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Some algebraic properties of hypergraphs

SOME ALGEBRAIC PROPERTIES OF HYPERGRAPHS

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Abstract. We consider Stanley-Reisner rings $k[x_1, \ldots, x_n]/I(\mathcal{H})$ where $I(\mathcal{H})$ is the edge ideal associated to some particular classes of hypergraphs. For instance, we consider hypergraphs that are natural generalizations of graphs that are lines and cycles, and for these we compute the Betti numbers. We also generalize some known results about chordal graphs and study a weak form of shellability.

Keywords: Betti numbers, chordal hypergraphs, connectivity, homologically connected hypergraphs, hypercycles, line hypergraphs, shellability

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1. Introduction

Stanley-Reisner rings associated to graphs have been widely studied since edge ideals were introduced by R. Villarreal in [22]. Chordal graphs have been in particular focus and indeed, many natural results and questions seem to be connected to this class of graphs. See for example [3], [14], [15], [17], [18], [11]. More recently, also edge ideals of hypergraphs have been studied and many results and familiar properties of graphs have got hypergraph analogue. See for example [7], [8], [15], [21]. Also [19] should be mentioned. In that paper the author computes Betti numbers of many classes of graphs.

Graphs may be generalized in several different ways of which hypergraphs is merely one. S. Faridi introduced ([9]) a way of viewing a simplicial complex as a generalization of a graph. The complexes considered are called facet complexes. The idea is to consider the facets of a complex as a kind of generalized edges. Since a graph may be considered as a one dimensional simplicial complex, in this way one indeed obtains a natural generalization of graphs. Many nice results may be found in [9],

[10], [17], [18], [23]. In [23], the author introduces the class of quasi-forests, which relates to chordal graph, see [18]. Also, in this paper, we show that they are closely related to chordal hypergraphs.

In Section 3 we consider natural hypergraph generalizations of graphs that are lines and cycles. We call them line hypergraphs and hypercycles. Here we generalize known results about Betti numbers from [19].

In Section 4 we consider chordal hypergraphs, as defined in [8]. Corollary 4.4 is known from [18], and Theorem 4.3 provides a partial generalization of this result.

The results in Section 5 are related to the concept of connectivity. Connectivity in graph theory is a well known concept and we explore it in a hypergraph context. Using (hypergraph) connectivity we are able to generalize some results on bounds on Betti numbers from [19]. Also, some results connecting the depth of certain Stanley-Reisner rings to connectivity, given in [14], can be seen in a wider context and explained a bit deeper.

In the last section, Section 6, we consider the notion of d-shellability, a weaker notion than ordinary shellability. We see that, to some extent, d-shellability behaves like ordinary shellability. In particular, we show that there is an Alexander dual notion called d-quotients, that provides a natural generalization of the concept of linear quotients that is known to be Alexander dual to shellability. For ideals with d-quotients we give a formula for the Betti numbers and show that these ideals admit Betti splittings.

2. Basics

We give some basics that will be used in the paper. A good reference for hypergraphs is Berge's book [1].

Let $\mathcal{X}$ be a finite set and $\mathcal{E} = \{E_1, \ldots, E_s\}$ a finite collection of non empty subsets of $\mathcal{X}$. The pair $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is called a *hypergraph*. The elements of $\mathcal{X}$ and $\mathcal{E}$, respectively, are called the *vertices* and the *edges*, respectively, of the hypergraph. If we want to specify what hypergraph we consider, we may write $\mathcal{X}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ for the vertices and edges respectively. A hypergraph is called *simple* if: (1) $|E_i| \geqslant 2$ for all $i = 1, \ldots, s$ and (2) $E_j \subseteq E_i$ only if $i = j$. If the cardinality of $\mathcal{X}$ is n we often just use the set $[n] = \{1, 2, \ldots, n\}$ instead of $\mathcal{X}$.

Let $\mathcal{H}$ be a hypergraph. A *subhypergraph* $\mathcal{K}$ of $\mathcal{H}$ is a hypergraph such that $\mathcal{X}(\mathcal{K}) \subseteq \mathcal{X}(\mathcal{H})$, and $\mathcal{E}(\mathcal{K}) \subseteq \mathcal{E}(\mathcal{H})$. If $\mathcal{Y} \subseteq \mathcal{X}$, the *induced hypergraph on* $\mathcal{Y}$, $\mathcal{H}_\mathcal{Y}$, is the subhypergraph with $\mathcal{X}(\mathcal{H}_\mathcal{Y}) = \mathcal{Y}$ and with $\mathcal{E}(\mathcal{H}_\mathcal{Y})$ consisting of the edges of $\mathcal{H}$ that lie entirely in $\mathcal{Y}$. A hypergraph $\mathcal{H}$ is said to be *d-uniform* if $|E_i| = d$ for every edge $E_i \in \mathcal{E}(\mathcal{H})$. By a uniform hypergraph we mean a hypergraph that is d-uniform for some d. Note that a simple 2-uniform hypergraph is just an ordinary simple

graph. In this paper we consider only simple uniform hypergraphs, and hence, by hypergraph we will always mean a simple hypergraph.

A *free vertex* v of a hypergraph is, if there is one, a vertex v that lies in at most one edge.

One type of hypergraphs of particular importance are the d-complete hypergraphs K_n^d: The *d-complete hypergraph*, K_n^d, on a set of n vertices, is defined by

$$\mathcal{E}(K_n^d) = \binom{[n]}{d}$$

where $\binom{F}{d}$ denotes the set of all subsets of F, of cardinality d. If $n < d$, we interpret K_n^d as n isolated points.

Recall that an *(abstract) simplicial complex* on vertex set $[n]$ is a collection, Δ, of subsets of $[n]$ with the property that $G \subseteq F$, $F \in \Delta \Rightarrow G \in \Delta$. The elements of Δ are called the *faces* of the complex and the maximal (under inclusion) faces are called *facets*. The set of facets of Δ is denoted by $\mathcal{F}(\Delta)$. The *dimension, $\dim F$, of a face F* in Δ, is defined to be $|F| - 1$, and the dimension of Δ is defined as $\dim \Delta = \max\{\dim F;\ F \in \Delta\}$. Note that the empty set $\emptyset$ is the unique -1 dimensional face of every complex that is not the void complex $\{\}$ which has no faces. The dimension of the void complex may be defined as $-\infty$. Let $V \subseteq [n]$. We denote by Δ_V the simplicial complex

$$\Delta_V = \{F \subseteq [n];\ F \in \Delta,\ F \subseteq V\}.$$

$\widetilde{H}_n(\Delta; k)$ will denote the reduced homology of (the chain complex of) Δ with coefficients in the field k.

To every d-uniform hypergraph on vertex set $[n]$ we associate two simplicial complexes, the *Independence complex of $\mathcal{H}$, $\Delta_{\mathcal{H}}$*, and the *Clique complex of $\mathcal{H}$, $\Delta(\mathcal{H})$*. These are defined as follows:

$$\Delta_{\mathcal{H}} = \{F \subseteq [n];\ E \nsubseteq F,\ \forall E \in \mathcal{E}(\mathcal{H})\}$$
$$\Delta(\mathcal{H}) = \left\{F \subseteq [n];\ \binom{F}{d} \subseteq \mathcal{E}(\mathcal{H})\right\}.$$

Throughout the paper R will denote some polynomial ring $k[x_1, \ldots, x_n]$. The number n will be the cardinality of the vertex set of some hypergraph considered. We use the convention that for a subset $F \subseteq [n]$, $x^F = \prod_{i \in F} x_i$. Now, let Δ be a simplicial complex on $[n]$. The *Stanley-Reisner ring R/I_Δ* of Δ is the quotient of the ring $R = k[x_1, \ldots, x_n]$ by the *Stanley-Reisner ideal*

$$I_\Delta = (x^F;\ F \notin \Delta)$$

generated by the nonfaces of Δ. Conversely, to every squarefree monomial ideal I one may associate a unique simplicial complex Δ_I in such way that its Stanley-Reisner ideal is precisely I. If Δ is a given simplicial complex on vertex set $[n]$, its *Alexander dual simplicial complex* is defined by

$$\Delta^* = \{F \subseteq [n]; \ [n] \setminus F \notin \Delta\}.$$

This yields a natural duality of squarefree monomial ideals as well. Hence we may denote the Stanley-Reisner ideal of Δ^* by I_Δ^*. More information about the relations between the ideals I_Δ and I_Δ^* can be found in [20].

If I is any monomial ideal, we denote by $\mathcal{G}(I)$ its unique set of minimal monomial generators.

If $\mathcal{H}$ is a hypergraph, the Stanley-Reisner ideal of $\Delta_\mathcal{H}$ is called the edge ideal of $\mathcal{H}$, and is denoted $I(\mathcal{H})$.

Recall that the projective dimension, $\mathrm{pd}_R(M)$ of an R-module M, is defined as

$$\mathrm{pd}_R(M) = \max\{i; \ \mathrm{Tor}_i^R(M, k) \neq 0\}.$$

This number may depend on the characteristic of k. Furthermore, recall that the number $\beta_i(M) = \dim_k \mathrm{Tor}_i^R(M, k)$ is called the i'th Betti number of M. Note that the Tor-modules and the Betti numbers are, in the cases we consider, naturally $\mathbb{N}^n$-graded. For details about the algebra used in connection with simplicial complexes, we refer the reader to the books [2] and [20].

3. Hypercycles and line hypergraphs

In [19] the author computes the Betti numbers of, among other things, graphs that are lines and cycles. When trying to lift these concepts to hypergraph analogue, one may handle the increased degree of freedom in potentially many different ways. We define line hypergraphs $L_n^{d,\alpha}$, and hypercycles $C_n^{d,\alpha}$ in a natural way and we compute their Betti numbers. In particular, we see that the formulas for the Betti numbers are independent of the characteristic of the field k.

Definition 3.1. For positive integers n, α and d, where $d \geqslant 2\alpha$, we define the *line hypergraph* $L_n^{d,\alpha}$ as the d-uniform hypergraph with edge set $\mathcal{E}(L_n^{d,\alpha}) = \{E_1, \dots, E_n\}$ and vertex set $\mathcal{X}(L_n^{d,\alpha}) = \bigcup\limits_{i=1}^{n} E_i$ such that:

(i) For any $i \neq j$, $E_i \cap E_j \neq \emptyset$ if and only if $|j - i| = 1$, $0 \leqslant i \leqslant n$.
(ii) $|E_i \cap E_{i+1}| = \alpha$ for all i, $1 \leqslant i \leqslant n - 1$.

The length of a line hypergraph is defined as the number of edges.

Definition 3.2. The *hypercycle* $C_n^{d,\alpha}$ is the d-uniform hypergraph with edge set $\mathcal{E}(C_n^{d,\alpha}) = \{E_1, \ldots, E_n\}$ and vertex set $\mathcal{X}(C_n^{d,\alpha}) = \bigcup\limits_{i=1}^{n} E_i$ such that:

(i) For any $i \neq j$ we have $E_i \cap E_j \neq \emptyset$ if and only if $|j - i| \equiv 1 \,(\bmod\, n)$.

(ii) $|E_i \cap E_{i+1}| = \alpha$ for all i, $1 \leqslant i \leqslant n - 1$ and $|E_1 \cap E_n| = \alpha$.

3.1. Betti numbers of line hypergraphs.

Assume that in a d-uniform hypergraph $\mathcal{H}$ the following holds: 1) If E_i, E_j are two intersecting edges, then $|E_i \cap E_j| = \alpha$ for some fixed α. 2) Every edge has a free vertex. For such hypergraph it is easy to see that $2\alpha < d$ must hold. This is the motivation of the following theorem, and the reason that it is natural to divide our further considerations into two cases, namely the case when $2\alpha < d$ and the case when $2\alpha = d$.

Theorem 3.3. *Let $\mathcal{H}$ be a hypergraph such that each edge of $\mathcal{H}$ has a free vertex. Then $\beta_i(R/I(\mathcal{H})) = \binom{n}{i}$, where n is the number of edges of $\mathcal{H}$.*

P r o o f. Let $\mathcal{E}(\mathcal{H}) = \{E_1, \ldots, E_n\}$ and $v_i \in E_i$ be a free vertex for any i, $1 \leqslant i \leqslant n$. Then, for any j, $1 \leqslant j \leqslant n$, x^{E_j} does not divide $\mathrm{lcm}(x^{E_1}, \ldots, x^{E_{j-1}}, x^{E_{j+1}}, \ldots, x^{E_n})$, since $v_j \nmid \mathrm{lcm}(x^{E_1}, \ldots, x^{E_{j-1}}, x^{E_{j+1}}, \ldots, x^{E_n})$. Therefore the Taylor resolution of $R/I(\mathcal{H})$ is minimal and $\beta_i(R/I(\mathcal{H})) = \binom{n}{i}$. □

In the following theorem, we give a combinatorial interpretation of the graded Betti numbers of a hypergraph in which all edges have a free vertex.

Theorem 3.4. *Let $\mathcal{H}$ be a hypergraph with edges $E_1, \ldots, E_n$ such that each edge has a free vertex. Then*

$$\beta_{i,j}(R/I(\mathcal{H})) = \left| \left\{ F \subseteq [n]; \ |F| = i, \left| \bigcup_{k \in F} E_k \right| = j \right\} \right|.$$

P r o o f. Since each edge of $\mathcal{H}$ has a free vertex, it is enough to find the number of basis elements $e_{k_1, \ldots, k_i}$ of degree j in the Taylor resolution of $R/I(\mathcal{H})$. We have $\deg(e_{k_1, \ldots, k_i}) = \deg(\mathrm{lcm}(x^{E_{k_1}}, \ldots, x^{E_{k_i}}))$. Since all $x^{E_{k_l}}$'s are squarefree, $\deg(\mathrm{lcm}(x^{E_{k_1}}, \ldots, x^{E_{k_i}})) = \left| \bigcup\limits_{l=1}^{i} E_{k_l} \right|$, which completes the proof. □

We have the following corollary:

Corollary 3.5. *Let $L_n^{d,\alpha}$ be a line hypergraph such that $d > 2\alpha$. Then*

$$\beta_{n,j}(R/I(L_n^{d,\alpha})) = \begin{cases} 0 & \text{if } j \neq n(d-\alpha)+\alpha, \\ 1 & \text{if } j = n(d-\alpha)+\alpha. \end{cases}$$

P r o o f. Since $d > 2\alpha$, each edge has a free vertex. Thus by Theorem 3.4,

$$\beta_{n,j}(R/I(L_n^{d,\alpha})) = \left| \left\{ F \subseteq [n]; \ |F| = n, \left| \bigcup_{k \in F} E_k \right| = j \right\} \right|.$$

Therefore it equals 1 if $j = \left| \bigcup_{i=1}^{n} E_i \right|$ and 0 if $j \neq \left| \bigcup_{i=1}^{n} E_i \right|$. Since $\left| \bigcup_{i=1}^{n} E_i \right| = n(d-\alpha)+\alpha$, the assertion holds. Note that $n(d-\alpha)+\alpha = |\mathcal{X}(L_n^{d,\alpha})|$. $\square$

To compute the Betti numbers of a line hypergraph, when $d > 2\alpha$ we need to prove a lemma. Let s_i $(1 \leqslant i \leqslant r)$ be positive integers and $L_n^{d,\alpha}$ be a line hypergraph of length n. Set $E(s_1, \ldots, s_r, n) = \{\mathcal{H}; \ \mathcal{H}$ is a subhypergraph of $L_n^{d,\alpha}$, which is comprised of r disjoint line hypergraphs of lengths $s_1, \ldots, s_r$ with no isolated vertex$\}$. Then we have the following lemma:

Lemma 3.6. *Let $s_1, \ldots, s_r$ be positive integers such that $1 \leqslant s_1 = \ldots = s_{l_1} < s_{l_1+1} = \ldots = s_{l_1+l_2} < s_{l_1+l_2+1} = \ldots = s_{l_1+l_2+l_3} < \ldots < s_{l_1+\ldots+l_{t-1}+1} = \ldots = s_{l_1+\ldots+l_t} = s_r$ and $s_1 + \ldots + s_r = i$, then $|E(s_1, \ldots, s_r, n)| = (r!/l_1! \ldots l_t!)\binom{n-i+1}{r}$.*

P r o o f. Let $L_n^{d,\alpha}$ be a line hypergraph of length n and S be the set of hypergraphs $\mathcal{H} \in E(s_1, \ldots, s_r, n)$ such that $\mathcal{H}$ is comprised of line hypergraphs $Q_1, \ldots, Q_r$ such that the length of Q_i is s_i and for any $1 \leqslant i < j \leqslant r$, if $v_\lambda \in \mathcal{X}(Q_i)$ and $v_\gamma \in \mathcal{X}(Q_j)$, then $\lambda < \gamma$. It can be seen that $|E(s_1, \ldots, s_r, n)| = (r!/l_1! \ldots l_t!)|S|$. We claim that there is a bijection between S and the set $\{(t_1, \ldots, t_{r+1}), t_1, t_{r+1} \geqslant 0, t_2, \ldots, t_r \geqslant 1, \sum_{l=1}^{r+1} t_l = n - i\}$. For any $\mathcal{H} \in S$, let $\mathcal{H}'$ be the subhypergraph of $L_n^{d,\alpha}$ with edge set $\mathcal{E}(L_n^{d,\alpha}) \setminus \mathcal{E}(\mathcal{H})$. Assume $\mathcal{H}'$ is comprised of l line hypergraphs $Q'_1, \ldots, Q'_l$ such that for any $1 \leqslant i < j \leqslant l$, if $v_\lambda \in \mathcal{X}(Q'_i)$ and $v_\gamma \in \mathcal{X}(Q'_j)$, then $\lambda < \gamma$. We have $r - 1 \leqslant l \leqslant r + 1$. Set $v_{\mathcal{H}} = (t_1, \ldots, t_{r+1})$, where t_i $(1 \leqslant i \leqslant r + 1)$ are as follows:

(i) If $l = r - 1$, set $t_1, t_{r+1} = 0$ and $t_i = l(Q'_{i-1})$ for any i $(2 \leqslant i \leqslant r)$.
(ii) If $l = r$ and $v_0 \in \mathcal{X}(Q_1)$, set $t_1 = 0$ and $t_i = l(Q'_i)$ for any i $(2 \leqslant i \leqslant r + 1)$.
(iii) If $l = r$ and $v_0 \in \mathcal{X}(Q'_1)$, set $t_{r+1} = 0$ and $t_i = l(Q'_i)$ for any i $(1 \leqslant i \leqslant r)$.
(iv) If $l = r + 1$, set $t_i = l(Q'_i)$ for any i $(1 \leqslant i \leqslant r + 1)$.

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We have $\sum_{k=1}^{r+1} t_k = \sum_{k=1}^{l} l(Q'_k) = n - i$. Define a function

$$\varphi\colon S \longrightarrow \left\{(t_1,\ldots,t_{r+1}),\ t_1, t_{r+1} \geqslant 0,\ t_2,\ldots,t_r \geqslant 1,\ \sum_{l=1}^{r+1} t_l = n - i\right\}$$

by $\varphi(\mathcal{H}) = v_{\mathcal{H}}$. It is easy to see that φ is a bijection and hence

$$|S| = \left|\left\{(t_1,\ldots,t_{r+1}),\ t_1, t_{r+1} \geqslant 0, t_2,\ldots,t_r \geqslant 1,\ \sum_{l=1}^{r+1} t_l = n - i\right\}\right|$$

$$= \left|\left\{(t'_1,\ldots,t'_{r+1}),\ t'_1,\ldots,t'_{r+1} \geqslant 1,\ \sum_{l=1}^{r+1} t'_l = n - i + 2\right\}\right| = \binom{n - i + 1}{r}.$$

Thus $|E(s_1,\ldots,s_r,n)| = (r!/l_1!\ldots l_t!)\binom{n-i+1}{r}$. $\square$

We now give the graded Betti numbers of a line hypergraph $L_n^{d,\alpha}$ in the case when $d > 2\alpha$.

Theorem 3.7. *Let $i < n$ be an integer and let $L_n^{d,\alpha}$ be a line hypergraph such that $d > 2\alpha$. Then $\beta_{i,id-\alpha(i-r)}(R/I(L_n^{d,\alpha})) = \binom{i-1}{r-1}\binom{n-i+1}{r}$ for any r, $1 \leqslant r \leqslant i$ and $\beta_{i,j}(R/I(L_n^{d,\alpha})) = 0$ for all other j.*

P r o o f.   Let $i < n$ and j be integers such that $\beta_{i,j}(R/I(L_n^{d,\alpha})) \neq 0$. Since $d > 2\alpha$, each edge has a free vertex. Thus as was shown in Theorem 3.4, $\beta_{i,j}(R/I(L_n^{d,\alpha})) = \left|\left\{F \subseteq [n];\ |F| = i,\ \left|\bigcup_{k \in F} E_k\right| = j\right\}\right|$. Let $E_{l_1},\ldots,E_{l_i}$ be some edges of $L_n^{d,\alpha}$ such that $\left|\bigcup_{t=1}^{i} E_{l_t}\right| = j$ and let $\mathcal{H}$ be the subhypergraph of $L_n^{d,\alpha}$ with edge set $\{E_{l_1},\ldots,E_{l_i}\}$ and assume that $\mathcal{H}$ is comprised of r line hypergraphs which are of lengths $s_1,\ldots,s_r$. Then $s_1 + \ldots + s_r = i$. Let $L_t^{d,\alpha} \subseteq \mathcal{H}$ be the line hypergraph of length s_t, so that $|\mathcal{X}(L_t^{d,\alpha})| = s_t d - \alpha(s_t - 1)$. Therefore $|\mathcal{X}(\mathcal{H})| = \sum_{t=1}^{r}(s_t d - \alpha(s_t - 1)) = id - \alpha(i - r)$ and $j = id - \alpha(i - r)$ for some $1 \leqslant r \leqslant i$. Hence

$$\beta_{i,id-\alpha(i-r)}(R/I(L_n^{d,\alpha})) = \sum_{1 \leqslant s_1 \leqslant \ldots \leqslant s_r, s_1 + \ldots + s_r = i} |E(s_1,\ldots,s_r,n)|$$

and $\beta_{i,j}(R/I(L_n^{d,\alpha})) = 0$ for those j that can not be written in the form $id - \alpha(i-r)$ for some r. Let $l_1,\ldots,l_m$ be positive integers and $P_{(l_1,\ldots,l_m)}$ be the number of solutions of $x_1 + \ldots + x_r = i$ such that $x_j \geqslant 1$ $(1 \leqslant j \leqslant r)$ and l_i components of $(x_1,\ldots,x_r)$ are equal for all i, $1 \leqslant i \leqslant m$. Then $\sum_{m \geqslant 1, l_i \geqslant 1} P_{(l_1,\ldots,l_m)} = \binom{i-1}{r-1}$. Also the number of

solutions of $x_1 + \ldots + x_r = i$ such that $1 \leqslant x_1 = \ldots = x_{l_1} < x_{l_1+1} = \ldots = x_{l_1+l_2} < x_{l_1+l_2+1} = \ldots = x_{l_1+l_2+l_3} < \ldots < x_{l_1+\ldots+l_{m-1}+1} = \ldots = x_{l_1+\ldots+l_m} = x_r$ is equal to $P_{(l_1,\ldots,l_m)} l_1! \ldots l_m!/r!$. Thus using Lemma 3.6 we see that

$$\beta_{i,id-\alpha(i-r)}(R/I(L_n^{d,\alpha})) = \sum_{m \geqslant 1, l_i \geqslant 1} P_{(l_1,\ldots,l_m)} \frac{l_1! \ldots l_m!}{r!} \frac{r!}{l_1! \ldots l_m!} \binom{n-i+1}{r}$$

$$= \sum_{m \geqslant 1, l_i \geqslant 1} P_{(l_1,\ldots,l_m)} \binom{n-i+1}{r} = \binom{i-1}{r-1}\binom{n-i+1}{r}.$$

The proof is complete. $\qquad\square$

We now consider $L_n^{d,\alpha}$ in the case when $d = 2\alpha$.

Theorem 3.8. *Let $L_n^{d,\alpha}$ be a line hypergraph such that $d = 2\alpha$. Then the non zero Betti numbers of $R/I(L_n^{d,\alpha})$ are, for $2j \geqslant i$, as follows:*

$$\beta_{i,j\alpha}(R/I(L_n^{d,\alpha})) = \binom{j-i}{2i-j}\binom{n+1-2j+2i}{j-i} + \binom{j-i-1}{2i-j}\binom{n+1-2j+2i}{j-i-1}.$$

P r o o f. Let $\mathcal{E}(L_n^{d,\alpha}) = \{E_1, \ldots, E_n\}$, where $E_i = \{x_{1,i}, \ldots, x_{d,i}\}$. Set $X_i = x_{1,i} \ldots x_{\alpha,i}$ and $X_{i+1} = x_{\alpha+1,i} \ldots x_{d,i}$ for any i, $1 \leqslant i \leqslant n$. Here $\{x_{\alpha+1,i} \ldots x_{d,i}\}$ are, for every $i = 1, \ldots, n-1$, the vertices in the intersection $E_i \cap E_{i+1}$. Then $I(L_n^{d,\alpha}) = (X_1 X_2, \ldots, X_n X_{n+1})$. Since the X_i's are independent variables and $\deg(X_i) = \alpha$, we have $\beta_{i,j\alpha}(R/I(L_n^{d,\alpha})) = \beta_{i,j}(R/(X_1 X_2, \ldots, X_n X_{n+1}))$. The result now follows, using Theorem 7.7.34 of [19]. $\qquad\square$

3.2. Betti numbers of hypercycles.
We start by giving a corollary that is similar to Corollary 3.5.

Corollary 3.9. *Let $C_n^{d,\alpha}$ be a hypercycle such that each edge has a free vertex. Then*

$$\beta_{n,j}(R/I(C_n^{d,\alpha})) = \begin{cases} 0 & \text{if } j \neq n(d-\alpha), \\ 1 & \text{if } j = n(d-\alpha). \end{cases}$$

P r o o f. By Theorem 3.4, $\beta_{n,j}(R/I(C_n^{d,\alpha})) = \left| \left\{ F \subseteq [n]; \ |F| = n, \ \left| \bigcup_{k \in F} E_k \right| = j \right\} \right|$. Therefore it is equal to 1 if $j = \left| \bigcup_{t=1}^{n} E_t \right|$ and 0 if $j \neq \left| \bigcup_{t=1}^{n} E_t \right|$. Since $\left| \bigcup_{t=1}^{n} E_t \right| = n(d-\alpha)$, the assertion holds. $\qquad\square$

We compute the Betti numbers of the hypercycle $C_n^{d,\alpha}$ in the same manner as in the previous section. That is, we consider the two cases $d > 2\alpha$ and $d = 2\alpha$ separately.

Let s_i ($1 \leqslant i \leqslant r$) be integers and $C_n^{d,\alpha}$ be a cycle of length n. Set $F(s_1, \ldots, s_r, n) = \{H; \ H$ is a subhypergraph of $C_n^{d,\alpha}$, which is comprised of r disjoint line hypergraphs of lengths $s_1, \ldots, s_r$ and with no isolated vertex$\}$, then we have the following lemma:

Lemma 3.10. *Let $s_1, \ldots, s_r$ be positive integers such that $s_1 = \ldots = s_{l_1} < s_{l_1+1} = \ldots = s_{l_1+l_2} < s_{l_1+l_2+1} = \ldots = s_{l_1+l_2+l_3} < \ldots < s_{l_1+\ldots+l_{t-1}+1} = \ldots = s_{l_1+\ldots+l_t} = s_r$ and $s_1 + \ldots + s_r = i$, then $|F(s_1, \ldots, s_r, n)| = n(r-1)!(l_1! \ldots l_t!)^{-1} \times \binom{n-i-1}{r-1}$.*

P r o o f.　Let $E_1, \ldots, E_{s_r} \in \mathcal{E}(C_n^{d,\alpha})$. The number of subgraphs of $C_n^{d,\alpha}$, which are comprised of r line hypergraphs $Q_1, \ldots, Q_r$ of lengths $s_1, \ldots, s_r$ and $\mathcal{E}(Q_r) = \{E_1, \ldots, E_{s_r}\}$ is equal to the number of subhypergraphs of $L_{n-s_r-2}^{d,\alpha}$, which are comprised of $r-1$ line hypergraphs of lengths $s_1, \ldots, s_{r-1}$. Therefore $|F(s_1, \ldots, s_r, n)| = n|E(s_1, \ldots, s_{r-1}, n - s_r - 2)|$, since the line hypergraph Q_r can start from any of the n edges. By Lemma 3.6, $|E(s_1, \ldots, s_{r-1}, n - s_r - 2)| = ((r-1)!/l_1! \ldots l_t!)\binom{n-i-1}{r-1}$, which completes the proof.　□

We now give the Betti numbers of $C_n^{d,\alpha}$ in the case when $d > 2\alpha$.

Theorem 3.11. *Let $i < n$ be an integer and let $C_n^{d,\alpha}$ be a hypercycle such that $d > 2\alpha$. Then $\beta_{i,id-\alpha(i-r)}(R/I(C_n^{d,\alpha})) = \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}$ for any r, $0 \leqslant r \leqslant i$ and $\beta_{i,j}(R/I(C_n^{d,\alpha})) = 0$ for other j's.*

P r o o f.　Since $d > 2\alpha$, each edge has a free vertex. Therefore as was shown in Theorem 3.4, $\beta_{i,j}(R/I(C_n^{d,\alpha})) = \left|\left\{\bigcup_{t=1}^{i} E_{l_t}; \ \left|\bigcup_{t=1}^{i} E_{l_t}\right| = j\right\}\right|$. Let $E_{l_1}, \ldots, E_{l_i}$ be some edges of $C_n^{d,\alpha}$ such that $\left|\bigcup_{t=1}^{i} E_{l_t}\right| = j$. Let $\mathcal{H}$ be a subhypergraph of $C_n^{d,\alpha}$ with edges $\{E_{l_1}, \ldots, E_{l_i}\}$ and assume that $\mathcal{H}$ is comprised of r line hypergraphs which are of lengths $s_1, \ldots, s_r$. Then $s_1 + \ldots + s_r = i$. Let $L_t^{d,\alpha} \subseteq \mathcal{H}$ be a line hypergraph of length s_t, then, as in the proof of Theorem 3.7, $|\mathcal{X}(L_t^{d,\alpha})| = s_t d - \alpha(s_t - 1)$ and therefore $|\mathcal{X}(\mathcal{H})| = \sum_{t=1}^{r} (s_t d - \alpha(s_t - 1)) = id - \alpha(i - r)$. Thus $j = id - \alpha(i - r)$ and we see that

$$\beta_{i,id-\alpha(i-r)}(R/I(C_n^{d,\alpha})) = \sum_{1 \leqslant s_1 \leqslant \ldots \leqslant s_r, s_1 + \ldots + s_r = i} |F(s_1, \ldots, s_r, n)|$$

and $\beta_{i,j}(R/I(C_n^{d,\alpha})) = 0$ for all j that can not be written in the form $id - \alpha(i - r)$ for some r.

Construct the numbers $P_{(l_1,\ldots,l_m)}$ and $P_{(l_1,\ldots,l_m)}l_1!\ldots l_m!/r!$ in the same way as in the proof of Theorem 3.7. Thus using Lemma 3.10 we see that

$$\beta_{i,id-\alpha(i-r)}(R/I(C_n^{d,\alpha})) = \sum_{m\geqslant 1, l_i \geqslant 1} P_{(l_1,\ldots,l_m)}\frac{l_1!\ldots l_m!}{r!}\frac{n(r-1)!}{l_1!\ldots l_m!}\binom{n-i-1}{r-1} =$$

$$= \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1},$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now consider the case when $d = 2\alpha$.

Theorem 3.12. *Let $C_n^{d,\alpha}$ be a hypercycle such that $d = 2\alpha$. Then the non zero Betti numbers of $R/I(C_n^{d,\alpha})$ are all in degree αj, where $j \leqslant n$ and are as follows:*

(i) *If $j < n$ and $2i \geqslant j$, then $\beta_{i,\alpha j}(R/I(C_n^{d,\alpha})) = n(n-2(j-i))^{-1}\binom{j-i}{2i-j}\binom{n-2(j-i)}{j-i}$.*
(ii) *If $n \equiv 1 \bmod 3$, $\beta_{\frac{2n+1}{3},\alpha n}(R/I(C_n^{d,\alpha})) = 1$.*
(iii) *If $n \equiv 2 \bmod 3$, $\beta_{\frac{2n-1}{3},\alpha n}(R/I(C_n^{d,\alpha})) = 1$.*
(iv) *If $n \equiv 0 \bmod 3$, $\beta_{\frac{2n}{3},\alpha n}(R/I(C_n^{d,\alpha})) = 2$.*

P r o o f. Let $\mathcal{E}(C_n^{d,\alpha}) = \{E_1,\ldots,E_n\}$, where $E_i = \{x_{1,i},\ldots,x_{d,i}\}$. Set $X_i = x_{1,i}\ldots x_{\alpha,i}$ and $X_{i+1} = x_{\alpha+1,i}\ldots x_{d,i}$ for any i, $1 \leqslant i \leqslant n-1$, where X_i and X_{i+1} denote the same things as they did in the proof of Theorem 3.8. Then

$$I(C_n^{d,\alpha}) = (X_1X_2,\ldots,X_{n-1}X_n,X_nX_1).$$

Since X_i are independent variables and $\deg(X_i) = \alpha$, we have $\beta_{i,j\alpha}(R/I(C_n^{d,\alpha})) = \beta_{i,j}(X_1X_2,\ldots,X_{n-1}X_n,X_nX_1)$. Using Theorem 7.6.28 of [19], the result follows.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We recall from [19] that a star graph is a complete bipartite graph $K_{1,n}$ for some n. One generalization of this graph is the d-complete bipartite hypergraph $K_{1,n}^d$. This kind of hypergraphs are considered in [7]. Another way of generalizing the star graph is to focus more on its appearance. Considering the following picture of $K_{1,4}$ (here on vertex set $\{y\} \sqcup \{x_1, x_2, x_3, x_4\}$):
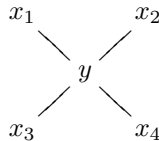
it is motivated to say that the hypergraphs $\mathcal{H}$ considered in the next theorem also are natural generalizations of the star graph.

Theorem 3.13. *Let $\mathcal{H}$ be a d-uniform hypergraph with edges $E_1, \ldots, E_n$ such that for any $i \neq j$, $|E_i \cap E_j| = \left| \bigcap_{l=1}^{n} E_l \right| = \alpha$ and each edge has a free vertex. Then $\beta_{i,j}(R/I(\mathcal{H})) \neq 0$ if and only if $j = id - \alpha(i-1)$ and $\beta_{i,id-\alpha(i-1)}(R/I(\mathcal{H})) = \binom{n}{i}$.*

P r o o f.   For any i and any edges $E_{l_1}, \ldots, E_{l_i}$, we have $\left| \bigcup_{t=1}^{i} E_{l_t} \right| = id - \alpha(i-1)$ and the number of elements of the form $\bigcup_{t=1}^{i} E_{l_t}$ is $\binom{n}{i}$. Therefore using Theorem 3.4, we have $\beta_{i,j}(R/I(\mathcal{H})) \neq 0$ if and only if $j = id - \alpha(i-1)$ and $\beta_{i,id-\alpha(i-1)}(R/I(\mathcal{H})) = \binom{n}{i}$. $\square$

4. Chordal hypergraphs

Chordal graphs have been considered more or less extensively for some time now. A core result in this area is the following theorem by R. Fröberg, [14]:

Theorem 4.1. *A graph $\mathcal{G}$ is chordal if and only if $R/I(\mathcal{G}^c)$ has linear resolution.*

From this result Fröberg easily concluded that the complexes $\Delta(\mathcal{G})$, $\mathcal{G}$ chordal, that are also Cohen-Macaulay, are the ones in which $\mathcal{G}$ is a generalized d-tree. See [14] for details. More nice results associated to chordal graphs are given in [18] and [11]. In [8] a notion of chordal hypergraph is given. We will use this to generalize some previously known results on chordal graphs.

The following definition is from [8]:

Definition 4.2. A *chordal hypergraph* is a d-uniform hypergraph, obtained inductively as follows:

- K_n^d is a chordal hypergraph, $n, d \in \mathbb{N}$.
- If $\mathcal{G}$ is chordal, then so is $\mathcal{H} = \mathcal{G} \cup_{K_j^d} K_i^d$, for $0 \leqslant j < i$. (We attach K_i^d to $\mathcal{G}$ in a common (under identification) K_j^d.)

In connection to this definition we mention two facts: First, it is easy to see that every line hypergraph $L_n^{d,\alpha}$ is a chordal hypergraph. Indeed, it may be written as

$$L_n^{d,\alpha} = K_d^d \cup_{K_\alpha^d} K_d^d \cup_{K_\alpha^d} K_d^d \cup_{K_\alpha^d} \ldots \cup_{K_\alpha^d} K_d^d \cup_{K_\alpha^d} K_d^d.$$

Also, as in chordal graphs, a chordal hypergraph does not contain any induced hypercycle $C_n^{d,\alpha}$. This is because an induced hypergraph of a chordal hypergraph is again chordal (see [8]). However, a hypercycle is not chordal.

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In the following theorem we will use the well known fact that the Alexander dual notion of shellability, is the concept of linear quotients (see [18, Theorem 1.4]). This also follows from Theorem 6.8.

Theorem 4.3. *If $\mathcal{H}$ is chordal hypergraph, then $I_{\Delta(\mathcal{H})}$ has linear quotients.*

Proof. Assume that $\mathcal{H}$ is chordal and $\Delta = \Delta(\mathcal{H})$, then inductively according to the definition of chordal hypergraph we show that Δ^* is shellable. We have $\Delta^* = \langle \mathcal{X}(\mathcal{H}) \setminus E, |E| = d, E \notin \mathcal{E}(\mathcal{H}) \rangle$. If $\mathcal{H} = K_n^d$, then $\Delta^* = \emptyset$ and the result holds. Assume that $\mathcal{H} = \mathcal{L} \cup_{K_j^d} K_i^d$, where $\mathcal{L}$ is chordal. By induction hypothesis we have $\Delta_{\mathcal{L}}^* = \langle \mathcal{X}(\mathcal{L}) \setminus E, |E| = d, E \notin \mathcal{E}(\mathcal{L}) \rangle$ is shellable. Assume that $H_1 < \ldots < H_n$ is a shelling for $\Delta_{\mathcal{L}}^*$. Let $S_1 = \mathcal{X}(\mathcal{L}) \setminus \mathcal{X}(K_j^d)$ and $S_2 = \mathcal{X}(K_i^d) \setminus \mathcal{X}(K_j^d)$, then for any subset $E \subseteq \mathcal{X}(\mathcal{H})$ such that $|E| = d$ and $E \notin \mathcal{E}(\mathcal{H})$, one has $|E \cap S_1| \geqslant 1$. Let $S_1 = \{x_1, \ldots, x_r\}$, $S_2 = \{y_1, \ldots, y_s\}$ and $S_3 = \mathcal{X}(K_j^d) = \{z_1, \ldots, z_j\}$. For integers $1 \leqslant l \leqslant d - 1$ and $1 \leqslant t \leqslant l$, consider the simplicial complex

$$A_{l,t} = \langle E^c; \ E \subseteq \mathcal{X}(\mathcal{H}), \ |E| = d, \ |E \cap S_2| = d - l, \ |E \cap S_1| = t, \ |E \cap S_3| = l - t \rangle.$$

It is easy to see that F is a facet of Δ^* if and only if F is a facet of $A_{l,t}$ for some $1 \leqslant l \leqslant d - 1$ and $1 \leqslant t \leqslant l$ or $F = H_i \cup S_2$ for some facet H_i of $\Delta_{\mathcal{L}}^*$. For any $1 \leqslant l \leqslant d - 1$ and $1 \leqslant t \leqslant l$ the facets of $A_{l,t}$ are all the sets of the form $\{y_{i_1}, \ldots, y_{i_{d-l}}\}^c \cap \{x_{j_1}, \ldots, x_{j_t}\}^c \cap \{z_{k_1}, \ldots, z_{k_{l-t}}\}^c$. First we give a shelling for $A_{l,t}$ for any $1 \leqslant l \leqslant d - 1$ and $1 \leqslant t \leqslant l$. From [??, Theorem 2.9], it is known that any skeleton of a shellable simplicial complex is again shellable. Thus $\langle F^c, |F| = d-l, F \subseteq \{y_1, \ldots, y_s\} \rangle$ and $\langle F^c, |F| = t, F \subseteq \{x_1, \ldots, x_r\} \rangle$ and $\langle F^c, |F| = l-t, F \subseteq \{z_1, \ldots, z_j\} \rangle$ are shellable simplicial complexes. Let $A_1, \ldots, A_m$ be a shelling for the simplicial complex $\langle F^c, |F| = d - l, F \subseteq \{y_1, \ldots, y_s\} \rangle$, $B_1, \ldots, B_n$ be a shelling for the simplicial complex $\langle F^c, |F| = t, F \subseteq \{x_1, \ldots, x_r\} \rangle$ and $C_1, \ldots, C_k$ be a shelling for the simplicial complex $\langle F^c, |F| = l-t, F \subseteq \{z_1, \ldots, z_j\} \rangle$. Consider the ordering on the facets of $A_{l,t}$ for which $A_i \cap B_j \cap C_k < A_{i'} \cap B_{j'} \cap C_{k'}$ if

(i) $k < k'$ or

(ii) $k = k'$ and $j < j'$ or

(iii) $k = k'$ and $j = j'$ and $i < i'$.

Let $A_i \cap B_j \cap C_k < A_{i'} \cap B_{j'} \cap C_{k'}$ be two facets of $A_{l,t}$. If $k < k'$, then there exists $v \in C_{k'} \setminus C_k$ and $k'' < k'$ such that $C_{k'} \setminus C_{k''} = \{v\}$. So $v = z_n$ for some n and $v \in A_{i'} \cap B_{j'} \cap C_{k'} \setminus A_i \cap B_j \cap C_k$ and $A_{i'} \cap B_{j'} \cap C_{k'} \setminus A_{i'} \cap B_{j'} \cap C_{k''} = \{v\}$. Since $k'' < k'$, we have $A_{i'} \cap B_{j'} \cap C_{k''} < A_{i'} \cap B_{j'} \cap C_{k'}$. If $k = k'$ and $j < j'$, then by the same way there exists $v \in A_{i'} \cap B_{j'} \cap C_k \setminus A_i \cap B_j \cap C_k$ and $j'' < j'$ such that $A_{i'} \cap B_{j'} \cap C_k \setminus A_{i'} \cap B_{j''} \cap C_k = \{v\}$. Now let $k = k'$, $j = j'$ and $i < i'$. Then there exists $v \in A_{i'} \setminus A_i$ and $i'' < i'$ such that $A_{i'} \setminus A_{i''} = \{v\}$. Therefore $v = y_n$ for some

n and $v \in A_{i'} \cap B_j \cap C_k \setminus A_i \cap B_j \cap C_k$ and $A_{i'} \cap B_j \cap C_k \setminus A_{i''} \cap B_j \cap C_k = \{v\}$ and $A_{i''} \cap B_j \cap C_k < A_{i'} \cap B_j \cap C_k$. So the above ordering is a shelling for $A_{l,t}$. Now consider an ordering for Δ^* as follows:

(i) For $F \in A_{l,t}$ and $G \in A_{l',t'}$, set $F < G$ if $l < l'$ or $l = l'$ and $t > t'$ or $(l,t) = (l',t')$ and in the ordering for $A_{l,t}$ discussed above, one has $F < G$.

(ii) For any $F \in A_{l,t}$ and $G = H_j \cup S_2$, set $F < G$.

(iii) Set $H_1 \cup S_2 < \ldots < H_n \cup S_2$.

We claim that this ordering is a shelling for Δ^*. Let F and G be two facets of Δ^* such that $F < G$ and let F be a facet of $A_{i,t}$ and G be a facet of $A_{j,s}$. Then $i \leqslant j$. The case $(i,t) = (j,s)$ is discussed above. Assume that $(i,t) \neq (j,s)$ and $i = j$. Then $t > s$.

Let $F = \{y_{k_1}, \ldots, y_{k_{d-i}}\}^c \cap \{x_{k_1'}, \ldots, x_{k_t'}\}^c \cap \{z_{k_1''}, \ldots, z_{k_{i-t}''}\}^c$ and $G = \{y_{l_1}, \ldots, y_{l_{d-i}}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{i-s}''}\}^c$. Since $t > s$, $i - t < i - s$. Also there exists $x_{k_\lambda'} \notin \{x_{l_1'}, \ldots, x_{l_s'}\}$ for some $1 \leqslant \lambda \leqslant t$. So $x_{k_\lambda'} \in G \setminus F$ and $G \setminus (\{y_{l_1}, \ldots, y_{l_{d-i}}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}, x_{k_\lambda'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{i-s-1}''}\}^c) = \{y_{l_1}, \ldots, y_{l_{d-i}}, x_{l_1'}, \ldots, x_{l_s'}, x_{k_\lambda'}, z_{l_1''}, \ldots, z_{l_{i-s-1}''}\} \setminus \{y_{l_1}, \ldots, y_{l_{d-i}}, x_{l_1'}, \ldots, x_{l_s'}, z_{l_1''}, \ldots, z_{l_{i-s}''}\} = \{x_{k_\lambda'}\}$. Also $\{y_{l_1}, \ldots, y_{l_{d-i}}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}, x_{k_\lambda'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{i-s-1}''}\}^c < G$.

Now let $i < j$ and $G = \{y_{l_1}, \ldots, y_{l_{d-j}}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{j-s}''}\}^c$. Then there exists $y_{k_\lambda} \in G \setminus F$. Since $j \geqslant 2$, we have $s \geqslant 2$ or $j - s \geqslant 1$. If $s \geqslant 2$, then $G \setminus \{y_{l_1}, \ldots, y_{l_{d-j}}, y_{k_\lambda}\}^c \cap \{x_{l_1'}, \ldots, x_{l_{s-1}'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{j-s}''}\}^c = \{y_{k_\lambda}\}$. Since $\{y_{l_1}, \ldots, y_{l_{d-j}}, y_{k_\lambda}\}^c \cap \{x_{l_1'}, \ldots, x_{l_{s-1}'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{j-s}''}\}^c \in A_{j-1,s-1}$, then in the ordering it appears before G. If $j - s \geqslant 1$, then $G \setminus \{y_{l_1}, \ldots, y_{l_{d-j}}, y_{k_\lambda}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{j-s-1}''}\}^c = \{y_{k_\lambda}\}$ and $\{y_{l_1}, \ldots, y_{l_{d-j}}, y_{k_\lambda}\}^c \cap \{x_{l_1'}, \ldots, x_{l_s'}\}^c \cap \{z_{l_1''}, \ldots, z_{l_{j-s-1}''}\}^c \in A_{j-1,s}$.

Now let $F < G$, where $F = \{y_{k_1}, \ldots, y_{k_{d-i}}\}^c \cap \{x_{k_1'}, \ldots, x_{k_t'}\}^c \cap \{z_{k_1''}, \ldots, z_{k_{i-t}''}\}^c$ and $G = H_j \cup S_2 = \{x_{l_1}, \ldots, x_{l_\lambda}, z_{l_1'}, \ldots, z_{l_{d-\lambda}'}\}^c$. Then there exists $y_m \in G \setminus F$. If $\lambda > 1$, then $G \setminus \{y_m, x_{l_1}, \ldots, x_{l_{\lambda-1}}, z_{l_1'}, \ldots, z_{l_{d-\lambda}'}\}^c = \{y_m\}$. Otherwise $\lambda = 1$ and $d - \lambda \geqslant 1$. Then $G \setminus \{y_m, x_{l_1}, \ldots, x_{l_\lambda}, z_{l_1'}, \ldots, z_{l_{d-\lambda-1}'}\}^c = \{y_m\}$. For two facets $H_i \cup S_2 < H_j \cup S_2$, the result is clear. $\qquad\square$

Corollary 4.4. *The graph $\mathcal{G}$ is chordal if and only if $I_{\Delta(\mathcal{G})}$ has linear quotients.*

P r o o f.   The fact that the edge ideal of a chordal graph has linear quotients, follows from the theorem. Assume $R/I(\Delta(\mathcal{G}))$ has linear quotients. Then it has linear resolution and thus, $\mathcal{G}$ is chordal. $\qquad\square$

In [14] Fröberg considers a class of chordal graphs called n-trees.

Definition 4.5. A n-tree is a chordal graph defined inductively as follows:

• K_{n+1} is a n-tree.

- If $\mathcal{G}$ is a n-tree, then so is $\mathcal{H} = \mathcal{G} \cup_{K_n} K_{n+1}$. (We attach K_{n+1} to $\mathcal{G}$ in a common (under identification) K_n).

Now, consider the corresponding subclass $\mathcal{T}_d$ of the class of chordal hypergraphs. That is, $\mathcal{T}_d$ is the class of chordal hypergraphs described as follows:

- K_{n+1}^d belongs to $\mathcal{T}_d$.
- If $\mathcal{G}$ belongs to $\mathcal{T}_d$, then so does $\mathcal{H} = \mathcal{G} \cup_{K_n^d} K_{n+1}^d$. (We attach K_{n+1}^d to $\mathcal{G}$ in a common (under identification) K_n^d.)

We get the following results:

Theorem 4.6. *For any hypergraph $\mathcal{H}$ in $\mathcal{T}_d$, the clique complex $\Delta(\mathcal{H})$ is pure shellable and hence Cohen-Macaulay.*

P r o o f. The proof is by induction. If $\mathcal{H} = K_{n+1}^d$, then $\Delta(\mathcal{H})$ is a simplex and pure shellable. Let $\mathcal{H} = \mathcal{G} \cup_{K_n^d} K_{n+1}^d$ and $F_1 < \ldots < F_r$ be a shelling for $\Delta(\mathcal{G})$. Then $\Delta(\mathcal{H}) = \langle F_1, \ldots, F_r, F_{r+1} \rangle$, where $F_{r+1} = \mathcal{X}(K_{n+1}^d)$. Let $\mathcal{X}(K_n^d) = L$. Then $L \subseteq F_i$ for some $1 \leqslant i \leqslant r$. We claim that $F_1 < \ldots < F_r < F_{r+1}$ is a shelling for $\Delta(\mathcal{H})$. Let $F_{r+1} = L \cup \{v\}$. Then for any $j \leqslant r$, one has $v \in F_{r+1} \setminus F_j$ and $F_{r+1} \setminus F_i = \{v\}$. $\square$

Corollary 4.7. *For any d-tree $\mathcal{G}$, the clique complex $\Delta(\mathcal{G})$ is pure shellable and hence Cohen-Macaulay.*

In the proof of the following proposition, we will use the fact that the Stanley-Reisner ideal of the complex $\Delta^*_{I(K_n^d)}$ is shellable. One may in fact use a lexicographic shelling, so, by symmetry, one may start the shelling with any facet of the complex.

Proposition 4.8. *Let $\mathcal{H} = K_m^d \cup_{K_j^d} K_i^d$, $m \geqslant d$ be given. Then $I(\mathcal{H})$ has linear quotients precisely when*
(i) $i, j < d$, *or*
(ii) $i \geqslant d$, *and* $j = m - 1$ *or* $j = i - 1$.

P r o o f. Put $A = \mathcal{X}(\mathcal{H}) \setminus \mathcal{X}(K_m^d)$ and $B = \mathcal{X}(\mathcal{H}) \setminus \mathcal{X}(K_i^d)$. We will show that the Alexander dual complex of $\mathcal{H}$ is shellable precisely in the cases mentioned above. Assume that $F_1' < \ldots < F_t'$ is a shelling of $\Delta_{I^*(K_m^d)}$. Set $F_i = F_i' \cup A$ for $i = 1, \ldots, t$. In the case (i) the sequence $F_1 < \ldots < F_t$ is a shelling of $\Delta_{I^*(\mathcal{H})}$.

In the case (ii) we find a shelling when $i \geqslant d$ and $j = m - 1$. The case $j = i - 1$ is similar. Let $i \geqslant d$, $j = m - 1$ and $\mathcal{X}(K_m^d) \setminus \mathcal{X}(K_j^d) = \{v\}$. Let $G_1' < \ldots < G_s'$ be a shelling of $\Delta_{I^*(K_i^d)}$. Set $G_i = G_i' \cup B$ for $i = 1, \ldots, s$. It is easy to see that the set of facets of $\Delta_{I^*(\mathcal{H})}$ is $\{F_i\}_{i=1}^t \cup \{G_j\}_{j=1}^s$. We claim that the ordering $G_1 < \ldots < G_s < F_1 < \ldots < F_t$ is a shelling of $\Delta_{I^*(\mathcal{H})}$. Let $G_i < F_j$, where

$G_i = \mathcal{X}(\mathcal{H}) \setminus E_1$ and $F_j = \mathcal{X}(\mathcal{H}) \setminus E_2$ for some edges E_1 of K_i^d and E_2 of K_m^d. Let $E_1 = \{w_1, \ldots, w_d\}$ and $E_2 = \{v, v_1, \ldots, v_{d-1}\}$, where $v_1, \ldots, v_{d-1} \in \mathcal{X}(K_j^d)$. Then there exists $1 \leqslant l \leqslant d$ such that $w_l \in F_j \setminus G_i$. Set $E_3 = \{w_l, v_1, \ldots, v_{d-1}\}$, then $\mathcal{X}(\mathcal{H}) \setminus E_3 = G_k$ for some k and $F_j \setminus G_k = \{w_l\}$.

Now, assume (i) and (ii) do not hold. Then $i \geqslant d$, $m - j \geqslant 2$ and $i - j \geqslant 2$. We first claim that if $j \leqslant d - 2$, there is no shelling: Consider the intersection $G_j \cap F_i$ (same notation as above). The two facets here correspond to two edges in $\mathcal{H}$, one from K_m^d and one from K_i^d. These two edges can at most have j elements in common. Hence, by considering the set complements of these edges we realize that the two facets can at most have $|\mathcal{X}(\mathcal{H})| - d - 2$ vertices in common. This shows that no ordering of the F_i's and the G_j's can be a shelling, since $\dim(F_i) = \dim(G_j) = \mathcal{X}(\mathcal{H}) - d - 1$ for every $i = 1, \ldots, t$, $j = 1, \ldots, s$.

So, we assume $j \geqslant d - 2$. Let $v_1, v_2 \in \mathcal{X}(K_m^d) \setminus \mathcal{X}(K_j^d)$, $w_1, w_2 \in \mathcal{X}(K_i^d) \setminus \mathcal{X}(K_j^d)$ and $u_1, \ldots, u_{d-2} \in \mathcal{X}(K_j^d)$. To finish the proof by contradiction, we assume $\Delta_{I^*(\mathcal{H})}$ is shellable. Consider two edges $E_1 = \{u_1, \ldots, u_{d-2}, v_1, v_2\}$ and $E_2 = \{u_1, \ldots, u_{d-2}, w_1, w_2\}$. Then $F_i = \mathcal{X}(\mathcal{H}) \setminus E_i$ for $i = 1, 2$ are facets of $\Delta_{I^*(\mathcal{H})}$. Without loss of generality may assume $F_1 < F_2$. Hence there exists vertex $v \in F_2 \setminus F_1$ and facet F_3 such that $F_2 \setminus F_3 = \{v\}$. Let $F_3 = \mathcal{X}(\mathcal{H}) \setminus E_3$ for some edge E_3. Since $F_2 \setminus F_1 = \{v_1, v_2\}$, we have $v = v_1$ or $v = v_2$. Therefore $E_3 \subseteq \mathcal{X}(K_m^d)$. Also $E_3 \setminus \{v\} \subseteq E_2$. Thus $w_1 \in E_3$ or $w_2 \in E_3$, which is a contradiction. $\square$

We end this section with a result on the diameter of the complement of a chordal graph. Recall that the diameter of a connected graph $\mathcal{G}$ is defined as

$$\operatorname{diam}(\mathcal{G}) = \max\{\operatorname{dist}(u, v); \; u, v \in \mathcal{X}(\mathcal{G})\},$$

where $\operatorname{dist}(u, v)$ is the number of edges in a shortest path between u to v. If $\mathcal{G}$ is not connected we set the diameter to be ∞.

Proposition 4.9. *Let $\mathcal{G}$ be a connected chordal graph. Then the diameter of the complementary graph $\mathcal{G}^c$ is at most 3.*

P r o o f. If $\operatorname{diam}(\mathcal{G}^c) > 3$ we find vertices u and v with $\operatorname{dist}(u, v) = 4$. Then the induced graph of $\mathcal{G}^c$ on $\{u, v_1, v_2, v_3, v\}$ is the path $uv_1, v_1v_2, v_2v_3, v_3v$. The graph complement of $\mathcal{G}_{\{u, v_1, v_2, v_3, v\}}^c$ contains a 4-cycle without any chord. This contradiction gives our result. $\square$

4.1. d-uniform hypergraphs and quasi forests. It is known that a certain class of simplicial complexes, called quasi-trees (see below), and chordal graphs, in some sense contain the same information.

Remark 4.10. Recall that a flag complex is a simplicial complex in which every minimal non face consists of precisely 2 elements. As one easily sees, such complex is determined by its 1-skeleton.

The following is the content of Lemma 3.1 in [18].

Lemma 4.11. *Let* Δ *be a simplicial complex. Then* Δ *is a quasi-forest precisely when* $\Delta = \Delta(\mathcal{G})$ *for some chordal graph* $\mathcal{G}$*. In particular, a quasi-forest is a flag complex.*

In this section we will see that there is also a close connection between quasi-trees and the class of chordal hypergraphs.

Definition 4.12 [Faridi, [9], Zheng, [23]]. Let Δ be a simplicial complex. A *subcollection* Γ, of Δ, is a subcomplex of Δ such that $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$. A facet F of Δ is called a *leaf* if either F is the only facet of Δ, or there exists a facet G in Δ, $G \neq F$, such that $F \cap H \subseteq F \cap G$ for any facet H in Δ, $H \neq F$.

Assume Δ is connected. Then Δ is called a *tree* if every subcollection of Δ has a leaf, and Δ is called a *quasi-tree* if there exists an order $F_1, \ldots, F_t$ of the facets of Δ such that for each $i = 1, \ldots, t$, F_i is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$, whose facets are $F_1, \ldots, F_i$. The order $F_1, \ldots, F_t$ is called a *leaf order*. A simplicial complex with the property that every connected component is a (quasi-)tree is called a (quasi-)*forest*.

Remark 4.13. A tree is a quasi-tree, but the converse need not hold.

Let Δ be a simplicial complex. Denote by $\mathcal{R}_d(\Delta)$ the simplicial complex obtained from Δ by removing every facet F with $1 \leqslant \dim F \leqslant d-2$, and all faces $G \subseteq F$, with $1 \leqslant \dim G \leqslant \dim F$, that are not faces of some facet of dimension greater than $d-2$. Conversely, denote by $\mathcal{A}_d(\Delta)$ the simplicial complex obtained from Δ by adding, as a facet, every face of dimension $d-2$ that is not already in the complex.

Lemma 4.14. *Let* $\Delta(\mathcal{H})$ *and* $\Delta(\mathcal{G})$ *be the clique complexes of a d-uniform hypergraph* $\mathcal{H}$*, and a graph* $\mathcal{G}$*, respectively. Then the following holds:*

- $\mathcal{A}_d(\mathcal{R}_d(\Delta(\mathcal{H}))) = \Delta(\mathcal{H})$;
- $\mathcal{R}_{d'}(\mathcal{A}_{d'}(\Delta(\mathcal{G}))) = \Delta(\mathcal{G})$ *for all* $d' < \min\{\dim F;\ F$ *facet in* $\Delta(\mathcal{G})\}$.

P r o o f. This follows immediately from the definition of $\Delta(\mathcal{H})$. $\qquad\square$

Lemma 4.15. *Let $\mathcal{H} = \mathcal{G} \cup_{K_j^d} K_i^d$ be a chordal hypergraph. If $i < d$ (that is K_i^d consists of i isolated vertices), we may exchange the attaching of K_i^d to K_j^d, with $i - j$ attachings of the form*

$$\mathcal{H}' = \mathcal{G}' \cup_{K_0^d} K_1^d.$$

P r o o f. This is clear, since either way, we are just adding a number of isolated vertices. $\qquad\square$

Proposition 4.16. *Let $\mathcal{H}$ be chordal hypergraph, and let $\mathcal{G}$ be a chordal graph. Then the following holds:*

(i) *$\mathcal{R}_d(\Delta(\mathcal{H}))$ is the clique complex of a chordal graph.*
(ii) *$\mathcal{A}_{d'}(\Delta(\mathcal{G}))$ is, for any d', the clique complex of a d'-uniform chordal hypergraph.*

P r o o f. (i) A chordal hypergraph $\mathcal{H}$ may, according to its inductive construction, be represented by a sequence of pairs of d-complete hypergraphs

$$(K_0^d, K_{i_1}^d), \ldots, (K_{j_t}^d, K_{i_t}^d),$$

where in each step of the construction of $\mathcal{H}$, $K_{i_s}^d$ is attached to $K_{j_s}^d$. We assume that in the construction of $\mathcal{H}$, Lemma 4.15 has been used if necessary. Then every d-complete hypergraph in the sequence $(K_0^d, K_{i_1}^d), \ldots, (K_{j_t}^d, K_{i_t}^d)$ yields a complete graph, and, by considering the facets, it is clear that $\mathcal{R}_d(\Delta(\mathcal{H}))$ is the complex of the chordal graph that is represented by the sequence of pairs $(K_0, K_{i_1}), \ldots, (K_{j_t}, K_{i_t})$. This proves (i).

Now, let $(K_0, K_{j_1}), \ldots, (K_{j_t}, K_{i_t})$ denote a chordal graph $\mathcal{G}$. If $d' - 2 \geqslant \dim \Delta(\mathcal{G})$, then the claim (ii) is trivial, so we assume $d' - 2 < \dim \Delta(\mathcal{G})$. It is obvious that $\mathcal{A}_{d'}(\Delta(\mathcal{G}))$ will be the complex of a d'-uniform hypergraph $\mathcal{H}$, since every minimal nonface has dimension $d' - 1$. We now show that $\mathcal{H}$ is chordal. We do this by constructing a sequence of pairs $(K_0^d, K_{i_1}^d), \ldots, (K_{j_r}^d, K_{i_r}^d)$, $r \geqslant t$, from the sequence $(K_0, K_{i_1}), \ldots, (K_{j_t}, K_{i_t})$, and showing that this sequence actually defines $\mathcal{H}$.

First note that if $i_s \geqslant d'$, a complete graph K_{i_s} immediately yields a d'-complete hypergraph $K_{i_s}^{d'}$. For such i_s, we get a pair $(K_{j_s}^{d'}, K_{i_s}^{d'})$, corresponding to the pair (K_{j_s}, K_{i_s}) in the sequence representing $\mathcal{G}$. If $i_s < d'$, we may instead associate to the pair (K_{j_s}, K_{i_s}) a sequence of "trivial pairs", as in Lemma 4.15. Continuing in this way, we obtain a sequence $(K_{j_1}^d, K_{i_i}^d), \ldots, (K_{j_r}^d, K_{i_r}^d)$, representing a d'-uniform chordal hypergraph $\mathcal{H}'$.

The d'-uniform chordal hypergraph that corresponds to the constructed sequence yields the same complex as $\mathcal{H}$, and hence we conclude that they must be the same. $\qquad\square$

Corollary 4.17. *To every chordal hypergraph $\mathcal{H}$ we may associate a quasi-forest Δ, and vice versa.*

P r o o f. If Δ is a quasi-forest, then $\Delta = \Delta(\mathcal{G})$ for some chordal graph ([18], Lemma 3.1). Then, according to the proposition, we may associate to Δ the chordal hypergraph $\mathcal{H}$ whose clique complex is the complex $\mathcal{A}_{d'}(\Delta(\mathcal{G}))$ in the proposition. Conversely, given a chordal hypergraph $\mathcal{H}$ we may associate to it the quasi-forest $\mathcal{R}_d(\Delta(\mathcal{H}))$ from the proposition. $\square$

5. Homologically connected hypergraphs, connectivity, and depth

For graphs and simplicial complexes there is a natural notion of being connected. This property may be described purely in terms of 0-homologies of certain chain complexes. Furthermore, the notion of being connected is very well behaved in the sense that if we choose the coefficients in the associated chain complex from a field k, it does not depend on the characteristic of k. This is one reason that arguments involving connectedness sometimes are very useful if one is trying to prove something about a graph or a simplicial complex. In [19], S. Jacques deduces some lower bounds on Betti numbers of graph algebras. The arguments used there are based on the connectedness property of graphs. In this section we define in a homological fashion a concept of connected hypergraph.

Definition 5.1. Let $\mathcal{H}$ be a d-uniform hypergraph and k be a field. The *connectivity of $\mathcal{H}$ over k*, con($\mathcal{H}$), is defined as

$$\mathrm{con}(\mathcal{H}) = \min\{|V|; V \subseteq [n], \dim \widetilde{H}_{d-2}((\Delta(\mathcal{H}))_{[n] \setminus V}; k) \neq 0\}.$$

Definition 5.2. Let k be a field. If $\mathcal{H}$ is a d-uniform hypergraph with non zero connectivity over k, we say that $\mathcal{H}$ is *homologically connected over k*. If $\mathcal{H}$ is homologically connected over every field, we say that $\mathcal{H}$ is homologically connected.

Note that in the case of graphs, this is the usual notion of connectedness. Also, in terms of homological connectedness, the connectivity of a d-uniform hypergraph $\mathcal{H}$, is the cardinality of a minimal disconnecting set of vertices.

Proposition 5.3. *If $\mathcal{H}$ is homologically connected over $\mathbb{Q}$, it is homologically connected over every field k.*

P r o o f. By the Universal Coefficient Theorem we have

$$\widetilde{H}_i(\Delta(\mathcal{H}); k) \cong \widetilde{H}_i(\Delta(\mathcal{H}); \mathbb{Q}) \otimes k \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\widetilde{H}_{i-1}(\Delta(\mathcal{H})), k).$$

One should note that when we consider a complex $\Delta_{\mathcal{H}}$ of a non empty d-uniform hypergraph, $\widetilde{H}_l(\Delta(\mathcal{H}); k) = 0$ for every $l \leqslant d - 3$ over every field k. $\square$

Recall *Hochster's formula*.

Theorem 5.4 [Hochster's formula]. *Let R/I_Δ be the Stanley-Reisner ring of a simplicial complex Δ. The non zero Betti numbers, $\beta_{i,\mathbf{j}}(R/I_\Delta) = \dim \operatorname{Tor}_i^R(R/I_\Delta, k)_{\mathbf{j}}$, of R/I_Δ, are only in squarefree degrees $\mathbf{j}$ and may be expressed as*

$$\beta_{i,\mathbf{j}}(R/I_\Delta) = \dim_k \widetilde{H}_{|\mathbf{j}|-i-1}(\Delta_{\mathbf{j}}; k).$$

Hence the total i'th Betti number may be expressed as

$$\beta_i(R/I_\Delta) = \sum_{V \subseteq [n]} \dim \widetilde{H}_{|V|-i-1}(\Delta_V; k).$$

P r o o f.   See [2], Theorem 5.5.1.   □

From this it follows that

$$\beta_{i,j}(R/I_\Delta) = \sum_{\substack{V \subseteq [n] \\ |V|=j}} \dim \widetilde{H}_{|V|-i-1}(\Delta_V; k).$$

Proposition 5.5. *If $\mathcal{G}$ is an induced hypergraph of a d-uniform hypergraph $\mathcal{H}$, such that $\mathcal{G}$ is not homologically connected over k, then*

$$\beta_{|\mathcal{X}(\mathcal{G})|-d+1}(\mathcal{H}) \neq 0.$$

P r o o f.   We will use the fact that $(\Delta(\mathcal{H}))_V = \Delta(\mathcal{H}_V)$. Consider Hochster's formula with $i = |\mathcal{X}(\mathcal{G})| - d + 1$;

$$\beta_{|\mathcal{X}(\mathcal{G})|-d+1}(R/I_{\Delta(\mathcal{H})}) = \sum_{V \subseteq \mathcal{X}(\mathcal{H})} \dim_k \widetilde{H}_{|V|-|\mathcal{X}(\mathcal{G})|+d-2}(\Delta(\mathcal{H}_V); k)$$

$$\geqslant \dim_k \widetilde{H}_{d-2}(\Delta_{\mathcal{X}(\mathcal{G})}; k) > 0.$$

□

Recall the *Auslander-Buchsbaum* formula: If M is a finitely generated R-module with $\operatorname{pd}_R(M) < \infty$, then $\operatorname{pd}_R(M) + \operatorname{depth}_R(M) = \operatorname{depth}_R(R)$. For a proof, see [2], Theorem 1.3.3.

Corollary 5.6. *If $\mathcal{G}$ is an induced hypergraph of a d-uniform hypergraph $\mathcal{H}$, such that $\mathcal{G}$ is not homologically connected over k, then*

$$|\mathcal{X}(\mathcal{G})| - d + 1 \leqslant \mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) \leqslant n,$$
$$0 \leqslant \mathrm{depth}_R(R/I_{\Delta(\mathcal{H})}) \leqslant n - |\mathcal{X}(\mathcal{G})| + d - 1,$$

where $n = |\mathcal{X}(\mathcal{H})|$.

P r o o f.   It is well know that (Hilbert's Syzygy Theorem) $n \geqslant \mathrm{pd}_R(R/I_{\Delta(\mathcal{H})})$. Furthermore, according to the lemma, $\beta_{|\mathcal{X}(\mathcal{G})|-d+1}(\mathcal{H}) > 0$. This gives the first assertion. The second follows from the first using the Auslander-Buchsbaum formula.

$\square$

Corollary 5.7. *If $\mathcal{H}$ is a d-uniform hypergraph that is not homologically connected over k, then*

$$n - d + 1 \leqslant \mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) \leqslant n,$$
$$0 \leqslant \mathrm{depth}_R(R/I_{\Delta(\mathcal{H})}) \leqslant d - 1,$$

where $n = |\mathcal{X}(\mathcal{H})|$.

If $\mathcal{H}$ is a d-uniform hypergraph that is not homologically connected, we will see in Theorem 5.11 below, that the two inequalities in the above corollary may in fact be replaced by two equalities. First, we prove the following theorem, which connects the depth of the Stanley-Reisner ring $R/I_{\Delta(\mathcal{H})}$, with the connectivity of $\mathcal{H}$.

Theorem 5.8. *Let $\Delta(\mathcal{H})$ be the complex of a d-uniform hypergraph $\mathcal{H}$ and put $g = \mathrm{depth}_R(R/I_{\Delta(\mathcal{H})})$. Then,*

$$\mathrm{con}(\mathcal{H}) = g - d + r + 1,$$

where r is the minimal number such that $\beta_{n-g-r,n-g-r+d-1}(R/I_{\Delta(\mathcal{H})}, k) \neq 0$. That is, r is the minimal number such that there exists a $V \subseteq [n]$, $|V| = n - (g-d+r+1)$ with $\widetilde{H}_{d-2}(\Delta_V; k) \neq 0$.

Remark 5.9. If $\mathcal{H}$ is a d-uniform hypergraph, recall that the *linear strand* of a resolution of $R/I_{\Delta(\mathcal{H})}$ (or for short, the linear strand of $R/I_{\Delta(\mathcal{H})}$) is the part of the resolution that is of degrees $(i, i + d - 1)$. Note that $r = \mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) - \max\{i; \beta_{i,i+d-1}(R/I_{\Delta(\mathcal{H})}) \neq 0\}$.

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P r o o f. We know that $\mathrm{Tor}^R_{n-g}(R/I_{\Delta(\mathcal{H})}, k) \neq 0$, but $\mathrm{Tor}^R_{n-i}(R/I_{\Delta(\mathcal{H})}; k) = 0$ for every $i < g$. In particular, $\mathrm{Tor}^R_{n-i}(R/I_{\Delta(\mathcal{H})}; k)_j = 0$ in every degree j if $i < g$. This gives, via Hochster's formula, that $\widetilde{H}_{|V|-(n-i+1)}(\Delta_V; k) = 0$ for every $V \subseteq [n]$, $i < g$, and that there exists a $V \subseteq [n]$ such that $\widetilde{H}_{|V|-(n-g+1)}(\Delta_V; k) \neq 0$. Let $r \geqslant 0$ be the minimal number such that $\mathrm{Tor}^R_{n-(g+r)}(R/I_{\Delta(\mathcal{H})}; k)_j \neq 0$ for $j = n-(g-d+r+1)$. This is the same thing as saying that there exists a $V \subseteq [n]$, $|V| = n-(g-d+r+1)$, such that $\widetilde{H}_{d-2}(\Delta_V; k) \neq 0$ but at the same time, for any $V \subseteq [n]$, $|V| > n-(g-d+r+1)$, the homology of Δ_V in degree $d-2$ is zero. This means precisely that $\mathrm{con}(\mathcal{H}) = g - d + r + 1$. $\qquad\square$

If $\mathcal{H}$ is 2-uniform (that is, if $\mathcal{H}$ is an ordinary simple graph) and we have linear resolution, the following is Lemma 3 in [14].

Corollary 5.10. *Let $\mathcal{H}$ be a d-uniform hypergraph and suppose the length of the linear strand of $R/I_{\Delta(\mathcal{H})}$ is maximal. Then*

$$\mathrm{depth}_R(R/I_{\Delta(\mathcal{H})}) = \mathrm{con}(\mathcal{H}) + d - 1.$$

Theorem 5.11. *Let $\mathcal{H}$ be a d-uniform hypergraph. Then $\mathcal{H}$ is not homologically connected over k precisely when*

$$\mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) = n - d + 1,$$
$$\mathrm{depth}_R(R/I_{\Delta(\mathcal{H})}) = d - 1,$$

where $n = |\mathcal{X}(\mathcal{H})|$, and the length of the linear strand of $R/I_{\Delta(\mathcal{H})}$ is maximal.

P r o o f. We know that $n - d + 1 \leqslant \mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) \leqslant n$. Put $\mathrm{pd}_R(R/I_{\Delta(\mathcal{H})}) = n - r$, $0 \leqslant r \leqslant d - 1$. Hochster's formula gives

$$\beta_{n-r}(R/I_{\Delta(\mathcal{H})}) = \sum_{V \subseteq \mathcal{X}(\mathcal{H})} \dim_k \widetilde{H}_{|V|-(n-r)-1}(\Delta(\mathcal{H}_V); k).$$

If $r \leqslant d-2$ we get that $|V|-(n-r)-1 \leqslant |V|-n+d-3 \leqslant d-3$. But $\widetilde{H}_l(\Delta(\mathcal{H}_V); k) = 0$ for all $l \leqslant d - 3$ and for all $V \subseteq \mathcal{X}(\mathcal{H})$.

The last claim follows from Corollary 5.10, since being not homologically connected, is the same thing as having connectivity 0. $\qquad\square$

Example 5.12. Since homologically connected and connected are the same things for an ordinary simple graph $\mathcal{G}$, we have $\mathrm{pd}_R(R/I_{\Delta(\mathcal{G})}) = n - 1$ and $\mathrm{depth}_R(R/I_{\Delta(\mathcal{G})}) = 1$ for any simple graph $\mathcal{G}$ that is not connected. Furthermore, the length of the linear strand of $R/I_{\Delta_\mathcal{G}}$ is maximal. This special case of Theorem 5.11 is Theorem 4.2.6 in [19].

Corollary 5.13. *Let $\mathcal{H}$ be a d-uniform hypergraph. If $R/I_{\Delta(\mathcal{H})}$ is Cohen-Macaulay of dimension at least d, then $\mathcal{H}$ has non zero connectivity. Put another way, the only d-uniform hypergraph $\mathcal{H}$ with connectivity 0 such that $R/I_{\Delta(\mathcal{H})}$ is Cohen-Macaulay, is the discrete hypergraph.*

Example 5.14. In [7] we considered several kinds of complete hypergraphs. For example the d-complete multipartite hypergraph $K^d_{n_1,\ldots,n_t}$. This is the hypergraph on vertex set $\mathcal{X} = [n_1] \sqcup \ldots \sqcup [n_t]$, where $\sqcup$ denotes disjoint union, and with edge set consisting of every d-set of $\mathcal{X}$ (that is, every subset of $\mathcal{X}$ of cardinality d) that does not lie entirely inside one of the $[n_i]$'s. This is a natural generalization of the usual complete multipartite graph $K_{n_1,\ldots,n_t}$. Precisely as $(K_{n_1,\ldots,n_t})^c$, the complement $(K^d_{n_1,\ldots,n_t})^c$ of the d-complete multipartite hypergraph is not homologically connected.

In Propositions 3.9 and 3.20 in [7], we determined when a couple of such complete hypergraphs in addition to having linear resolutions also has the Cohen-Macaulay property. The conclusion there is that the only case in which this happens, is in the extremal case when the considered hypergraph in fact is isomorphic to a d-complete hypergraph. This fact now follows immediately from the above corollary, since it is easily seen (by computing the Betti numbers) that the considered hypergraphs are not homologically connected.

Proposition 5.15. *Let $\mathcal{H}$ be a d-uniform hypergraph on vertex set $[n]$. Then the Betti number $\beta_{n-d+1}(R/I_{\Delta(\mathcal{H})})$ can be non zero only in degree n. Furthermore, it determines whether $\mathcal{H}$ has non zero connectivity or not.*

P r o o f.   This follows from Hochster's formula and the fact that $\beta_{i,j}(R/I_{\Delta(\mathcal{H})}) = 0$ if $j < i + d - 1$. □

Remark 5.16. In case of ordinary simple graphs, by the above proposition, the number $\beta_{n-1}(R/I_{\Delta(\mathcal{G})}) + 1$ is the number of connected components of $\mathcal{G}$.

Example 5.17. Let $\mathcal{H}$ be the 3-uniform hypergraph on vertex set $\{a, b, c, d\}$ and with edge set $\mathcal{E}(\mathcal{H}) = \{abc, bcd\}$ (we let xyz denote the edge $\{x, y, z\}$). We may visualize $\mathcal{H}$ as follows:
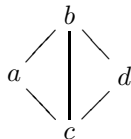
By computing the Betti numbers of $R/I_{\Delta(\mathcal{H})}$ using some suitable computer program, one sees that $\beta_2(R/I_{\Delta(\mathcal{H})}) = 1$. If we add to the edge set the edge abd, the resulting hypergraph has non zero connectivity.

Example 5.18. Let $\mathcal{H}$ be the 3-uniform hypergraph on vertex set $\{a, b, c, d, e\}$ and with edge set $\mathcal{E}(\mathcal{H}) = \{abc, cde\}$. $\mathcal{H}$ is illustrated below:
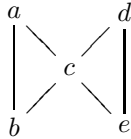
The Betti number $\beta_3(R/I_{\Delta(\mathcal{H})}) = 4$ shows that $\mathcal{H}$ has 0 connectivity. If we add to the edge set the edge acd, the resulting hypergraph has 4 homologically connected components. If we continue and add the edge bce, the resulting hypergraph still has 0 connectivity. Adding the edge abd does not create a hypergraph with non zero connectivity, but, finally, by adding the edge abe we arrive at a hypergraph with edge set $\{abc, cde, acd, bce, abd, abe\}$, that has non zero connectivity.

Example 5.19. Let $\mathcal{H} = (K_n^d)^c$. Then we know that (Theorem 3.1, [7]) $\beta_{i,j}(R/I_{\Delta(\mathcal{H})}) = \binom{n}{j}\binom{j-1}{d-1}$. Hence $\mathcal{H}$ certainly does not have non zero connectivity. This is quite natural since it generalizes the fact that the discrete graph on n vertices has $n = \binom{n-1}{2-1} + 1$ (homologically) connected components.

Example 5.20. One of the complete hypergraphs considered in [7] is the $d(a_1, \ldots, a_t)$-complete hypergraph $K_{n_1, \ldots, n_t}^{d(a_1, \ldots, a_t)}$. This hypergraph has vertex set the disjoint union $\mathcal{X} = [n_1] \sqcup \ldots \sqcup [n_t]$ and edge set consisting of all d-sets of $\mathcal{X}$ such that precisely a_i elements come from $[n_i]$. In [7] it is shown that $R/I(K_{n_1, \ldots, n_t}^{d(a_1, \ldots, a_t)})$ has linear resolution and projective dimension $n - d + 1$. Hence $(K_{n_1, \ldots, n_t}^{d(a_1, \ldots, a_t)})^c$ has connectivity 0.

Remark 5.21. The above examples show that the connectivity of an arbitrary d-uniform hypergraph is often is. This together with Corollary 5.13 shows that d-uniform hypergraphs $\mathcal{H}$ such that $R/I_{\Delta(\mathcal{H})}$ is Cohen-Macualay, indeed are very special.

In Example 5.17 and Example 5.18, we have $\dim \Delta(\mathcal{H}) = 2$. It is easy to see that if $\mathcal{H}$ is a d-uniform hypergraph such that $\dim \Delta(\mathcal{H}) = d - 1 \geqslant 1$, then the converse of Corollary 5.13 holds. That is, if $\dim \Delta(\mathcal{H}) = d - 1 \geqslant 1$, then $R/I_{\Delta(\mathcal{H}}$ is Cohen-Macaulay over k precisely when $\mathcal{H}$ has non zero connectivity over k. This will follow from the following result (Lemma 7 in [13]) of Fröberg.

Lemma 5.22. Let R/I be a Stanley-Reisner ring with $\dim R/I = e$ and embedding dimension n. Then R/I is Cohen-Macaulay if and only if $\widetilde{H}_i(\Delta_V; k) = 0$ for every i and $V \subseteq [n]$ such that $|V| = n - e + i + 2$.

Remark 5.23. An easy way to think of this lemma is as follows: First note that $|V| = n - e + i + 2$ gives $i = 0, \ldots, e - 2$. The claim in the lemma is now "symmetric" relative to this sequence of indexes. $\widetilde{H}_0(\Delta_V; k)$ should be zero for $|V| = n - (e - 2)$, $\widetilde{H}_1(\Delta_V; k)$ should be zero whenever $|V| = n - (e - 2) - 1$, $\widetilde{H}_2(\Delta_V; k)$ should be zero whenever $|V| = n - (e - 2) - 2$, a.s.o.

If $\Delta = \Delta(\mathcal{H})$, the lemma of Fröberg gives us the following, which is completely analogous to that considered right after Lemma 7 in [13]. If $\dim \Delta(\mathcal{H}) = d - 2$, the complex is always Cohen-Macaulay. This follows since the claim in the lemma in this case is that reduced homology in degree -1 is zero (we consider non empty complexes). The claim could also be easily verified by noting that $\Delta(\mathcal{H})$ in this case is the independence complex $\Delta_{K_n^d}$ of some d-complete hypergraph, see [7], Corollary 3.2. Assume $\dim \Delta(\mathcal{H}) = d - 1$. In this case the condition in the lemma is that $\mathcal{H}$ be homologically connected (i.e. that $\widetilde{H}_{d-2}(\Delta(\mathcal{H}); k) = 0$). If $\dim \Delta(\mathcal{H}) = d$ the condition in the lemma says that $\widetilde{H}_{d-1}(\Delta(\mathcal{H}); k) = 0$ and that $\widetilde{H}_{d-2}(\Delta(\mathcal{H}_V); k) = 0$ for every $V \subseteq [n]$ with $|V| = n - 1$.

Consider Lemma 5.22 for a complex $\Delta(\mathcal{H})$ with linear resolution. Since induced complexes $(\Delta(\mathcal{H}))_V = \Delta(\mathcal{H}_V)$ can only have homology in degree $d - 2$, one gets:

Corollary 5.24. Let $\mathcal{H}$ be a d-uniform hypergraph such that $\dim(R/I_{\Delta(\mathcal{H})}) = e$ and $R/I_{\Delta(\mathcal{H})}$ has linear resolution. Then it is also Cohen-Macaulay if and only if $\widetilde{H}_{d-2}(\Delta(\mathcal{H}_V); k) = 0$ for every $V \subseteq [n]$ with $|V| = n - (e - d)$. Furthermore, in this case we have

$$e = \text{con}(\mathcal{H}) + d - 1.$$

6. d-SHELLABILITY

Pure shellable simplicial complexes are somewhat of a cornerstone of combinatorial commutative algebra. This is perhaps mostly since in some situations they provide a nice non-technical (not always an easy though) way of showing that a complex Δ is Cohen-Macaulay. Also, the concept has many times been successfully used to prove, via Alexander duality, that certain rings have linear resolutions, indeed, even linear quotients.

We start by recalling the definition of shellability, pure and non-pure. We use the following notation: Given a finite collection $\{F_1, \ldots, F_t\}$ of non empty subsets of $[n]$, we denote by $\langle F_1, \ldots, F_t \rangle$ the simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$.

Definition 6.1. Let Δ be a simplicial complex on $[n]$ with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$. Δ is called pure shellable if

(i) $|F_i| = |F_j|$ for every pair of indices $1 \leqslant i < j \leqslant t$.

(ii) There exists an ordering $F_1, \ldots, F_t$ of the facets such that $\langle F_j \rangle \cap \langle F_1, \ldots F_{j-1} \rangle$ is generated by a non-empty set of proper maximal faces of $\langle F_j \rangle$ for every $j = 2, \ldots, t$.

A simplicial complex Δ is by definition called non-pure shellable if (ii) but not necessarily (i), holds in the above definition.

Henceforth unless otherwise stated, by shellable we mean shellable in the non-pure sense.

Remark 6.2. It is well known (see for example [2], Theorem 5.1.13) that pure shellability implies Cohen-Macaulayness. This follows from Corollary 6.19 below and the *Eagon-Reiner Theorem* ([4], Theorem 3).

In the following two definitions we introduce the concepts of d-shellability and d-quotients.

Definition 6.3. Let Δ be a simplicial complex on $[n]$. Δ is called d-shellable if its facets can be ordered $F_1, \ldots, F_t$, such that $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ is generated by a non-empty set of proper faces of $\langle F_j \rangle$ of dimension $|F_j| - d - 1$ for every $j = 2, \ldots, t$.

Remark 6.4. The concepts of being 1-shellable and shellable coincide. If Δ is a simplicial complex, a linear ordering of $\mathcal{F}(\Delta)$ satisfying the conditions of Definition 6.3 is called a *d-shelling* of Δ.

Definition 6.5. Let I be a monomial ideal. We say that I has d-quotients if there exists an ordering $x^{m_1} \leqslant \ldots \leqslant x^{m_t}$ of the minimal generators of I, such that if we for $s = 1, \ldots, t$, put $I_s = (x^{m_1}, \ldots, x^{m_s})$, then for every s there are monomials $x^{b_{s_i}}$, $i = 1, \ldots, r_s$, $\deg x^{b_{s_i}} = d$ for all i, such that

$$I_{s-1} : x^{m_s} = (x^{b_{s_1}}, \ldots, x^{b_{s r_s}}).$$

The motivation behind these definitions is the following well known theorem, which we generalize below.

Theorem 6.8. *Let $I = (x^{m_1}, \ldots, x^{m_t})$ be a squarefree monomial ideal. Then I has linear quotients (that is, 1-quotients) precisely when the Alexander dual ideal, I^*, is shellable. In particular, if I^* is shellable and $\deg x^{m_i} = \deg x^{m_j}$ for every pair i, j of indices, then I^* is Cohen-Macaulay.*

Example 6.7. The clique complexes of line hypergraphs $L_n^{d,\alpha}$ and of hypercycles $C_n^{d,\alpha}$ are both $(d - \alpha)$-shellable.

Theorem 6.8. *Let I be a squarefree monomial ideal. Then I has d-quotients precisely when the Alexander dual ideal, I^*, is d-shellable.*

P r o o f. Let $I = (x^{m_1}, \ldots, x^{m_t})$, where the x^{m_i} are the minimal generators. By definition, it is clear that the set of facets of Δ^*, the Stanley-Reisner complex of I^*, is $\mathcal{F}(\Delta^*) = \{F_1, \ldots, F_t\}$, where $F_i = [n] \setminus m_i$ for $i = 1, \ldots, t$. With the notation clear, the proof is almost tautological.

Assume I has d-quotients. If for every $1 \leqslant i < j \leqslant t$, $x^{a_{ji}}$ denotes the minimal generator of $(x^{m_i}) : x^{m_j}$, then (possibly after re-indexing) $I_{j-1} : x^{m_j}$ is minimally generated by the set $x^{a_{j_1}}, \ldots, x^{a_{j_r}}$, for some $r \leqslant j - 1$. This is equivalent to saying that the sets a_{j_α}, $\alpha = 1, \ldots, j_r$, that all have cardinality d by assumption, are precisely the minimal subsets of $[n]$ such that $F_j \setminus a_{j_\alpha} \subseteq F_i$ for some $1 \leqslant i < j$, and that $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ is pure of dimension $|F_j| - d - 1$ and equals $\langle F_j \setminus a_{j_1}, \ldots, F_j \setminus a_{j_r} \rangle$.

The converse is proved by a similar argument: Assume Δ^* is d-shellable, and let $\mathcal{F}(\Delta^*) = \{F_1, \ldots, F_t\}$. Put $m_i = [n] \setminus F_i$. Then the Alexander dual ideal I of I^*, is minimally generated by the monomials x^{m_i}, $i = 1, \ldots, t$. For every $j = 2, \ldots, t$, we let a_{j_α}, $\alpha = 1, \ldots, j_r$ denote the subsets of F_j that one has to remove in order for $F_j \setminus a_{j_\alpha}$ to be a generator of $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$. Then the monomials $x^{a_{j_\alpha}}$ are precisely the minimal generators of $I_{j-1} : x^{m_j}$. □

The following theorem occurs frequently in the literature. It shows that simplicial complexes that are 1-shellable may be defined in (at least) three equivalent ways:

Theorem 6.9. *Let Δ be a simplicial complex on vertex set $[n]$, with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$. Then the following conditions are equivalent:*

(i) *Δ is shellable and $F_1, \ldots, F_t$ is a shelling.*

(ii) *For all i, j, $1 \leqslant i < j \leqslant t$, there exist a vertex v and an integer k with $1 \leqslant k < j$, such that $v \in F_j \setminus F_i$ and $F_j \setminus F_k = \{v\}$.*

(iii) *The set $\{F \in [n]; \; F \in \langle F_1, \ldots, F_j \rangle, \; F \notin \langle F_1, \ldots, F_{j-1} \rangle\}$ has a unique minimal element for all $2 \leqslant i \leqslant t$.*

Two of these statements, slightly modified, remain equivalent in the case of d-shellable complexes also for $d > 1$.

Theorem 6.10. *Let Δ be a simplicial complex on vertex set $[n]$, with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$. Then the following conditions are equivalent:*

(i) *Δ is d-shellable and $F_1, \ldots, F_t$ a d-shelling.*

(ii) *For all i, j, $1 \leqslant i < j \leqslant t$, there exist some set $a_j \subseteq [n]$, $|a_j| = d$, and a k with $1 \leqslant k < j$, such that $a_j \subseteq F_j$, $a_j \cap F_i = \emptyset$ and $F_j \setminus F_k = a_j$.*

P r o o f. The implication (i) $\Rightarrow$ (ii) follows by considering the proof of Theorem 6.8. For the converse let F be a face of $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$. Then F lies in some $\langle F_i \rangle$, $i < j$. Let a_j be a set that fits the description in (ii). Then F is also a face of $\langle F_j \setminus a_j \rangle$ so $\langle F_j \rangle \cap \langle F_1, \ldots, F_{j-1} \rangle$ is pure of dimension $|F_j| - d - 1$. $\qquad\square$

As for shellable complexes, links of faces of d-shellable complexes stay d-shellable:

Proposition 6.11. *Let Δ be a d-shellable complex and F a face of Δ. Then* $\operatorname{lk}_\Delta(F)$ *is again d-shellable.*

P r o o f. Assume $F_1, \ldots, F_t$ is a d-shelling of Δ and that the face F lies is the facets $F_{i_1}, \ldots, F_{i_r}$, where $i_1 < \ldots < i_r$. Put $G_{i_j} = F_{i_j} \setminus F$. Then $\operatorname{lk}_\Delta(F) = \{G_{i_1}, \ldots, G_{i_r}\}$. If $j \leqslant r$ and G is a face of $G_{i_j} \cap \langle G_{i_1}, \ldots, G_{i_{j-1}} \rangle$, then $F \cup G$ is a face of $F_{i_j} \cap \langle F_1, \ldots, F_{i_{j-1}} \rangle$. Hence, if G is maximal we see that $|G| = |F_{j_i}| - |F| - d$, which is our result. $\qquad\square$

We now investigate the behavior of the Betti numbers of ideals with d-quotients. The following two results are more or less obvious. We record them just since they show that the notion of d-quotients is not empty.

Lemma 6.12. *Let $\mathbf{y} = y_1, \ldots, y_r$ be a sequence of monomials in $R = k[x_1, \ldots, x_n]$. Then $\mathbf{y}$ is an R-sequence precisely when $\gcd(y_i, y_j) = 1$ for every $i \neq j$.*

Proposition 6.13. *For every pair of integers $1 \leqslant d \leqslant d'$ there exist a squarefree monomial ideal $I = (x^{m_1}, \ldots, x^{m_t}) \subseteq k[x_1, \ldots, x_n]$, n sufficiently large, $\deg x^{m_i} = d'$ for every $i = 1, \ldots, t$, such that: if we put $I_s = (x^{m_1}, \ldots, x^{m_s})$, $s = 1, \ldots, t$, then every colon ideal $I_s : x^{m_{s+1}}$ is generated by an R-sequence $\mathbf{x}_s = x^{b_{s_1}}, \ldots, x^{b_{s_{r_s}}}$ of squarefree monomials of degree d.*

P r o o f. Let $M \subseteq [n]$ be a set such that $|M| = d' - d$. Choose the generators x^{m_i} such that $|m_i| = d'$ for every $i = 1, \ldots, t$ and $m_i \cap m_j = M$ for every $i \neq j$. $\quad\square$

Splittable monomial ideals, introduced by Eliahou and Kervaire in [6], have been studied for example in [12], [15], [16]. This class of ideals is well behaved in the sense that their Betti numbers satisfy the *Eliahou-Kervaire formula*, see [6] Proposition 3.1. The following definition (that is Definition 1.1 in [12]), captures the content of the Eliahou-Kervaire formula in an axiomatic way.

Definition 6.14. Let I, J and K be monomial ideals such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I = J + K$ is a *Betti splitting* if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

for all $i \in \mathbb{N}$ and (multi)degrees j.

It is easy to see that a monomial ideal I with linear quotients has a very natural Betti splitting. This is the core of the fact that the minimal free resolution of I is a mapping cone. The connection between "being a mapping cone" and "having a Betti splitting", is described in [12] Proposition 2.1.

Theorem 6.15. *Let* $I = (x^{m_1}, \ldots, x^{m_t})$, $\deg x^{m_i} = d'$ *for every* $i = 1, \ldots, t$, *be a squarefree monomial ideal with d-quotients,* $d \leqslant d'$, *and put* $I_s = (x^{m_1}, \ldots, x^{m_s})$, $s = 1, \ldots, t$. *Then*

(i) $\beta_{i,j}(R/(I_{s-1} : x^{m_s})(-d'))$ *and* $\beta_{i,j}(R/I_{s-1})$ *are not non zero in any common degree* j *for any* $i \geqslant 2$, $s = 2, \ldots, t$. *Hence* $I_s = I_{s-1} + (x^{m_s})$ *is a Betti splitting.*

(ii) *For all* i, $2 \leqslant i \leqslant \mathrm{pd}_R(R/I)$, *we have*

$$\beta_i(R/I) = \sum_{s=2}^{t} \beta_{i-1}(R/(I_{s-1} : x^{m_s})(-d')).$$

P r o o f.   (ii) is a consequence of (i) since if we assume that (i) holds, then for every $s = 2, \ldots, t$ we have an exact sequence

$$0 \to R/(I_{s-1} : x^{m_s})(-d') \xrightarrow{x^{m_s}} R/I_{s-1} \to R/I_s \to 0,$$

where the first map the is multiplication by x^{m_s}. It follows from the long exact Tor-sequence that $\beta_i(R/I_t) = \beta_i(R/I_{t-1}) + \beta_{i-1}(R/(I_{t-1} : x^{m_t})(-d'))$. Noting that $\beta_2(R/I_1) = 0$, (ii) now follows by induction.

To prove (i), let $2 \leqslant r \leqslant t$ and consider the following exact sequence

(6.1)
$$0 \to I_{r-1} \cap (x^{m_r}) \to I_{r-1} \oplus (x^{m_r}) \to I_r \to 0.$$

The non trivial maps are $x \mapsto (x, -x)$ and $(x, y) \mapsto x + y$. Let $F'.$ and $G'.$ be the minimal free resolutions of $I_{r-1} \cap (x^{m_r})$ and $I_{r-1} \oplus (x^{m_r})$ respectively. It follows from Proposition 2.1 in [12] that $I_r = I_{r-1} + (x^{m_r})$ is a Betti splitting precisely when the mapping cone, $\mathrm{cone}(\alpha)$, of the lifting $\alpha' \colon F'. \to G'.$ of the left map in the above exact sequence is the minimal free resolution of I_r. By looking at the generators of the ideals $I_{r-1} : x^{m_r}$ and $I_{r-1} \cap (x^{m_r})$, it is clear that we have a homogeneous R-module isomorphism

$$(I_{r-1} : x^{m_r})(-d') \to I_{r-1} \cap (x^{m_r}),$$

the map being the multiplication by x^{m_r}. Hence we wish to show that the mapping cone of the lifting of the inclusion

$$(I_{r-1} : x^{m_r})(-d') \to I_{r-1} \oplus (x^{m_r})$$

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gives a minimal free resolution of I_r. Note that the minimal free resolutions of I_{r-1} and $I_{r-1} \oplus (x^{m_r})$ only differ in a simple way at the bottom degrees. Hence the mapping cone of the lifting of the map $(I_{r-1} : x^{m_r})(-d') \to I_{r-1} \oplus (x^{m_r})$ is essentially the same (except possibly in the bottom degrees) as the mapping cone of the lifting of the injection $(I_{r-1} : x^{m_r})(-d') \to I_{r-1}$.

Let $F.$ and $G.$ be the minimal free resolutions of $R/(I_{r-1} : x^{m_r})(-d')$ and R/I_{r-1} respectively, and $\alpha \colon F. \to G.$ a lifting of the map $R/(I_{r-1} : x^{m_r})(-d') \to R/I_{r-1}$. Hence $\mathrm{cone}(\alpha)$ and $\mathrm{cone}(\alpha')$ are the same except possibly in the bottom degrees after shifting the homological degrees one step. It is well known that, see Exercise A 3.30 in [5], if we have an ideal J/I in a quotient ring R/I, the mapping cone of the lifting of the inclusion $J/I \to R/I$ provides a free resolution of R/J. Since $R/(I_{r-1} : x^{m_r}) \cong I_r/I_{r-1}$, the result now follows after separately considering the bottom degrees. $\square$

Example 6.16. $I = (abc, cde, bef, adf) \subseteq k[a, b, c, d, e, f]$ is an ideal with 2-quotients. The Betti numbers of R/I, in homological degrees $1, 2$, and 3, are $4, 6$ and 3. The corresponding Betti numbers for $R(-3)/(I_1 : cde)$, $R(-3)/(I_2 : bef)$, and $R(-3)/(I_3 : adf)$, are $1, 0$ and 0; $2, 1$ and 0; and $3, 2$ and 0 respectively. It is easily verified that these sum up, according to the proposition, to the Betti numbers of R/I.

Corollary 6.17. Let $I = (x^{m_1}, \ldots, x^{m_t})$, $\deg x^{m_i} = d'$ for every $i = 1, \ldots, t$, be a squarefree monomial ideal with d-quotients, $d \leqslant d'$, and assume the minimal generators of $I_{s-1} : x^{m_s}$ form an R-sequence for every $s = 1, \ldots, t$. Then $\beta_{i,j}(R/I)$ is non zero only for $j = i + d' - 1 + (i-1)(d-1)$, and for all i, $2 \leqslant i \leqslant \mathrm{pd}(R/I)$, we have

$$\beta_{i,j}(R/I) = \sum_{s=2}^{t} \binom{r_s}{i-1},$$

where r_s is the cardinality of the minimal generating set of $I_{s-1} : x^{m_s}$.

P r o o f.   Let $I_{s-1} : x^{m_s} = (x^{b_{s_1}}, \ldots, x^{b_{s_{r_s}}})$. It is then easy to see that $\beta_i(R/(I_{s-1} : x^{m_s})(-d')) = \binom{r_s}{i}$ in degree $j = id + d'$ and equals zero in all other degrees. By induction, $\beta_i(R/I_{s-1})$ is non zero only in degree $j = i + d' - 1 + (i-1)(d-1) = d' + id - d$. This shows that $\beta_i(R/I_s)$ may be non zero only in degree $j = i + d' - 1 + (i-1)(d-1)$ and that $\beta_i(R/I_s) = \beta_i(R/I_{s-1}) + \beta_{i-1}(R/(I_{s-1} : x^{m_s})(-d'))$. The result now follows by induction and Theorem 6.15. $\square$

Example 6.18. An example of such ideal is $I = (abc, cde, cfg, chi)$ in the polynomial ring $k[a, b, c, d, e, f, g, h, i]$. The Betti numbers in homological degrees $1, 2, 3$

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and 4 are $4, 6, 4$ and 1, and they lie in the degrees described in the corollary. It is easily seen that these are constructed from the Betti numbers of the colon ideals.

Corollary 6.19. *Let* $I = (x^{m_1}, \ldots, x^{m_t})$, $\deg x^{m_i} = d'$ *for every* $i = 1, \ldots, t$, *be a squarefree monomial ideal with linear quotients. If* $I_{s-1} : x^{m_s} = (x_{s_1}, \ldots, x_{s_{r_s}})$, $s = 2, \ldots, t$, *then for all* $2 \leqslant i \leqslant \mathrm{pd}_R(R/I)$, $\beta_{i,j}(R/I)$ *is non zero only in degree* $j = i + d' - 1$ *and we have*

$$\beta_{i,j}(R/I) = \sum_{s=2}^{t} \binom{r_s}{i-1}.$$

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