle i\rangle})^2 = 0$ and $\tilde{H}^r(j^r_0(t\mathrm{id}_{\mathbb{R}^m}))(a^{\langle i\rangle}) = t^{-1}a^{\langle i\rangle}$. Suppose that ν is such a quasi-morphism. Then $H^r(j^r_0(t\mathrm{id}_{\mathbb{R}^m}))(\nu(a^{\langle i\rangle})) = t^{-1}\nu(a^{\langle i\rangle})$ (as ν is $GL(m)$-invariant) and $(\nu(a^{\langle i\rangle}))^2 = 0$ and $\nu(a^{\langle i\rangle}) \in A^r = \mathcal{D}^r_m$. Hence $\nu(a^{\langle i\rangle}) = 0$ as $r \geqslant 2$. Then $\nu\colon \tilde{A}^r \to \mathbb{R} \subset A^r$ is the trivial algebra homomorphism. On the other hand, $t^r = \mathrm{id}\colon \mathcal{D}^r_m \tilde{=} A^r \to A^r$ and $t^r = \nu \circ \tilde{t}^r$, a contradiction. The additional part of the proposition is an immediate consequence of the first and Theorem 1. $\square$

Modifying accordingly (almost directly) the proof of Proposition 2 we can even get the following more strict result.

Proposition 3. *For* $r \geqslant 2$ *and* $m \geqslant 1$ *and* $n \geqslant 1$*, there is no* $\mathcal{FM}_{m,n}$*-natural transformation* $\eta_{(Y,\nabla)}\colon J^1J^{r-1}Y \to J^rY$ *depending on classical linear connections* ∇ *on the base of* Y.

Remark 1. It is an easy observation that for $r \geqslant 3$ and $m \geqslant 1$, the symmetrization $s\colon \bar{A}^r \to A^r$ is not G^r_m-invariant. So, for $r \geqslant 3$ and $m \geqslant 2$, there is no morphism ν of the admissible triples of $\overline{J}^r$ and J^r. (Otherwise, since $\nu \circ \bar{t}^r = t^r = \mathrm{id}\colon \mathcal{D}^r_m \tilde{=} A^r \to A^r$, we have $\nu_{|A^r} = \mathrm{id}_{A^r} = s_{|A^r}$, and then $\nu = s$ because of an obvious modification of Lemma 2.) Consequently, for $r \geqslant 3$ and $m \geqslant 2$ there is no natural transformation $\overline{J}^r \to J^r$. In other words, if $r \geqslant 3$ and $m \geqslant 2$, then to symmetrize semiholonomic jets of $p\colon Y \to M$, an auxiliary geometric object is unavoidable. This fact has been also deduced in [2] by using other arguments. For $r = 2$, the symmetrization $s\colon \bar{A}^2 \to A^2$ is G^2_m-invariant, and then s is a morphism of admissible triples (it corresponds to the classical independent of ∇ symmetrization $\overline{J}^2Y \to J^2Y$).

Remark 2. In [2] we proposed a symmetrization of nonholonomic jets by means of projectable classical linear connections ∇ on Y. From Proposition 2 it follows that using projectable classical linear connections on Y (or other objects different from classical linear connections on the base of Y) to symmetrize nonholonomic jets of Y is unavoidable.

Remark 3. The bundle $\overline{J}^{r,r-1}Y := J^1J^{r-1}Y \cap \overline{J}^rY$ is called the special r-jet prolongation of Y. In [10], Kolář presented a symmetrization $\overline{J}^{r,r-1}Y \to J^rY$ of special r-jets without using any additional geometric object. In contrast, because of

Proposition 3, to symmetrize jets from $J^1 J^{r-1}Y$ even classical linear connections on the base of Y do not suffice.

3. On holonomic prolongation of general connections

In [19], we constructed (in a rather complicated way) an r-th order holonomic connection $B^r(\Gamma,\nabla)\colon Y \to J^rY$ from a general connection $\Gamma\colon Y \to J^1Y$ by means of a torsion free classical linear connection ∇ on the base M.

Moreover, Theorem 1 in [19] says that there is only one canonical construction (i.e. the respective natural operator) of an r-th order holonomic connection $D(\Gamma,\nabla)\colon Y \to J^rY$ from a general connection $\Gamma\colon Y \to J^1Y$ by means of a torsion-free classical linear connection ∇ on the base of Y.

Now, using the symmetrization $\bar{s}_{(Y,\nabla)}\colon \overline{J}^rY \to J^rY$ from Example 2 we can construct (reobtain) the r-th order holonomic connection $B^r(\Gamma,\nabla)\colon Y \to J^rY$ in the following elegant way.

Example 3. Let $\Gamma\colon Y \to J^1Y$ be a general connection on a fibred manifold $Y \to M$ and let ∇ be a classical linear connection on M. We define an r-th order holonomic connection

$$D^r(\Gamma,\nabla) := \bar{s}_{(Y,\nabla)} \circ \Gamma^{(r-1)}\colon Y \to J^rY \ ,$$

where $\bar{s}_{(Y,\nabla)}\colon \overline{J}^rY \to J^rY$ is the symmetrization (from Example 2) of semiholonomic r-jets and $\Gamma^{(r-1)}\colon Y \to \overline{J}^rY$ is the r-th order semiholonomic Ehresmann prolongation of Γ.

Because of the above mentioned uniqueness result from [19] we get

Proposition 4. *If ∇ is torsion-free, then $D^r(\Gamma,\nabla) = B^r(\Gamma,\nabla)$, where $D^r(\Gamma,\nabla)$ is as in Example 3 and $B^r(\Gamma,\nabla)$ is as in Example 5 in [19].*

4. A final remark

Let $p\colon Y \to M$ be an $\mathcal{FM}_m$-object and ∇ a classical linear connection on M. In [21], M. Modugno defined an involution $e_\nabla\colon J^1J^1Y \to J^1J^1Y$ depending on ∇. In [3] and [4], M. Doupovec and the author defined "exchange isomorphisms" $(A^{r,s}_\nabla)_Y\colon J^rJ^sY \to J^sJ^rY$ and $(B^{r,s}_\nabla)_Y\colon \tilde{J}^r\tilde{J}^sY \to \tilde{J}^s\tilde{J}^rY$ depending on ∇. Applying $(A^{r,s}_\nabla)_Y$, we can lift s-order holonomic connections $\Gamma\colon Y \to J^sY$ to s-th order holonomic connections $A(\Gamma,\nabla) := (A^{r,s}_{\nabla})_Y \circ J^r\Gamma\colon J^rY \to J^sJ^rY$ on $J^rY \to M$, see [3].

It seems to be very probable that using our general construction of $\overline{\nu}_{(Y,\nabla)}$ from Example 1, we could produce "exchange isomorphisms" $FJ^sY \to J^sFY$ or (even) $FGY \to GFY$ depending on ∇ for many fppb-functors F and G on $\mathcal{FM}_m$. Maybe the "exchange isomorphism" (or eventually some other isomorphism) $A^F \otimes A^G \to A^G \otimes A^F$ is a quasi-morphism of admissible triples of FG and GF for many fppb-functors F and G. At this moment, we do not know if it is really true. The admissible triple of FG depends on the admissible triples of F and G in a rather complicated way, see [1] (or [11]). Clearly, the (hypothetic) "exchange isomorphisms" $FJ^sY \to J^sFY$ could be used to produce s-th order holonomic connections on $FY \to M$ from s-th order holonomic connections on $Y \to M$ by means of ∇.

In [19], we proposed a quite different general construction of r-th order holonomic connections $\mathcal{F}^r_q(\Theta,\nabla)\colon FY \to J^rFY$ on $FY \to M$ from q-th order holonomic connections $\Theta\colon Y \to J^qY$ on $Y \to M$ by means of ∇ for any bundle functor F of order k on the category $\mathcal{FM}_{m,n}$ of (m,n)-dimensional fibred manifolds and their local fibred diffeomorphisms.

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Author's address: W. M. Mikulski, Institute of Mathematics, Jagiellonian University, Lojasiewicza 6, Kraków, Poland, e-mail: Wlodzimierz.Mikulski@im.uj.edu.pl.