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A NEW NON-INTERIOR CONTINUATION METHOD FOR P_0-NCP BASED ON A SSPM-FUNCTION*

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Abstract. In this paper, we consider a new non-interior continuation method for the solution of nonlinear complementarity problem with P_0-function (P_0-NCP). The proposed algorithm is based on a smoothing symmetric perturbed minimum function (SSPM-function), and one only needs to solve one system of linear equations and to perform only one Armijo-type line search at each iteration. The method is proved to possess global and local convergence under weaker conditions. Preliminary numerical results indicate that the algorithm is effective.

Keywords: non-interior continuation method, nonlinear complementarity, P_0-function, coercivity, quadratic convergence

MSC 2010: 90C25, 90C30, 90C48, 65K05

1. Introduction

Throughout this paper, we consider the following nonlinear complementarity problem with P_0 function f (for short, denoted by P_0-NCP(f)) which is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad f(x) \geqslant 0, \quad x^\top f(x) = 0, \tag{1}$$

where $f\colon\ \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable P_0-function.

Nonlinear complementarity problems (NCPs) have attracted much attention due to their wide range of applications in many fields, such as operations research, engineering design, economics equilibrium and so on. We refer the interested readers

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to the survey papers by Pang [12], Ferris and Pang [7], Ferris, Mangasarian, and Pang [6], Harker and Pang [8], and the references therein. Different methods have been proposed to treat NCPs. Recently, there has been strong interest in non-interior continuation methods for NCPs [3], [2], [1], [5], [9], [14], [15]. The idea of non-interior continuation method is to use a smooth function to reformulate the problem concerned as a family of parameterized smooth equations and to solve the smooth equations approximately at each iteration. By reducing the parameter to zero, it is hoped that a solution of the original problem can be found. However, many of these algorithms strongly depend on the assumptions of strict complementarity and uniform nonsingularity [3], [15]. Without uniform nonsingularity assumptions, Tseng [15] studied the local quadratic convergence of general predictor-corrector-type path-following methods for monotone NCP via the error bound theory. However, the algorithms given in [3], [15] usually need to solve two linear systems of equations and to perform two or three line searches per iteration and depend strongly on strict complementarity.

Motivated by this direction, in this paper, based on a SSPM-function, we reformulate the P_0-NCP(f) as a system of nonlinear equations and propose a non-interior continuation method for its solution. It is shown that our algorithm has the following nice properties:

(i) The algorithm is well-defined and a solution of P_0-NCP(f) can be obtained from any accumulation point of the iteration sequence generated by the method.

(ii) It can start from an arbitrary point.

(iii) It need to solve only one system of linear equations and to perform only one Armijo-type line search at each iteration.

(iv) The boundedness of the level set can be obtained due to the coercivity of the smoothing function.

(v) The global and superlinear convergence of the algorithm are obtained without strict complementarity. Moreover, the algorithm has locally quadratic convergence if f' is Lipschitz continuous.

The rest of this paper is organized as follows. In the next section, we introduce some preliminaries to be used in the subsequent sections, and based on the minimum function, a SSPM-function and its properties are presented. In Section 3, we present a new non-interior continuation method for solving the P_0-NCP(f) and show its well-definedness. The global convergence and local convergence of the algorithm are investigated in Section 4. Numerical experiments and conclusions are given in Section 5 and 6, respectively.

The following notation will be used throughout this paper. All vectors are column vectors, $A^\top$ denotes the transpose of a matrix A, $\mathbb{R}^n$ denotes the space of

n-dimensional real column vectors (for $n = 1$, $\mathbb{R} \equiv \mathbb{R}^1$ stands for the set of real numbers). Symbols $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the respective nonnegative and positive orthants of $\mathbb{R}^n$, while $\mathbb{R}_+$ and $\mathbb{R}_{++}$ are used for the nonnegative and positive real numbers, respectively. We define $N := \{1, 2, \ldots, n\}$. For any vector $u \in \mathbb{R}^n$, we denote by $\operatorname{diag}\{u_i \colon i \in N\}$ the diagonal matrix whose ith diagonal element is u_i and $\operatorname{vec}\{u_i \colon i \in N\}$ the vector u. The matrix I represents the identity matrix of suitable dimension. The symbol $\|\cdot\|$ stands for the 2-norm. For any differentiable function $f \colon \mathbb{R}^n \to \mathbb{R}^n$, $f'(x)$ denotes the Jacobian of f at x. We denote the solution set of P_0-NCP(f) by $\Theta := \{x \in \mathbb{R}^n \colon x \geqslant 0,\ f(x) \geqslant 0,\ x^\top f(x) = 0\}$. For any $\alpha, \beta \in \mathbb{R}_{++}$, $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) means α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$. $\mathbb{R}^n \times \mathbb{R}^m$ is identified with $\mathbb{R}^{n+m}$. For any matrix $A \in \mathbb{R}^{n\times n}$, $A \succcurlyeq 0$ ($A \succ 0$) means A is positive semi-definite (positive definite, respectively).

2. Preliminaries and a SSPM-function

2.1. Preliminaries

In this subsection, we recall some useful definitions that will be used in the subsequent sections.

Definition 2.1. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be a P_0-matrix if all its principal minors are nonnegative.

Definition 2.2. A function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be a P_0-function if for all $x, y \in \mathbb{R}^n$ with $x \neq y$, there exists an index $i_0 \in N$ such that

$$x_{i0} \neq y_{i0}, \quad (x_{i0} - y_{i0})[f_{i0}(x) - f_{i0}(y)] \geqslant 0.$$

Definition 2.3. Let $\mathcal{D}$ be a closed, convex subset of $\mathbb{R}^n$ and $f \colon \mathcal{D} \to \mathbb{R}^n$ a continuous mapping. If there exists a point $u \in \mathcal{D}$ such that

$$\lim_{\|x\|\to+\infty} \frac{(x-u)^\top f(x)}{\|x\|} = \infty, \quad x \in \mathcal{D},$$

then the mapping f is called satisfying the coercivity condition in $\mathcal{D}$.

The following concept of semi-smoothness plays an important role in the design of higher-order Newton-type methods.

Definition 2.4. Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous around $x \in \mathbb{R}^n$. We call f to be semi-smooth at x if f is directionally differentiable at x and

$$\lim_{V\in\partial f(x+th'),\, h'\to h,\, t\to 0^+} Vh' \text{ exists for all } h\in\mathbb{R}^n,$$

where $\partial f(\cdot)$ denotes the generalized Jacobian as defined in Clarke [4].

The concept of semi-smoothness was originally introduced by Mifflin for functions [10]. Qi and Sun extended the definition of semi-smooth functions to vector-valued functions [16]. Convex functions, smooth functions, and piecewise linear functions are examples of semi-smooth functions. A function is semi-smooth at x if and only if all its component functions are semi-smooth. The composition of semi-smooth functions is still a semi-smooth function.

2.2. A SSPM-function and its properties

In this subsection, we give a SSPM-function and state its properties.

For any $(a,b)\in\mathbb{R}^2$ consider the minimum function

$$g(a,b) := \min\{a,b\}. \tag{2}$$

By introducing a parameter $\mu\in\mathbb{R}$, we perturb symmetrically (2) as

$$g(\mu,a,b) := \min\{\mu a + (1+\mu)b, (1+\mu)a+\mu b\}.$$

By smoothing $g(\mu,a,b)$, we obtain the following smoothing function, i.e., SSPM-function

$$\varphi(\mu,a,b) := (1+2\mu)(a+b) - \sqrt{(a-b)^2+4\mu^2}. \tag{3}$$

The following lemma gives two simple properties of the smoothing function φ defined by (3). Its proof is obvious.

Lemma 2.5. *Let $(\mu,a,b)\in\mathbb{R}^3$ and $\varphi(\mu,a,b)$ be defined by (3). Then the following results hold:*

(i) *We have*

$$\varphi(0,a,b) = 0 \Longleftrightarrow a\geqslant 0,\ b\geqslant 0,\ ab=0. \tag{4}$$

(ii) *$\varphi(\mu,a,b)$ is globally Lipschitz continuous for any $\mu>0$.*

(iii) *$\varphi(\mu,a,b)$ is continuously differentiable at all points in $\mathbb{R}^3$ different from $(0,c,c)$ for arbitrary $c\in\mathbb{R}$. In particular, if $\mu>0$, then $\varphi(\mu,a,b)$ is continuously differentiable at arbitrary $(a,b)\in\mathbb{R}^2$.*

From (3), for any $\mu \neq 0$, a straightforward calculation yields

$$\varphi'_\mu(\mu, a, b) = 2(a+b) - \frac{4\mu}{\sqrt{(a-b)^2 + 4\mu^2}}, \tag{5}$$

$$\varphi'_a(\mu, a, b) = 1 + 2\mu - \frac{a-b}{\sqrt{(a-b)^2 + 4\mu^2}}, \tag{6}$$

$$\varphi'_b(\mu, a, b) = 1 + 2\mu + \frac{a-b}{\sqrt{(a-b)^2 + 4\mu^2}}. \tag{7}$$

It is not difficult to see that for any $\mu > 0$, φ'_μ, φ'_a, and φ'_b are continuous, and $0 < \varphi'_a < 2(1+\mu)$, $0 < \varphi'_b < 2(1+\mu)$.

For any $z := (\mu, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ we denote

$$G(z) := \begin{pmatrix} \mathrm{e}^\mu - 1 \\ \Phi(z) \end{pmatrix} \tag{8}$$

where $\Phi\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\Phi(z) := \begin{pmatrix} \varphi(\mu, x_1, f_1(x)) \\ \varphi(\mu, x_2, f_2(x)) \\ \vdots \\ \varphi(\mu, x_n, f_n(x)) \end{pmatrix}. \tag{9}$$

Obviously, Φ is continuously differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$.

Define the merit function $\Psi\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$\Psi(z) := \|G(z)\|^2 = (\mathrm{e}^\mu - 1)^2 + \|\Phi(z)\|^2. \tag{10}$$

From (4), we know that the P_0-NCP(f) is equivalent to the equation $G(z) = 0$ in the sense that their solutions coincide.

Theorem 2.6. *Let $z := (\mu, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $G(z)$ be defined by* (8) *and* (9)*. Then the following results hold.*

(i) *$G(z)$ is continuously differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$ with Jacobian*

$$G'(z) = \begin{pmatrix} \mathrm{e}^\mu & 0 \\ B(z) & C(z) \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}
B(z) &:= \operatorname{vec}\Big\{2(x_i+f_i(x)) - \frac{4\mu}{\sqrt{(x_i-f_i(x))^2+4\mu^2}}\colon\ i\in N\Big\},\\
C(z) &:= C_1(z)+C_2(z)f'(z),\\
C_1(z) &:= (1+2\mu)I - \operatorname{diag}\Big\{\frac{x_i-f_i(x)}{\sqrt{(x_i-f_i(x))^2+4\mu^2}}\colon\ i\in N\Big\},\\
C_2(z) &:= (1+2\mu)I + \operatorname{diag}\Big\{\frac{x_i-f_i(x)}{\sqrt{(x_i-f_i(x))^2+4\mu^2}}\colon\ i\in N\Big\}.
\end{aligned}$$

(ii) *If f is a P_0 function, then $G'(z)$ is nonsingular on $\mathbb{R}_{++}\times\mathbb{R}^n$.*

P r o o f. (i) Note that $\Phi(\mu,x)$ is continuously differentiable at any $(\mu,x)\in\mathbb{R}_{++}\times\mathbb{R}^n$. It is not hard to show that $G(z)$ defined by (8) is also continuously differentiable at any $z=(\mu,x)\in\mathbb{R}_{++}\times\mathbb{R}^n$. For any $\mu>0$, a direct calculation from (8) yields (11).

Next, we prove (ii). By (6) and (7), we obtain $C_1(z)\succ 0$ and $C_2(z)\succ 0$. In order to prove that $G'(z)$ is non-singular, we need only to show that the matrix $C(z)$ is non-singular. In fact, since f is a P_0-function, then $f'(x)$ is a P_0-matrix for all $x\in\mathbb{R}^n$ by Theorem 2.8 in [11]. Taking into account the fact that $C_2(z)$ is a positive diagonal matrix, by a straightforward calculation we have that all principal minors of the matrix $C_2(z)f'(z)$ are non-negative. By Definition 2.1, we know that the matrix $C_2(z)f'(z)$ is a P_0-matrix. Hence, by Theorem 3.3 in [2], the matrix $C_1(z)+C_2(z)f'(z)$ is invertible, which implies that the matrix $G'(z)$ is non-singular. $\square$

3. The algorithm and its well-definedness

We are now in the position to describe our algorithm formally.

A l g o r i t h m 3.1 (A new non-interior continuation method for P_0-NCP(f)).

Step 0. Choose constants $\delta\in(0,1)$, $\sigma\in(0,1)$, and an arbitrary initial point $z^0:=(\mu_0,x^0)\in\mathbb{R}_{++}\times\mathbb{R}^n$. Let $\eta=\sqrt{\Psi(z^0)}+1$ and $\bar\mu=\mu_0$, $\bar z:=(\bar\mu,0)\in\mathbb{R}_{++}\times\mathbb{R}^n$. Choose $\gamma\in(0,1)$ such that

$$\gamma\bar\mu\eta<\frac{1}{2}. \tag{12}$$

Set $k:=0$.

Step 1. If $\Psi(z^k)=0$, then stop. Else, let

$$\beta_k:=\beta(z^k)=\mathrm{e}^{\mu_k}\gamma\min\{1,\Psi(z^k)\}. \tag{13}$$

Step 2. Compute $\Delta z^k := (\Delta\mu_k, \Delta x^k) \in \mathbb{R} \times \mathbb{R}^n$ by

$$G(z^k) + G'(z^k)\Delta z^k = \beta_k \bar{z}. \tag{14}$$

Step 3. Let ν_k be the smallest nonnegative integer ν such that

$$\Psi(z^k + \delta^\nu \Delta z^k) \leqslant [1 - \sigma(1 - 2\gamma\eta\bar{\mu})\delta^\nu]\Psi(z^k). \tag{15}$$

Let $\lambda_k := \delta^{\nu_k}$.

Step 4. Set $z^{k+1} := z^k + \lambda_k \Delta z^k$ and $k := k+1$. Go to Step 1.

Remark 3.2. Notice that the algorithm has to solve only one system of linear equations and performs only one Armijo-type line search. If $\Psi(z^k) = 0$, then (x^k, y^k) is the solution of the P_0-NCP(f). So, the stopping criterion in Step 1 is reasonable.

Next, we show the well-definedness of Algorithm 3.1. To this end, we need the following lemma.

Lemma 3.3 ([9], Lemma 4.2)**.** *For any $\mu \geqslant 0$,*

$$-\mu \leqslant \frac{1 - \mathrm{e}^\mu}{\mathrm{e}^\mu} \leqslant -\mu \mathrm{e}^{-\mu}. \tag{16}$$

Define the set

$$\Omega := \{z = (\mu, x) \in \mathbb{R}_+ \times \mathbb{R}^n \colon\ \mu \geqslant \gamma \min\{1, \Psi(z)\}\bar{\mu}\}.$$

The following theorem shows that Algorithm 3.1 is well-defined.

Theorem 3.4. *Algorithm 3.1 is well-defined and generates an infinite sequence $\{z^k := (\mu_k, x^k)\}$ with $\mu_k > 0$ and $z^k \in \Omega$ for all $k \geqslant 0$.*

Proof. If $\mu_k > 0$, since f is a continuously differentiable P_0-function, it follows from Theorem 2.6 that the matrix $G'(z^k)$ is non-singular. Hence, Step 2 is well-defined at the kth iteration.

For any $\alpha \in (0, 1]$, from (14) we have

$$\Delta\mu_k = \frac{1 - \mathrm{e}^{\mu_k}}{\mathrm{e}^{\mu_k}} + \frac{\beta_k \bar{\mu}}{\mathrm{e}^{\mu_k}}. \tag{17}$$

From Lemma 3.3 and (17), for any $\alpha \in (0, 1]$, we have

$$\begin{aligned} \mu_{k+1} = \mu_k + \alpha\Delta\mu_k = \mu_k + \alpha\Big(\frac{1 - \mathrm{e}^{\mu_k}}{\mathrm{e}^{\mu_k}} + \frac{\beta_k \bar{\mu}}{\mathrm{e}^{\mu_k}}\Big) \\ \geqslant (1 - \alpha)\mu_k + \alpha\gamma\bar{\mu}\min\{1, \Psi(z^k)\} > 0. \end{aligned}$$

By the Taylor expansion and (17), we have

$$
\begin{aligned}
\mathrm{e}^{\mu_k+\alpha\Delta\mu_k}-1 &= \mathrm{e}^{\mu_k}[1+\alpha\Delta\mu_k+O(\alpha^2)]-1 \\
&= (\mathrm{e}^{\mu_k}-1)+\alpha\mathrm{e}^{\mu_k}\Delta\mu_k+O(\alpha^2) \\
&= (1-\alpha)(\mathrm{e}^{\mu_k}-1)+\alpha\beta_k\bar{\mu}+O(\alpha^2).
\end{aligned} \tag{18}
$$

It follows from $\beta_k^2=\mathrm{e}^{2\mu_k}\gamma^2(\min\{1,\Psi(z^k)\})^2\leqslant \mathrm{e}^{2\mu_k}\gamma^2\Psi(z^k)$ that

$$
\beta_k\leqslant \mathrm{e}^{\mu_k}\gamma\sqrt{\Psi(z^k)}. \tag{19}
$$

From (11) and (13), we obtain

$$
\mathrm{e}^{\mu_k}-1\leqslant\sqrt{\Psi(z^k)}\quad\text{and}\quad \mathrm{e}^{\mu_k}\leqslant\eta. \tag{20}
$$

Thus, we have

$$
\begin{aligned}
(\mathrm{e}^{\mu_k+\alpha\Delta\mu_k}-1)^2 &= (1-\alpha)^2(\mathrm{e}^{\mu_k}-1)^2 \\
&\quad +2\alpha(1-\alpha)\beta_k(\mathrm{e}^{\mu_k}-1)\bar{\mu}+\alpha^2\beta_k^2\bar{\mu}^2+O(\alpha^2) \\
&\leqslant (1-2\alpha)(\mathrm{e}^{\mu_k}-1)^2+2\alpha\gamma\sqrt{\Psi(z^k)}\mathrm{e}^{\mu_k}(\mathrm{e}^{\mu_k}-1)\bar{\mu}+O(\alpha^2) \\
&\leqslant (1-\alpha)(\mathrm{e}^{\mu_k}-1)^2+2\alpha\gamma\eta\Psi(z^k)\bar{\mu}+O(\alpha^2).
\end{aligned} \tag{21}
$$

On the other hand, from (14) we find that

$$
\Phi(z^k)+\Phi'(z^k)\Delta z^k=0.
$$

Therefore, we get

$$
\begin{aligned}
\|\Phi(z^k+\alpha\Delta z^k)\|^2 &= \|\Phi(z^k)+\alpha\Phi'(z^k)\Delta z^k+o(\alpha)\|^2 \\
&= \|(1-\alpha)\Phi(z^k)+o(\alpha)\|^2 \\
&= (1-\alpha)^2\|\Phi(z^k)\|^2+o(\alpha) \\
&= (1-2\alpha)\|\Phi(z^k)\|^2+o(\alpha).
\end{aligned} \tag{22}
$$

It follows from (8), (21), and (22) that

$$
\begin{aligned}
\Psi(z^k+\alpha\Delta z^k) &= (\mathrm{e}^{\mu_k+\alpha\Delta\mu_k}-1)^2+\|\Phi(z^k+\alpha\Delta z^k)\|^2 \\
&= (\mathrm{e}^{\mu_k+\alpha\Delta\mu_k}-1)^2+(1-2\alpha)\|\Phi(z^k)\|^2+o(\alpha) \\
&\leqslant (1-\alpha)(\mathrm{e}^{\mu_k}-1)^2+2\alpha\gamma\eta\Psi(z^k)\bar{\mu}+(1-\alpha)\|\Phi(z^k)\|^2+o(\alpha) \\
&\leqslant (1-\alpha)\Psi(z^k)+2\alpha\gamma\eta\Psi(z^k)\bar{\mu}+o(\alpha) \\
&= [1-(1-2\gamma\eta\bar{\mu})\alpha]\Psi(z^k)+o(\alpha).
\end{aligned}
$$

Since $\gamma\eta\bar{\mu} < 1/2$, there exists a positive constant $\overline{\alpha} \in (0,1]$ such that $\alpha \in (0,\overline{\alpha}]$, and $\Psi(z^k + \Delta z^k) \leqslant [1 - \sigma(1 - 2\gamma\eta\bar{\mu})\alpha]\Psi(z^k)$. Then the non-negative integer ν is found. Thus, Step 3 is well-defined. Therefore, Algorithm 3.1 is well-defined and generates an infinite sequence $\{z^k := (\mu_k, x^k)\}$ with $\mu_k > 0$ for all $k \geqslant 0$.

Next, we prove $z^k \in \Omega$ for all $k \geqslant 0$ by induction on k. Obviously, $\mu_0 \geqslant \gamma \min\{1, \Psi(z^k)\}\bar{\mu}$. Suppose that $z^k \in \Omega$, i.e., $\mu_k \geqslant \gamma \min\{1, \Psi(z^k)\}\bar{\mu}$. Then it follows from (15)–(17) that

$$\begin{aligned}
\mu_{k+1} &= \mu_k + \alpha\Delta\mu_k \\
&= \mu_k + \alpha\Big(\frac{1 - \mathrm{e}^{\mu_k}}{\mathrm{e}^{\mu_k}} + \frac{\beta_k\bar{\mu}}{\mathrm{e}^{\mu_k}}\Big) \\
&\geqslant \mu_k + \alpha\Big(-\mu_k + \frac{\gamma \mathrm{e}^{\mu_k} \min\{1, \Psi(z^k)\}\bar{\mu}}{\mathrm{e}^{\mu_k}}\Big) \\
&\geqslant (1-\alpha)\mu_k + \alpha\gamma\bar{\mu}\min\{1, \Psi(z^k)\} \\
&\geqslant (1-\alpha)\gamma\bar{\mu}\min\{1, \Psi(z^k)\} + \alpha\gamma\bar{\mu}\min\{1, \Psi(z^k)\} \\
&= \gamma\bar{\mu}\min\{1, \Psi(z^k)\} \geqslant \gamma\bar{\mu}\min\{1, \Psi(z^{k+1})\}.
\end{aligned}$$

□

4. Convergence analysis

In this section, we analyze the global and local convergence properties of Algorithm 3.1. It is shown that any accumulation point of the iteration sequence $\{z^k := (\mu_k, x^k)\}$ is a solution of the system $G(z) = 0$. If the accumulation point z^* satisfies a nonsingularity assumption, then the iteration sequence $\{z^k\}$ superlinearly converges to z^* without strict complementarity. Moreover, if f' is Lipschitz continuous on $\mathbb{R}^n$, then $\{z^k\}$ quadratically converges to z^*.

In order to analyze the global convergence properties of Algorithm 3.1, we need the following results.

Lemma 4.1. *Let $\Phi(\mu, x)$ be defined by (9). For any $\mu > 0$ and $c > 0$, define level set*

$$L_\mu(c) := \{x \in \mathbb{R}^n \colon \ \|\Phi(\mu, x)\| \leqslant c\}. \tag{23}$$

Then, for any $\mu_2 \geqslant \mu_1 > 0$ and $c > 0$, the set $L(c) = \bigcup_{\mu_1 \leqslant \mu \leqslant \mu_2} L_\mu(c)$ is bounded.

From Lemma 4.1, we know that the set $L_\mu(c)$ is bounded for any $\mu > 0$. We can immediately get the following result.

Lemma 4.2. *Suppose that f is a P_0-function and $\mu > 0$. Then the function $\|\Phi(\mu, x)\|^2$ is coercive, i.e., $\lim\limits_{\|x\|\to\infty} \|\Phi(\mu, x)\|^2 = \infty$.*

Lemma 4.3. *Let $\Psi(\cdot)$ be defined by* (8) *and $\{z^k := (\mu_k, x^k)\}$ be the iteration sequence generated by Algorithm* 3.1*. Then the sequence $\{\Psi(z^k)\}$ is convergent. If it does not converge to zero, then $\{z^k := (\mu_k, x^k)\}$ is bounded.*

Proof. From Step 3 and Theorem 3.4 we know that $\{\Psi(z^k)\}$ is monotonically decreasing and $\{z^k\} \in \Omega$. So, $\{\Psi(z^k)\}$ is convergent. Then there exists Ψ^* such that $\Psi(z^k) \to \Psi^*$ as $k \to \infty$. If $\{\Psi(z^k)\}$ does not converge to zero, we have $\Psi^* > 0$. From $\{z^k\} \subset \Omega$ and $\mu_k \leqslant \mathrm{e}^{\mu_k} - 1 \leqslant f(z^k) \leqslant f(z^0)$, we know that $\{\mu_k\}$ is bounded. Obviously, there exist $\mu_1, \mu_2 > 0$ such that $0 < \mu_1 \leqslant \mu_k \leqslant \mu_2$ for all $k \geqslant 0$. Let $c_0 := \|\Psi(z^0)\|$ and $L(c_0) := \bigcup\limits_{\mu_1 \leqslant \mu_k \leqslant \mu_2} L_{\mu_k}(c_0)$, where $L_{\mu_k}(c_0)$ is defined by (23). It is not difficult to see that $x^k \in L(c_0)$, since $x^k \in L_{\mu_k}(c_0)$. It follows from Lemma 4.1 that the set $L(c_0)$ is bounded and hence $\{x^k\}$ is bounded. Therefore, $\{z^k\}$ is bounded. $\square$

Now we are in the position to give the main results. First, we give the global convergence.

Theorem 4.4 (Global convergence)**.** *Suppose that f is a continuously differentiable P_0-function, the sequence $\{z^k = (\mu_k, x^k)\}$ is generated by Algorithm* 3.1*, and the solution set Θ is non-empty and bounded. Then $\{z^k\}$ has at least one accumulation point $\{z^* = (\mu_*, x^*)\}$ with $x^* \in \Theta$, and any accumulation point of $\{z^k\}$ is a solution of $G(z) = 0$.*

Proof. From Lemma 4.2, the SSPM-function defined by (3), and $G(z)$ defined by (8) we get coerciveness. So, the level set $L(c)$ is bounded and the infinite sequence $\{z^k\}$ generated by Algorithm 3.1 has at least one accumulation point. Without loss of generality, we assume that $z^* = (\mu_*, x^*)$ is the limit point of the sequence $z^k = (\mu_k, x^k)$ as $k \to \infty$. It follows from the continuity of $G(\cdot)$ that $\|G(z^k)\|$ converges to a non-negative number $\|G(z^*)\|$. From the definition of $\beta(\cdot)$, we obtain that β_k is monotonically decreasing, and converges to $\beta_* = \mathrm{e}^{\mu_*}\gamma \min\{1, \Psi(z^*)\}$.

Now, we prove $G(z^*) = 0$ by contradiction. In fact, if $G(z^*) \neq 0$, then $\|G(z^*)\| > 0$. For $\mu_k \in \Omega$, we have $0 < \beta_*\mu_0 \leqslant \mu_*$. By Theorem 2.6, there exists a closed neighborhood $\mathscr{N}(z^*)$ of z such that for any $z \in \mathscr{N}(z^*)$, we have $\mu \in \mathbb{R}_{++}$ and

$G'(z)$ is invertible. Then, for any $z \in \mathscr{N}(z^*)$, let $\Delta z := (\Delta\mu, \Delta x) \in \mathbb{R} \times \mathbb{R}^n$ be the unique solution of the system of equations:

$$G(z) + G'(z)\Delta z = \beta(z)\overline{z},$$

then we can find a positive number $\overline{\alpha} \in (0, 1]$ such that

$$\Psi(z + \alpha\Delta z) \leqslant [1 - \sigma(1 - 2\gamma\eta\bar{\mu})\alpha]\Psi(z)$$

for any $\alpha \in (0, \overline{\alpha}]$ and $z \in \mathscr{N}(z)$. Therefore, for a nonnegative integer ν such that $\delta^\nu \in (0, \overline{\alpha}]$, we have $\nu^k \leqslant \nu$ for all sufficiently large k. Since $\delta^{\nu^k} \geqslant \delta^\nu$, it follows from (15) that

$$\Psi(z^{k+1}) \leqslant [1 - \sigma(1 - 2\gamma\eta\bar{\mu})\delta^{\nu^k}]\Psi(z^k) \leqslant [1 - \sigma(1 - 2\gamma\eta\bar{\mu})\delta^\nu]\Psi(z^k).$$

This contradicts the fact that the sequence $\{\Psi(z^k)\}$ converges to $\Psi(z^*) = \|G(z^*)\|^2 > 0$. The proof is completed. $\square$

To establish the locally Q-quadratic convergence of Algorithm 3.1, we need the following assumption:

Assumption 4.5. Assume that z^* satisfies the nonsingularity condition, i.e., all $V \in \partial G(z^*)$ are nonsingular.

Next we give the rate of convergence for Algorithm 3.1.

Theorem 4.6 (Local convergence)**.** *Suppose that f is a continuously differentiable P_0-function and z^* is an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm* 3.1*. If Assumption* 4.5 *holds, then*

(i) $\lambda_k \equiv 1$ *for all z^k sufficiently close to z^*.*

(ii) *The whole sequence $\{z^k\}$ superlinearly converges to z^*, i.e.,*

$$\|z^{k+1} - z^*\| = o(\|z^k - z^*\|), \tag{24}$$

and

$$\mu_{k+1} = o(\mu_k). \tag{25}$$

Furthermore, if f' is Lipschitz continuous on $\mathbb{R}^n$, then

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^2), \tag{26}$$

and

$$\mu_{k+1} = O(\mu_k^2). \tag{27}$$

P r o o f. (i) By Theorems 2.5 and 4.4, G is semi-smooth at z^*. From Theorem 4.4 we see that z^* is a solution of $G(z) = 0$. Then, from Proposition 4.1 of [16], for all z^k sufficiently close to z^*,

$$\|G'(z^k)^{-1}\| = O(1).$$

Hence, under the assumption that G is semi-smooth at z^*, for z^k sufficiently close to z^*, we have

$$\begin{aligned}\|z^k + \Delta z^k - z^*\| &= \|z^k + G'(z^k)^{-1}[-G(z^k) + \beta_k \overline{z}] - z^*\| \\ &= O(\|G(z^k) - G(z^*) - G'(z^k)(z_k - z^*)\| + \beta_k \|\bar{\mu}\|) \\ &= o(\|z^k - z^*\|) + O(\Psi(z^k)).\end{aligned} \tag{28}$$

Then, because G is semi-smooth at z^*, G is locally Lipschitz continuous near z^* (if f' is Lipschitz continuous on $\mathbb{R}^n$, then G is strongly semi-smooth), for all z_k close to z^*,

$$\Psi(z^k) = \|G(z^k)\|^2 = O(\|z^k - z^*\|^2). \tag{29}$$

Therefore, from (28) and (29), if G is semi-smooth (strongly semi-smooth, respectively) at z^*, for all z_k sufficiently close to z^*,

$$\|z^k + \Delta z^k - z^*\| = o(\|z^k - z^*\|) = O(\|z^k - z^*\|^2). \tag{30}$$

By following the proof of Theorem 3.1 of [13], for all z_k sufficiently close to z^*, we have

$$\|z^k - z^*\| = O(\|G(z^k) - G(z^*)\|). \tag{31}$$

Hence, if G is semi-smooth (strongly semi-smooth, respectively) at z^*, for all z^k sufficiently close to z^*, we have

$$\begin{aligned}\Psi(z^k + \Delta z^k) &= \|G(z^k + \Delta z^k)\|^2 \\ &= O(\|z^k + \Delta z^k - z^*\|^2) \\ &= o(\|z^k - z^*\|^2) \\ &= o(\|G(z^k) - G(z^*)\|^2) \\ &= o(\Psi(z^k)) \quad (= O(\Psi(z^k)^2)).\end{aligned} \tag{32}$$

Therefore, for all z_k sufficiently close to z^*, we have $z^{k+1} = z^k + \Delta z^k$, i.e., $\lambda_k \equiv 1$.

Next, we prove (ii). By (i) and (30), we get (24), and if G is strongly semi-smooth at z^*, (26) is proved. From the definition of β_k and the fact that $z^k \to z^*$ as $k \to \infty$, for all k sufficiently large,

$$\beta_k = \gamma \Psi(z^k) = \gamma \|G(z^k)\|^2.$$

Also, because for all k sufficiently large, $z^{k+1} = z^k + \Delta z^k$, we have for all k sufficiently large that

$$\mu_{k+1} = \mu_k + \Delta\mu_k = \beta_k \mu_0.$$

Hence, for all k sufficiently large, $\mu^{k+1} = \gamma \|G(z^k)\|^2 \mu_0$, which together with (26), (29), and (31), gives

$$\lim_{k\to\infty} \frac{\mu_i^{k+1}}{\mu_i^k} = \lim_{k\to\infty} \frac{\|G(z^k)\|^2}{\|G(z^{k-1})\|^2} = \lim_{k\to\infty} \frac{\|G(z^k) - G(z^*)\|^2}{\|G(z^{k-1}) - G(z^*)\|^2} = 0, \quad i \in N.$$

This proves (25). If G is strongly semi-smooth at z^*, then from the above argument we can easily get (27). □

5. Numerical results

In this section, we present the results of some numerical experiments with Algorithm 3.1. All these experiments were performed on the personal computer with Intel(R) Pentium(R) 4 CPU 2.00 GHz and 512 MB memory. The operating system was Windows XP (SP2) and the implementations were done in MATLAB 7.0.1.

n	Iter	Res	CPU time (s)
80	32	1.8775e−21	0.7910
	30	2.1075e−16	0.7010
	26	9.9435e−22	0.5200
	31	2.7201e−19	0.7510
120	38	1.7741e−19	1.7120
	32	1.4904e−19	1.5120
	30	2.8043e−23	1.1920
	31	4.4845e−20	1.2720
160	47	5.5494e−23	3.8350
	38	1.2552e−20	2.6640
	49	2.2640e−24	3.8550
	44	3.3695e−24	3.6550
200	53	9.4888e−23	6.9800
	47	2.8573e−21	5.8790
	58	3.4899e−23	7.3300
	56	1.0675e−23	7.0700

Table 5.1. Numerical results of Algorithm 3.1.

We tested some NCPs with $f(x) = P(x) + Mx + q$, where $P(x)$ and $Mx + q$ are the non-linear and linear parts of $f(x)$, respectively. We consider the problem P_0-NCP(f) and we form the matrix P and the vector q as follows. The matrix $M = A^\top A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-2, 2)$ and a skew symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-10, 10)$. The components of $P(x)$, the non-linear part of $f(x)$, are $P_j(x) = p_j \cdot \arctan(x_j)$, where p_j is a random variable in $(0, 2)$.

The initial point x^0 is generated from a uniform distribution in the interval $(0, 2)$ and μ_0 is a random number in $(0, 2)$. Throughout the computational experiments, the parameters used in the algorithm were $\sigma = 0.6$, $\gamma = 0.0005$, $\delta = 0.95$. We use $\|\Psi(x)\|^2 \leqslant 10^{-10}$ to be the termination criterion. We choose $n = 80$, 120, 160, 200 as the dimension of the problem, respectively. The results are listed in Tab. 5.1. Iter stands for the numbers of iterations. CPU time (s) denotes the CPU time in second needed for obtaining optimum. Res represents the value of $\|\Psi(x^k)\|^2$ when our stop rule is satisfied.

The results in Tab. 5.1 show the feasibility and efficiency of our Algorithm 3.1. We also obtained similar results for other random examples.

6. Conclusions

In this paper, the P_0-NCP(f) was discussed in detail by combining the virtues of the SSPM-function and the non-interior continuation method. The boundedness of the level sets was obtained under the assumption of the P_0 property of f. Without strict complementarity, we provided a weaker condition to guarantee the global convergence and local convergence of the Algorithm. The proposed algorithm does not have restrictions on its starting point. Compared to many previous works, our method has stronger convergence properties under milder assumptions. We also report some preliminary computational results. The numerical experiments show that our algorithm has good convergence properties.

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