Bálint Farkas

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ADJOINT BI-CONTINUOUS SEMIGROUPS AND SEMIGROUPS ON THE SPACE OF MEASURES

Bálint Farkas, Darmstadt

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Abstract. For a given bi-continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X we define its adjoint on an appropriate closed subspace X° of the norm dual X'. Under some abstract conditions this adjoint semigroup is again bi-continuous with respect to the weak topology $\sigma(X^{\circ}, X)$. We give the following application: For Ω a Polish space we consider operator semigroups on the space $\mathrm{C_b}(\Omega)$ of bounded, continuous functions (endowed with the compact-open topology) and on the space $\mathrm{M}(\Omega)$ of bounded Baire measures (endowed with the weak*-topology). We show that bi-continuous semigroups on $\mathrm{M}(\Omega)$ are precisely those that are adjoints of bi-continuous semigroups on $\mathrm{C_b}(\Omega)$. We also prove that the class of bi-continuous semigroups on $\mathrm{C_b}(\Omega)$ with respect to the compact-open topology coincides with the class of equicontinuous semigroups with respect to the strict topology. In general, if Ω is not a Polish space this is not the case.

Keywords: not strongly continuous semigroups, bi-continuous semigroups, adjoint semigroup, mixed-topology, strict topology, one-parameter semigroups on the space of measures

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1. Introduction

The probably simplest example of a semigroup on the space $\mathrm{C_b}(\mathbb{R})$, namely the shift semigroup, fails to be strongly continuous and even measurable with respect to the sup-norm. To overcome this and a probably more important fact, namely the failure of strong continuity of many transition semigroups, several different approaches have been developed, such as those of π-semigroups by Priola [28] or weakly continuous semigroups by Cerrai [5]. One could even simply consider other locally convex topologies on $\mathrm{C_b}(\mathbb{R})$ than the sup-norm-topology as, e.g., was done by Dorroh, Neuberger [6]. A more recent abstract approach is that of *integrable semigroups on norming dual pairs* due to Kunze, see [16], [17], and [18]. In this paper, we give preference to the notion of *bi-continuous semigroups* initiated by Kühnemund [19], [20].

The reason for prefering this class of semigroups is the fact that there is already a vast abstract theory developed for them: there are generation, approximation and perturbation results (see Kühnemund [19], [20] and Farkas [11], [12]). Even the Hille-Phillips functional calculus was formulated in this setting, and was used to prove convergence rates for rational approximation schemes and for efficient Laplace inversion formulas, see Jara [15]. More recently mean-ergodic theorems for bi-continuous semigroups have been studied by Albanese, Lorenzi, Manco [1]. On the other hand, bi-continuous semigroups have appeared in several applications concerning parabolic equations and equations with unbounded coefficients on the space of bounded continuous functions. We mention here the papers: Albanese, Lorenzi, Manco [1], Albanese, Mangino [2], Es-Sarhir, Farkas [9], [10], Farkas, Lorenzi [13], Lorenzi, Zamboni [23], Metafune, Pallara, Wacker [26]. It is the aim of the present note to complement the existing theory of bi-continuous semigroups by the construction of the *adjoint semigroup* (this is done in Section 2).

Beside these, semigroups on spaces of measures have been attracting much attention recently, see Manca [24], Lant, Thieme [22]. It is not surprising, however, that such questions were addressed much earlier, for example by Sentilles [31], who studied operator semigroups of $\mathrm{C_b}(\Omega)$ (bounded continuous functions) and on $\mathrm{M}(\Omega)$ (bounded measures), where Ω is a locally compact space. The important construction there was that of a *strict-topology*, which will be also crucial in this paper (it is the topology considered by Dorroh, Neuberger in [6]). In this respect we will rely on the paper by Sentilles [30]. In Section 3 below, the adjoint bi-continuous semigroup construction mentioned above will be applied to studying bi-continuous semigroups on the space $\mathrm{M}(\Omega)$ of bounded Baire measures. We further show that bi-continuous semigroups in some cases may be included in the theory of equicontinuous semigroups on locally convex spaces, but we also give an example, a rather pathological one, showing that this is not always possible (Section 4). Our result on the automatic equicontinuity of semigroups (Theorem 3.4) has been obtained independently by M. Kunze even in a more general situation, see [17].

Let us first recall some terminology and set up the framework (see Kühnemund [19], [20]). For considering bi-continuous semigroups one needs a Banach space $(X, \|\cdot\|)$ which is endowed with an additional locally convex topology τ. The two topologies need somehow be connected, hence we assume the following:

Hypothesis A.

(i) τ is Hausdorff and coarser than the norm-topology.

(ii) The locally convex space (X, τ) is sequentially complete on τ-closed, norm-bounded sets.

(iii) The dual space $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, i.e.,

$$\|x\| = \sup_{\substack{\varphi\in(X,\tau)'\\ \|\varphi\|\leqslant 1}} |\varphi(x)|.$$

Now, τ-bi-continuous semigroups are defined as follows:

Definition 1.1. A one-parameter semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on a Banach space X is called a (τ-)*bi-continuous semigroup*, if

(i) the orbit $\mathbb{R}_+ \ni t \mapsto T(t)x$ is τ-continuous for all $x \in X$ (τ-strongly continuous),

(ii) $t \mapsto T(t)$ is a norm-bounded function, say, on $[0, 1]$, in which case it is exponentially bounded on $\mathbb{R}_+$.

(iii) $(T(t))_{t\geqslant 0}$ is locally-bi-equicontinuous, which means that for a norm-bounded τ-null sequence x_n the convergence $T(t)x_n \to 0$ holds in the topology τ and uniformly in compact intervals.

The main feature of this definition is that it mixes *properties with respect to two topologies*: the norm-topology of Banach spaces (thus it allows for norm-estimates) and a weaker notion of convergence. As in (iii) above, for functions $T\colon \mathbb{R}_+ \to \mathscr{L}(X)$ we use the term "*locally...*", if the property "..." is satisfied for operators $T(t)$, t ranging over compact intervals of $\mathbb{R}_+$.

Let us first present some examples for bi-continuous semigroups.

Example 1.2. Illustrative examples of bi-continuous semigroups are those on the space $\mathrm{C_b}(\Omega)$ when this space is endowed with the compact-open topology τ_c. To be more specific, consider a locally compact Hausdorff or a metrisable topological space Ω (or even more generally a completely regular k_f-space, i.e., a space Ω for which the continuity of a function $f\colon \Omega \to \mathbb{R}$ is decided already on compact sets). The linear space of continuous and bounded functions $f\colon \Omega \to \mathbb{R}$ becomes a Banach space when endowed with the supremum-norm $\|\cdot\|_\infty$. The additional topology that we consider is the *compact-open topology* τ_c generated by the family of semi-norms

$$\{p_K(f) := \sup_{x\in K} |f(x)|\colon\ K \subseteq \Omega \text{ compact}\}.$$

It is trivial that for $X = \mathrm{C_b}(\Omega)$ and $\tau = \tau_\mathrm{c}$ Hypothesis A is satisfied. Now some important examples of bi-continuous semigroups in this settings:

1. The shift semigroup on $\mathrm{C_b}(\mathbb{R})$ is bi-continuous for the compact-open topology.
2. If Ω is a Polish space Dorroh and Neuberger have studied semigroups $(T(t))_{t\geqslant 0}$ on $\mathrm{C_b}(\Omega)$ induced by jointly-continuous flows, see [6], [7]. Kühnemund [19, Sec. 3.2] has shown that these semigroups are bi-continuous with respect to the topology τ_c.

3. Given a separable Hilbert space H, Ornstein-Uhlenbeck semigroups on $\mathrm{C_b}(H)$ have proved to be bi-continuous semigroups with respect to the compact-open topology, see Kühnemund [19, Sec. 3.3] (or [12]).
4. Metafune, Pallara, Wacker proved in [26] that solutions of certain second order parabolic equations with unbounded coefficients give rise to τ_c-bi-continuous semigroups on $\mathrm{C_b}(\mathbb{R}^d)$.

There are many other instances of bi-continuous semigroups of this kind. Without claiming completeness we mention the following references: Albanese, Lorenzi, Manco [1], Lorenzi, Zamboni [23], Es-Sarhir, Farkas [9], Farkas, Lorenzi [13].

The next example concerns the weak*-topology on the dual of a Banach space (and includes, among others, the shift semigroup on $\mathrm{L}^\infty(\mathbb{R})$).

Example 1.3. Let E be a Banach space, $X = E'$ and $\tau = \sigma(E', E)$ the weak*-topology. If $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on E with respect to the norm, then its adjoint $(T'(t))_{t\geqslant 0}$ is bi-continuous on X with respect to the weak*-topology (see [19, Sec. 3.5]). It is a consequence of the Krein-Šmulian Theorem (see [29, Sec. IV.6]) and it is shown in [11] that if E is separable, then every τ-bi-continuous semigroup on X is of this form. We show in Section 4 that one cannot drop the separability assumption.

The last example illustrates that bi-continuous semigroups naturally appear in operator theory, too.

Example 1.4. Let $(T(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$ be C_0-semigroups (strongly continuous for the norm) on a Banach space E. Consider $X = \mathscr{L}(E)$ endowed besides the operator norm also with the strong operator topology τ_{stop}. It is obvious that Hypothesis A is satisfied. The *implemented semigroup* $(U(t))_{t\geqslant 0}$ is defined by

$$U(t) := L_{S(t)} R_{T(t)},$$

where L and R stand for the left and the right multiplication, respectively by the indicated linear operator. It is easy to see that $(U(t))_{t\geqslant 0}$ is a semigroup on X which is bi-continuous with respect to τ_{stop} (see Kühnemund [19, Sec. 3.4] and Alber [3]).

The mixed topology

To close the introduction we present a construction for the so-called mixed topology, which allows us to handle the "two-topologies feature" of bi-continuous semigroups by means of a single locally convex topology. Let (X, τ) be as in Hypothesis A and let $\mathscr{P}$ be a family of seminorms determining τ such that $p \leqslant \|\cdot\|$ for all $p \in \mathscr{P}$, and $\|\cdot\| = \sup_{p\in\mathscr{P}} p(x)$. For $(p_n) \subseteq \mathscr{P}$ and $(a_n) \in \mathrm{c}_0$, $a_n \geqslant 0$ (positive null-sequence) consider the seminorm

$$\tilde{p}_{(p_n,a_n)}(x) := \sup_{n\in\mathbb{N}} a_n p_n(x).$$

Let τ_{m} be a locally convex topology, called the *mixed topology*, determined by the family of seminorms

$$\tilde{\mathscr{P}} := \bigl\{\tilde{p}_{(p_n,a_n)}\colon\ (p_n) \in \mathscr{P},\ (a_n) \in \mathrm{c}_0,\ a_n \geqslant 0\bigr\}.$$

It is clear that τ is coarser and the norm-topology is finer than τ_{m}. Wiweger in [33] presents a very general construction for the mixed topology (without the assumptions in Hypothesis A and even on not necessarily locally convex spaces). As a consequence of Examples D) and E) and of Theorem 3.1.1 of [33] one obtains that τ_{m} is the finest locally convex topology on X that coincides with τ on norm-bounded sets, for instance, if

1) $X = \mathrm{C_b}(\Omega)$ is endowed with the sup-norm, Ω is a completely regular space, and $\tau = \tau_{\mathrm{c}}$ is the compact-open topology on $\mathrm{C_b}(\Omega)$ (cf. Example 1.2);
2) or, $X = E'$, E is a Banach space and $\tau = \sigma(E', E)$ is the weak*-topology (cf. Example 1.3).

By a routine argument one proves the following lemma (or see [33, Theorem 2.3.1]):

Lemma 1.5. *A sequence $x_n \in X$ is convergent in the topology τ_{m} if and only if it is norm-bounded and τ convergent.*

The following result translates the notion of bi-continuous semigroups to the language of mixed topologies.

Proposition 1.6. *The class of τ-bi-continuous semigroups and the class of τ_{m}-strongly continuous and locally sequentially τ_{m}-equicontinuous semigroups coincide.*

Proof. Let $(T(t))_{t\geqslant 0}$ be a τ-bi-continuous semigroup. Since $[0,1] \ni h \to T(h)x$ is norm-bounded and τ-continuous for all x, we obtain by Lemma 1.5 that these orbits are also τ_{m}-continuous. The sequential τ_{m}-equicontinuity of the family

$$\{T(t)\colon\ t \in [0, t_0]\}$$

is simply a reformulation of Definition 1.1 (iii) in view of Lemma 1.5.

For the converse, let $(T(t))_{t\geqslant 0}$ be a τ_{m}-strongly continuous and locally sequentially equicontinuous semigroup. Then the orbits $[0,1] \ni h \to T(h)x$ are norm-bounded since they are τ_{m}-continuous. The τ-strong continuity is immediate, while the local bi-equicontinuity follows again by Lemma 1.5. □

Given a bi-continuous semigroup $(T(t))_{t\geqslant 0}$ we would like to define its adjoint in such a way that it be again a bi-continuous semigroup hence fitting in this theory. First, we have to specify the space with the properties listed in Hypothesis A. The next proposition offers a candidate.

Proposition 2.1. *Given a Banach space X with the additional topology τ satisfying Hypothesis A, denote by X° the set of all norm-bounded linear functionals which are τ-sequentially continuous on norm-bounded sets of X. Then X° is a closed linear subspace of the norm-dual X', hence it is a Banach space.*

Proof. That X° is a linear subspace is trivial. Take $\varphi_n \in X^\circ$ with $\|\varphi_n - \varphi\| \to 0$, where $\varphi \in X'$. We have to show that $\varphi \in X^\circ$. To this end, consider a norm-bounded τ-null sequence (x_n). Then $\varphi(x_n) \to 0$ follows from

$$\begin{aligned}|\varphi(x_n)| &\leqslant |\varphi(x_n) - \varphi_k(x_n)| + |\varphi_k(x_n)|\\ &\leqslant K\|\varphi - \varphi_k\| + |\varphi_k(x - x_n)| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

first by taking $k \in \mathbb{N}$ sufficiently large and then for fixed k using the continuity assumptions on φ_k. □

So the norm inherited from X' makes X° a Banach space. We equip X° additionally with the weak topology $\tau^\circ := \sigma(X^\circ, X)$. It is our aim to consider bi-continuous semigroups with respect to this topology on X°. For this purpose we have to verify the validity of Hypothesis A. Clearly, τ° is Hausdorff since X separates the points of X°, and also X is norming. Trivially, τ° is coarser than the norm-topology on X°. It remains to check only the τ°-sequential completeness on closed, norm-bounded sets, but this generally may fail to hold, so we incorporate this requirement into our hypotheses.

Hypothesis B. Suppose that $X^\circ \cap \overline{B(0,1)}$ is sequentially complete for $\sigma(X^\circ, X)$.

The next example shows that this assumption is indeed restrictive, i.e. it is not a consequence of the general framework of Hypothesis A.

Example 2.2. Let E be a non-reflexive Banach space, $X = E'$ its norm dual and $\tau = \sigma(E', E)$ the weak*-topology. Suppose also that E' is separable, whence $\sigma(E'', E')$ is metrisable on bounded sets. Then $X^\circ = (X, \tau)' = E$ and $\tau^\circ = \sigma(E, E')$, the weak*-topology. Take $y \in E'' \setminus E$ arbitrary with $\|y\|_{E'} \leqslant 1$, and take $y_n \in E$ with $\|y_n\|_E \leqslant 1$ converging to y in the weak*-topology $\sigma(E'', E')$ (such a sequence exists by Goldstine's Theorem). This shows that $\tau^\circ = \sigma(E, E')$ (being the restriction of $\sigma(E'', E')$ to E) is not complete on $\overline{B_{X^\circ}(0,1)}$. For a concrete example take $E = c_0$, $E' = \ell^1$, $E'' = \ell^\infty$, $y = \mathbf{1}$ the constant 1 sequence, and y_n the sequence with first n members 1 the others 0. So $X = \ell^1$ with $\tau = \sigma(\ell^1, c_0)$, and $y_n \to y$ in ℓ^∞ for $\sigma(\ell^\infty, \ell^1)$.

The framework of Hypothesis A is now established. We are ready to define adjoint bi-continuous semigroups. The proof of the next proposition is straightforward.

Proposition 2.3. *Let $B \in \mathscr{L}(X)$ be a norm-bounded linear operator which is also τ-sequentially continuous on norm-bounded sets. Then the adjoint $B' \in \mathscr{L}(X')$ leaves X° invariant.*

For a linear operator $B \in \mathscr{L}(X)$ that is also τ-sequentially continuous on norm bounded sets, denote by B° the restriction of $B' \in \mathscr{L}(X')$ to X°.

Take now a bi-continuous semigroup $(T(t))_{t\geqslant 0}$ on (X, τ). Then the operators $T(t)^\circ$ obviously form a semigroup $(T^\circ(t))_{t\geqslant 0}$, which is τ°-strongly continuous by definition. The exponential boundedness of $(T^\circ(t))_{t\geqslant 0}$ is trivial. To establish the local bi-equicontinuity, we assume the following on the underlying space.

Hypothesis C. Every norm-bounded τ°-null sequence $(\varphi_n) \subset X^\circ$ is τ equicontinuous on norm bounded sets.

Under this and the previous hypotheses we have the following result.

Proposition 2.4. *Let $(T(t))_{t\geqslant 0}$ be a τ-bi-continuous semigroup on (X, τ), and suppose that Hypotheses B and C are satisfied. Then the semigroup $(T^\circ(t))_{t\geqslant 0}$ is a τ°-bi-continuous semigroup on X° (recall: by definition $\tau^\circ = \sigma(X^\circ, X)$).*

P r o o f. By what is said in the paragraph preceding Hypothesis C it remains to show the local τ°-bi-equicontinuity of $(T^\circ(t))_{t\geqslant 0}$. To this end take a norm-bounded τ°-null sequence and $x \in X$. For $t_0 > 0$ one has the τ-compactness of $\{T(t)x \colon t \in [0, t_0]\}$, thus by the equicontinuity of φ_n we have

$$[T^\circ(t)\varphi_n](x) = \varphi_n(T(t)x) \to 0$$

uniformly on $[0, t_0]$ (use here that for the equicontinuous family φ_n the pointwise convergence is the same as the uniform convergence on compact sets; actually what is needed here is an adaptation of Theorem III. 4.5. Schaefer [29] to our situation). This means precisely the τ°-bi-equicontinuity of $(T^\circ(t))_{t\geqslant 0}$. □

In this section, we would like to carry out the adjoint construction from the previous section in the particular case when $X = \mathrm{C_b}(\Omega)$, the space of bounded and continuous functions. For this purpose we first need to study the mixed topology on $\mathrm{C_b}(\Omega)$, and recall some results from Sentilles [31] and, in slightly modified form, from Farkas [12].

Let Ω be a Polish space or a σ-compact locally compact Hausdorff space. Consider the mixed topology τ_{m} which is the finest locally convex topology that coincides with τ_{c} on sup-norm bounded sets of $\mathrm{C_b}(\Omega)$ (see the end of Section 1). The dual of $(\mathrm{C_b}(\Omega), \tau_{\mathrm{m}})$ is the space $\mathrm{M}(\Omega)$ of bounded Baire measures in Ω. We briefly indicate a way to see this. By assumption Ω is completely regular. The dual of $\mathrm{C_b}(\Omega)$ (as a Banach space) is isomorphic to the space $\mathrm{M}(\beta\Omega)$ of all bounded Baire measures on the Stone-Čech compactification $\beta\Omega$ of Ω, and the isomorphism is given by $\varphi(f) = \int_{\beta\Omega} f \, \mathrm{d}\mu$. (One can represent a continuous linear function even by regular Borel measures, in this respect we refer to Knowles [21] and Mařík [25].)

If Ω is Polish then it is G_δ, and if Ω is locally compact it is open in $\beta\Omega$, see, e.g., Walker [32, Chap. 1]. Furthermore, as Ω is Lindelöf and G_δ in $\beta\Omega$, it is also a Baire set there (a space with this property was called absolute Baire by Negrepontis [27], see also Frolík [14]). Therefore it is possible to identify $\mathrm{M}(\Omega)$ with a subspace of $\mathrm{M}(\beta\Omega)$ in the following way:

$$\iota \colon \mathrm{M}(\Omega) \to \mathrm{M}(\beta\Omega),$$
$$[\iota(\nu)](B) := \nu(\Omega \cap B) \quad \text{for all} \quad \nu \in \mathrm{M}(\Omega) \quad \text{and} \quad B \subseteq \beta\Omega \quad \text{a Baire set.}$$

Then ι is an injection with

$$\operatorname{rg} \iota = \{\mu \colon \ \mu \in \mathrm{M}(\beta\Omega),\ |\mu|(\beta\Omega \setminus \Omega) = 0\}.$$

One can see that a measure $\mu \in \mathrm{M}(\beta\Omega)$ gives rise to a linear functional $\varphi \in \mathrm{C_b}(\Omega)'$ which is not only norm-continuous, but also τ_{c}-continuous on norm-bounded sets, if and only if it belongs to $\operatorname{rg} \iota$, i.e. if we use the above identification, it is a Baire measure on Ω (see, e.g., [12] or Sentilles [30]).

The mixed topology, also called the *strict topology* and denoted by $\beta_0 := \tau_{\mathrm{m}}$ in this setting, has the following remarkable properties.

Theorem 3.1 (Sentilles [30]). *Let Ω either be a σ-compact, locally compact space, or a Polish space. Then the following assertions are true:*

a) $\beta_0 = \mu(\mathrm{C_b}(\Omega), \mathrm{M}(\Omega))$, *the Mackey topology, where* $\mathrm{M}(\Omega)$ *denotes the space of all bounded Baire-measures on* Ω.

b) *A linear operator* $T\colon \mathrm{C_b}(\Omega) \to \mathrm{C_b}(\Omega)$ *is* β_0*-continuous, if and only if it is* β_0*-sequentially continuous. The same holds for linear functionals.*

c) *Every* $(\varphi_n) \subset (\mathrm{C_b}(\Omega), \beta_0)'$ $\sigma(\mathrm{M}(\Omega), \mathrm{C_b}(\Omega))$*-null sequence is* β_0*-equicontinuous.*

As a consequence of b) we have $\mathrm{C_b}(\Omega)^\circ = \mathrm{M}(\Omega)$. We now turn our attention to τ_{c}-bi-continuous semigroups.

Bi-continuous semigroups on $\mathrm{C_b}(\Omega)$

By Proposition 2.1, for a linear operator $B \in \mathscr{L}(\mathrm{C_b}(\Omega))$ which is τ_{c}-continuous on norm-bounded sets the (Banach space) adjoint B' leaves $\mathrm{M}(\Omega)$ invariant, its restriction is denoted by B°. The next lemma is proved in [12].

Lemma 3.2. *Let* Ω *be a Polish space, and let* $T\colon \mathbb{R}_+ \to \mathscr{L}(\mathrm{C_b}(\Omega))$ *be a* τ_{c}*-strongly continuous function consisting of operators that are* τ_{c}*-continuous on norm-bounded sets. For a norm-bounded, weak***-compact set* $\mathscr{K} \subseteq \mathrm{M}(\Omega)$ *and* $t_0 > 0$ *the set of measures*

$$\{T^\circ(t)\nu\colon\ t \in [0, t_0],\ \nu \in \mathscr{K}\}$$

is tight.

The next proposition is also taken from [12]. We repeat it here with a slight modification and the additional assertion concerning β_0-continuity.

Proposition 3.3. *Let* $T\colon \mathbb{R}_+ \to \mathscr{L}(\mathrm{C_b}(\Omega))$ *be* τ_{c}*-strongly continuous and locally norm-bounded. Suppose that* $T(t)$ *takes norm-bounded* τ_{c}*-null sequences into* τ_{c}*-null sequences. Then for all compact sets* $K \subseteq \Omega$ *and* $\varepsilon > 0$*, there exists* $M > 0$ *and* $K' \subseteq \Omega$ *compact such that*

$$\sup_{x\in K} |(T(t)f)(x)| \leqslant M \sup_{x\in K'} |f(x)| + \varepsilon \|f\|_\infty$$

holds uniformly for t *in compact intervals of* $\mathbb{R}_+$*. In particular, it is locally-*β_0*-equicontinuous, or, which is the same, it is* τ_{c}*-bi-equicontinuous.*

P r o o f. Let $\varepsilon > 0$, $t_0 > 0$ and let $K \subseteq \Omega$ be a compact set. Take a compact set $K' \subseteq \Omega$ such that $|T^\circ(t)\delta_x|(\Omega \setminus K') \leqslant \varepsilon$ for all $t \in [0, t_0]$ and $x \in K$. Such a compact set exists by Lemma 3.2. We then obtain

$$\begin{aligned} \sup_{x\in K} |T(t)f(x)| &= \sup_{x\in K} \left| \int_\Omega f \,\mathrm{d}T^\circ(t)\delta_x \right| \\ &\leqslant \sup_{x\in K} \int_{K'} |f| \,\mathrm{d}|T^\circ(t)\delta_x| + \sup_{x\in K} \int_{\Omega\setminus K'} |f| \,\mathrm{d}|T^\circ(t)\delta_x| \\ &\leqslant \sup_{t\in[0,t_0]} \|T(t)\| \cdot \sup_{x\in K'} |f(x)| + \varepsilon\|f\|, \end{aligned}$$

which is the assertion. □

A variant of this result has been obtained independently by M. Kunze, see [17, Theorem 4.4].

Let us summarise the above.

Theorem 3.4. *Let Ω be a Polish space and let us consider the set $\mathscr{F}$ of τ_c-strongly continuous semigroups $(T(t))_{t\geqslant 0}$ on $\mathrm{C_b}(\Omega)$ for which each $T(t)$ is τ_c-sequentially continuous on sup-norm bounded sets. Then the class of τ_c-bi-continuous semigroups and the class of β_0-locally equicontinuous, β_0-strongly continuous semigroups both coincide with $\mathscr{F}$.*

Bi-continuous semigroups on $\mathrm{M}(\Omega)$

We will now study the adjoint of a τ_c-bi-continuous semigroup. To do that we have to verify Hypotheses B and C. The validity of Hypothesis B, i.e., that $\mathrm{M}(\Omega)$ is $\sigma(\mathrm{M}(\Omega), \mathrm{C_b}(\Omega))$-sequentially complete if Ω is a Polish space, was already known to Alexandroff [4]. Hypothesis C is satisfied by Theorem 3.1 c). Thus the abstract results from Section 2 yield the following (recall: we abbreviate $\tau^\circ := \sigma(\mathrm{C_b}(\Omega), \mathrm{M}(\Omega))$.

Theorem 3.5. *Let Ω be a Polish space and $(T(t))_{t\geqslant 0}$ a τ_c-bi-continuous semigroup on $\mathrm{C_b}(\Omega)$. Then the semigroup $(T(t)^\circ)_{t\geqslant 0}$ defined as $T(t)^\circ := T(t)'|_{\mathrm{M}(\Omega)}$ is a τ°-bi-continuous semigroup on the space of bounded Baire measures $\mathrm{M}(\Omega)$.*

It is little surprising that the converse of this statement is also true:

Theorem 3.6. *Let Ω be a Polish space. Let $(S(t))_{t\geqslant 0}$ be a τ°-bi-continuous semigroup on the space $\mathrm{M}(\Omega)$. Then there is a τ_c-bi-continuous semigroup $(T(t))_{t\geqslant 0}$ on $\mathrm{C_b}(\Omega)$ with $T^\circ(t) = S(t)$.*

P r o o f. For $f \in \mathrm{C_b}(\Omega)$ we set

$$(T(t)f)(x) := \int_\Omega f \, \mathrm{d}(S(t)\delta_x),$$

where δ_x denotes the Dirac measure at the point $x \in \Omega$. We then have

$$\sup |(T(t)f)(x)| \leqslant \|S(t)\| \cdot \|f\|_\infty,$$

so $T(t)f$ is a bounded function. If $x_n \to x$ in Ω for $n \to \infty$, then $\delta_{x_n} \to \delta_x$ in $\mathrm{M}(\Omega)$ with respect to τ°. Since $S(t)$ is τ°-continuous on $\mathrm{M}(\Omega)$, we have $S(t)\delta_{x_n} \to S(t)\delta_x$ and hence the continuity of $T(t)f$ follows. Altogether we obtain that $T(t) \in \mathscr{L}(\mathrm{C_b}(\Omega))$. Obviously $(T(t))_{t\geqslant 0}$ is an exponentially bounded semigroup. We have to show that for each fixed $t \geqslant 0$ the operator is τ_c-bi-continuous, and that the semigroup $(T(t))_{t\geqslant 0}$ is τ_c-strongly continuous. Then by Theorem 3.4 $(T(t))_{t\geqslant 0}$ is a τ_c-bi-continuous semigroup, and by construction $T(t)^\circ = S(t)$ holds.

We first prove that for $t > 0$ fixed $T(t)$ is τ_c-bi-continuous. Assume the contrary, i.e., that there are a sup-norm bounded sequence $f_n \in \mathrm{C_b}(\Omega)$ τ_c-convergent to 0 (i.e. β_0-convergent to 0), a compact set $K \subseteq \Omega$ and $\varepsilon > 0$ such that

$$\sup_{x\in K} |(T(t)f_n)(x)| > \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ take a point $x_n \in K$ with

$$|(T(t)f_n)(x_n)| > \varepsilon.$$

We can suppose by virtue of compactness that $x_n \to x$ for some $x \in K$. Then $\delta_{x_n} \to \delta_x$ in τ°, and we obtain that $S(t)\delta_{x_n}$ is a τ°-convergent sequence. By Theorem 3.1 c) this sequence is β_0-equicontinuous. So by Schaefer [29, Sec. III.4.5] we can deduce

$$|(T(t)f_n)(x_n)| \leqslant \sup_{m\in\mathbb{N}} \left| \int_\Omega f_n \,\mathrm{d}S(t)\delta_{x_m} \right| \to 0 \quad \text{for } n \to \infty.$$

This is a contradiction.

To see the τ_c-strong continuity let $K \subseteq \Omega$ be a compact set. Assume by contradiction that there are $f \in \mathrm{C_b}(\Omega)$, $\varepsilon > 0$ and $t_n \in [0,1]$ with $t_n \to 0$ for $n \to \infty$ such that

$$\sup_{x\in K} |(T(t_n)f)(x) - f(x)| > \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ take a point $x_n \in K$ with

$$\left| \int_\Omega f \,\mathrm{d}(S(t_n)\delta_{x_n} - \delta_{x_n}) \right| = |(T(t_n)f)(x_n) - f(x_n)| > \varepsilon.$$

By compactness we can pass to a subsequence and assume that x_n converges to some $x \in K$. This means δ_{x_n} τ°-converges to δ_x. By the local τ°-bi-equicontinuity of $(S(t))_{t\geqslant 0}$, we have

$$\sup_{s\in[0,1]} \left| \int_\Omega f \,\mathrm{d}(S(s)\delta_{x_n} - \delta_{x_n}) \right| \to 0 \quad \text{for } n \to \infty,$$

a contradiction. $\square$

A surprising fact is that though τ_c is generally not metrisable, the continuity of norm-bounded linear operators on norm-bounded sets can be described by convergent sequences. It is clear that some kind of countability plays an important role here (cf. metric or σ-compact spaces). Indeed, the simplest non-countable space gives rise to a counterexample to Theorem 3.4, when Ω is not a Polish space. More specifically, we construct below a bi-continuous semigroup which is not β_0-locally-equicontinuous. For other illuminating, related examples we refer to Kunze [17, Sec. 3].

Example 4.1. Let $\Omega = \omega_1$ be the first uncountable ordinal number and υ the order topology. Suppose that $f_n \to 0$ in the topology τ_c. We claim that there exists $\alpha \in \omega_1$ such that $f_n \to 0$ uniformly on $[\alpha, \omega_1)$. Suppose the contrary, i.e., for all $\alpha < \omega_1$ there exists $k \in \mathbb{N}$, $k > 0$ such that for all $N \in \mathbb{N}$ there exist $n \geqslant N$ and $x \in [\alpha, \omega_1)$ with $|f_n(x)| > 1/k$. For all $\alpha \in \omega_1$ we have $k_\alpha \in \mathbb{N}$ and we may assume that $k_{\alpha_\xi} = k$ for a cofinal sequence $\alpha_\xi \in \omega_1$. By induction we choose a sequence

$$x_{\alpha_{\xi_1}} < x_{\alpha_{\xi_1}} \dot{+} 1 < x_{\alpha_{\xi_2}} < x_{\alpha_{\xi_2}} \dot{+} 1 < \ldots < x_{\alpha_{\xi_j}} < x_{\alpha_{\xi_j}} \dot{+} 1 < \ldots$$

with $f_{n_j}(x_{\alpha_{\xi_j}}) > 1/k$. Since $K := \big\{ \lim_{j\to\infty} x_{\alpha_{\xi_j}}, x_{\alpha_{\xi_j}} : j \in \mathbb{N} \big\}$ is compact and

$$\sup_{y\in K} |f_{n_j}(y)| \geqslant \frac{1}{k} \quad \text{for all } j \in \mathbb{N},$$

we have arrived to a contradiction. Thus we have the existence of $\alpha \in \omega_1$ as asserted above. Now, consider the family $\{[\xi, \omega_1) \colon \xi > \alpha\}$, which has the finite intersection property and thus by compactness possesses an accumulation point $x \in \beta\Omega$. All f_n extend to the Stone-Čech compactification $\beta\Omega$ and $|f_n(y)| < \varepsilon$ for all $y \in [\alpha, \omega_1)$ if $n \geqslant N$. Take $n \in \mathbb{N}$, $n \geqslant N$. By the continuity of f_n on $\beta\Omega$ we have a neighbourhood U of x such that for all $y \in U$

$$|f_n(y) - f_n(x)| \leqslant \varepsilon.$$

There exist $\xi \in (\alpha, \omega_1)$ with $\emptyset \neq U \cap [\xi, \omega_1) \ni z$, so

$$|f_n(x)| \leqslant |f_n(x) - f_n(z)| + |f_n(z)| \leqslant \varepsilon + \varepsilon.$$

Thus $f_n(x) \to 0$, which shows that δ_x is τ_c-sequentially-continuous (on norm-bounded sets). However, it is clear that it is not τ_c-continuous on norm-bounded sets.

Consider now the C_0-semigroup $(T(t))_{t\geqslant 0}$ generated by the bounded operator $A := \mathbf{1} \otimes \delta_x$. Since A is idempotent the semigroup T takes the form

$$T(t) = \mathrm{e}^{tA} = I - A + \mathrm{e}^t A.$$

This semigroup is bi-continuous but none of $T(t)$, $t > 0$ is τ_c-continuous on norm bounded sets.

We close the paper by the following counterexample complementing Example 1.3.

Example 4.2. We present a $\sigma(E', E)$-bi-continuous semigroup on $X = E'$ that is not the adjoint of a strongly continuous semigroup on E (where by Example 1.3 E is a fortiori non-separable). A Banach space E is said to have the *Mazur property* if every weakly*-sequentially continuous linear functional on E is weakly*-continuous. Not every Banach space has this property, for instance $E = \ell^\infty$ lacks it. (We refer to further details and examples, e.g., to Edgar [8].) Now let E be a Banach space without the Mazur property and let $X := E'$. Consider a weakly*-sequentially continuous functional φ on E' that is not weakly*-continuous, and let $x \in E'$ be an element with $\varphi(x) = 1$. Set $A := x \otimes \varphi$, which is obviously a bounded idempotent linear operator on E'. Now the semigroup with the asserted properties is given as in Example 4.1: $T(t) := e^{tA} = I - A + e^t A$.

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Author's address: B. Farkas, Technische Universität Darmstadt, Fachbereich Mathematik, Darmstadt, Germany, Schlossgartenstr. 7, D-64289, e-mail: `farkas@mathematik.tu-darmstadt.de`.