Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 3, 795–807

Persistent URL: http://dml.cz/dmlcz/141640

THE FRACTIONAL DIMENSIONAL THEORY IN LÜROTH EXPANSION

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(Received May 31, 2010)

Abstract. It is well known that every $x \in (0,1]$ can be expanded to an infinite Lüroth series in the form of

$$x = \frac{1}{d_1(x)} + \ldots + \frac{1}{d_1(x)(d_1(x)-1)\ldots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)} + \ldots,$$

where $d_n(x) \geqslant 2$ for all $n \geqslant 1$. In this paper, sets of points with some restrictions on the digits in Lüroth series expansions are considered. Mainly, the Hausdorff dimensions of the Cantor sets

$$F_\varphi = \{x \in (0,1] \colon d_n(x) \geqslant \varphi(n),\ \forall n \geqslant 1\}$$

are completely determined, where φ is an integer-valued function defined on $\mathbb{N}$, and $\varphi(n) \to \infty$ as $n \to \infty$.

Keywords: Lüroth series, Cantor set, Hausdorff dimension

MSC 2010: 11K55, 28A78, 28A80

1. Introduction

For each $x \in (0,1]$, let $d_1 = d_1(x) \in \mathbb{N}$ be the unique integer such that

$$\frac{1}{d_1(x)} < x \leqslant \frac{1}{d_1(x)-1}, \tag{1.1}$$

This work is supported by National Natural Science Foundation of China (Grant Nos. 10631040, 10926160), the Science and Nature Fund of Hunan Province (Grant No. 09JJ3001), and the Key Plan Project of Department of Science and Technology of Hunan Province (Grant No. 2009NK4038).

and let the transformation $T\colon (0,1] \to (0,1]$ be defined as

$$T(x) := d_1(x)(d_1(x)-1)\Big(x - \frac{1}{d_1(x)}\Big). \tag{1.2}$$

Then for every $x \in (0,1]$, the algorithm of (1.2) leads to an infinite series expansion in the form

$$x = \frac{1}{d_1(x)} + \sum_{n\geqslant 2} \frac{1}{d_1(x)(d_1(x)-1)\dots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)}, \tag{1.3}$$

where $d_n(x) = d_1(T^{n-1}(x)) \geqslant 2$ $(\forall n \geqslant 1)$ are called the digits of x. The infinite series (1.3) is called the Lüroth expansion of x, which was first introduced by J. Lüroth in 1883 ([11]). Lüroth series expansions play an important role in the representation theory of numbers, probability theory, and dynamical systems. For metrical properties, the digits $\{d_n, n \geqslant 1\}$ are stochastically independent but with infinite mean ([7], p. 66). For dynamical properties, the transformation T is invariant and ergodic with respect to the Lebesgue measure ([2], ([7], p. 80), [9], [13]). And for more research related to Lüroth expansions, we can refer to [3], [7], [12] and [14]. For the exceptional sets in Lüroth expansions, the earliest research was conducted by Šalát in [12], where the author obtained the Hausdorff dimension of the sets $M_k = \{x \in (0,1]\colon d_n(x) = k,\ n = 1,2,\dots\}$ for any $k \in \mathbb{N}$, and in the conformal system theory, K. J. Falconer ([5]) obtained the Hausdorff dimension for the general case of the above sets, i.e., the set $J_A = \{x \in (0,1]\colon d_n(x) \in A \text{ for all } n \geqslant 1\}$, where $A \subset \mathbb{N} \setminus \{1\}$. In recent years, the Lüroth expansions have been attaked with great importance once more. For given probability sequence $\vec{p} = (p_1, p_2, \dots)$, i.e., $p_j \geqslant 0$ for all $j \in \mathbb{N}$ and $\sum\limits_{j=1}^{\infty} p_j = 1$, A. H. Fan etc. ([6]) obtained the dimension of the Besicovitch-Eggleston set

$$E(\vec{p}) := \Big\{x \in (0,1]\colon \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{d_1(x)=j+1\}}(T^k(x)) = p_j,\ \text{for all } j \geqslant 1\Big\}.$$

For frequency of digits in the Lüroth expansion, L. Barreiraa and G. Iommi ([1]) computed the Hausdoff dimension of the set $F_\alpha = \{x \in (0,1]\colon \tau_k(x) = \alpha_k$ for each $k \in \mathbb{N}\}$, where $\tau_k(x,n) = \operatorname{card}\{i \in \{1,\dots,n\}\colon d_i(x) = k\}$, $\tau_k(x) = \lim\limits_{n\to\infty} \tau_k(x,n)/n$, and $\alpha = (\alpha_1, \dots, \alpha_2, \dots)$, $\sum\limits_{i=1}^{\infty} \alpha_i = 1$.

Since the digits $\{d_n(x)\colon x \in (0,1]\}_{n=1}^{\infty}$ are independent and identically distributed, an immediate consequence of the Borel-Cantelli lemma yields:

Fact: Let φ be an arbitrary positive function on natural numbers $\mathbb{N}$ and $E(\varphi) = \{x \in (0,1]\colon d_n(x) \geqslant \varphi(n) \text{ for all } n\}$. Then $\mathscr{L}(E(\varphi))$ is null or full according to whether the series $\sum_{n=1}^{\infty} 1/\varphi(n)$ converges or not, where $\mathscr{L}$ denotes the Lebesgue measure.

Since $\{\{d_n(x)\}_{n=1}^{\infty}\colon\ x \in (0,1]\}$ can assume arbitrarily large values, it is possible that there are points that deviate from the above fact, namely that $d_n(x) \geqslant \varphi(n)$ holds for all $n \geqslant 1$. Then it leads to the following questions.

Let $\varphi\colon\ \mathbb{N} \to \mathbb{N}$ with $\varphi(n) \to \infty$ as $n \to \infty$. It is of interest to determine the dimension of the set

$$F_\varphi = \{x \in (0,1]\colon\ d_n(x) \geqslant \varphi(n),\ \text{for all } n \geqslant 1\}.$$

Recall that

$$F_\varphi = \{x \in I\colon\ d_n(x) \geqslant \varphi(n),\ \forall n \geqslant 1\},$$

where $\varphi\colon\ \mathbb{N} \to \mathbb{N}$ with $\varphi(n) \to \infty$ as $n \to \infty$. We will prove

Theorem 1.1. *For any $\varphi\colon\ \mathbb{N} \to \mathbb{N}$ with $\varphi(n) \to \infty$ as $n \to \infty$, for any $b \geqslant 1$ write $\log b = \limsup_{n\to\infty} \log\log\varphi(n)/n$. Then*

$$\dim_H F_\varphi = \frac{1}{1+b}. \tag{1.4}$$

Throughout this paper, I denotes the interval $(0,1]$, $|\cdot|$ the diameter of a set, $\dim_{\mathrm{H}}$ the Hausdorff dimension and 'cl' the closure of a subset of I.

It should be mentioned that some ideas of the paper are derived from [8] and [15], as the authors proved the continued fraction case.

2. Preliminaries

In this section we present some elementary properties which are enjoyed by the Lüroth expansion and some lemmas that will be used later.

Proposition 2.1 ([7], p. 18)**.** *The series on the right hand side of* (1.3) *is the expansion of its sum by the algorithm* (1.1) *and* (1.2) *if and only if*

$$d_n \geqslant 2 \quad \text{for all } n \geqslant 1.$$

For any $d_1, \ldots, d_n \in \mathbb{N}$ with $d_k \geqslant 2$ ($\forall\ 1 \leqslant k \leqslant n$), we call

$$I(d_1, \ldots, d_n) = \{x \in I\colon\ d_k(x) = d_k,\ 1 \leqslant k \leqslant n\}$$

an n-th order basic cylinder.

Proposition 2.2 ([7], p. 67). *For any $d_1, \ldots, d_n \in \mathbb{N}$ with $d_k \geqslant 2$ $(1 \leqslant k \leqslant n)$, the n-th order basic interval $I(d_1, \ldots, d_n)$ is the interval with the endpoints*

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n},$$

and

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}.$$

As a consequence,

$$|I(d_1, \ldots, d_n)| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}. \tag{2.1}$$

To end this section, we present a simple result which confirms that any changes of the restrictions on the digits with only finite terms will not influence the final Hausdorff dimension. Namely, let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be two sequences of nonempty subsets of $\mathbb{N} \setminus \{1\}$ with $A_n = B_n$ when n is large. Set

$$\mathscr{A} = \{x \in I\colon\ d_n(x) \in A_n,\ \text{for all } n \geqslant 1\},$$
$$\mathscr{B} = \{x \in I\colon\ d_n(x) \in B_n,\ \text{for all } n \geqslant 1\}.$$

Lemma 2.1. $\dim_H \mathscr{A} = \dim_H \mathscr{B}$.

Proof. Assume that $A_n = B_n$ when $n > N$. Notice that

$$\mathscr{A} = \bigcup_{a_j \in A_j, 1 \leqslant j \leqslant N} \{x \in I\colon\ d_j(x) = a_j,\ 1 \leqslant j \leqslant N,\ d_n(x) \in A_n,\ \forall n > N\}.$$

We write the terms in the union as $\mathscr{A}(a_1, \ldots, a_N)$ for simplicity. Similarly, for $\mathscr{B}$, we write

$$\mathscr{B} = \bigcup_{b_j \in B_j, 1 \leqslant j \leqslant N} \mathscr{B}(b_1, \ldots, b_N).$$

Now we write $x = [d_1(x), d_2(x), \ldots]$, where $\{d_n(x)\}_{n=1}^{\infty}$ is the digits sequence of its Lüroth expansion. Then we define a map $\Gamma\colon \mathscr{A}(a_1, \ldots, a_N) \to \mathscr{B}(b_1, \ldots, b_N)$ as

$$\Gamma([a_1, \ldots, a_N, d_{N+1}, \ldots, d_n, \ldots]) = [b_1, \ldots, b_N, d_{N+1}, \ldots, d_n, \ldots],$$

which means Γ transform an n-th cylinder to another one.

For any $x, x' \in \mathscr{A}$, we can write $x = 1/a_1 + \ldots + T^N(x)/a_1(a_1-1)\ldots a_N(a_N-1)$ and $x' = 1/a_1 + \ldots + T^N(x')/a_1(a_1-1)\ldots a_N(a_N-1)$; then

$$|\Gamma(x) - \Gamma(x')| = \prod_{j=1}^{N} \frac{|T^N x - T^N x'|}{b_j(b_j-1)} = \prod_{j=1}^{N} \frac{a_j(a_j-1)}{b_j(b_j-1)} |x - x'|.$$

Thus, $\dim_H \mathscr{A}(a_1,\ldots,a_N) = \dim_H \mathscr{B}(b_1,\ldots,b_N)$, and $\dim_H \mathscr{A}(a_1,\ldots,a_N) \leqslant \dim_H \mathscr{B}$. So, $\dim_H \mathscr{A} \leqslant \dim_H \mathscr{B}$. □

3. Proof of Theorem 1.1

In fact, Theorem 1.1 is an immediate consequence of the following two theorems.

Theorem 3.1. *For any $a > 1$ and $b > 1$, set*

$$E(a,b) = \{x \in (0,1] \colon\ d_n(x) \geqslant a^{b^n},\ \forall n \geqslant 1\},$$
$$\widetilde{E}(a,b) = \{x \in (0,1] \colon\ d_n(x) \geqslant a^{b^n},\ \forall n\}.$$

Then $\dim_H E(a,b) = \dim_H \widetilde{E}(a,b) = 1/(1+b)$.

Theorem 3.2. *Set $F = \{x \in I \colon\ d_n(x) \to \infty$ as $n \to \infty\}$. Then $\dim_H F = \frac{1}{2}$.*

We will prove Theorem 1.1 by assuming Theorems 3.1 and 3.2 hold and the proof of Theorems 3.1 and 3.2 will be postponed to the end of this section.

P r o o f of Theorem 1.1. Recall that $\log b = \limsup\limits_{n\to\infty} \log\log \varphi(n)/n$, hence for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $\varphi(n) \leqslant e^{(b+\varepsilon)^n}$. Thus

$$F_\varphi \supset \bigl\{x \in I \colon\ d_j(x) \geqslant \varphi(j),\ 1 \leqslant j \leqslant N,\ d_n(x) \geqslant e^{(b+\varepsilon)^n},\ \forall n > N\bigr\}.$$

Therefore, by Lemma 2.1 and Theorem 3.1, we have $\dim_H F_\varphi \geqslant 1/(1+b+\varepsilon)$. Letting $\varepsilon \to 0$, we get

$$\dim_H F_\varphi \geqslant \frac{1}{1+b}.$$

For the lower bound, we will distinguish two cases according to whether $b = 1$ or $b > 1$.

(i) $b = 1$. It is evident that $F_\varphi \subset F$. Thus, by Theorem 3.2, we have

$$\dim_H F_\varphi \leqslant \frac{1}{2} = \frac{1}{1+b}.$$

(ii) $b > 1$. For any $\varepsilon > 0$, $\varphi(n) \geqslant e^{(b-\varepsilon)^n}$ holds for infinitely many n's. Thus, $F_\varphi \subset \widetilde{E}(e, b-\varepsilon)$. So, by Theorem 3.1, we have $\dim_H F_\varphi \leqslant 1/(1+b-\varepsilon)$. Letting $\varepsilon \to 0$, we obtain $\dim_H F_\varphi \leqslant 1/(1+b)$. □

4. Proof of Theorem 3.1

We will show $\dim_H \widetilde{E}(a,b) \leqslant 1/(b+1)$ and $\dim_H E(a,b) \geqslant 1/(b+1)$.

4.1. Upper bound. To get the upper bound, a suitable covering system is needed.

A covering system. We mention that the main idea is borrowed from Lúczak ([10]) for the case of continued fractions. First, we introduce some notation. For any $d_1, \ldots, d_n \in \mathbb{N}$, let

$$Q_n = Q_n(d_1, \ldots, d_n) = \prod_{k=1}^{n} d_k, \;\; q_n = q_n(d_1, \ldots, d_n) = \prod_{k=1}^{n} (d_k - 1),$$
$$P_n = P_n(d_1, \ldots, d_n) = Q_n q_{n-1} \Big(\frac{1}{d_1} + \sum_{j=2}^{n} \frac{1}{d_1(d_1-1)\ldots d_{j-1}(d_{j-1}-1)d_j} \Big).$$

If $d_j = d_j(x)$ $(1 \leqslant j \leqslant n)$ for some $x \in (0,1]$, we denote $q_n = q_n(x)$, $Q_n = Q_n(x)$ and $P_n = P_n(x)$ for simplicity. Then $P_n(x)/Q_n(x)q_{n-1}(x)$ is nothing but the n-th approximate term of x in its Lüroth expansion, i.e.,

$$x = \frac{P_n(x)}{q_{n-1}(x)Q_n(x)} + \frac{T^n(x)}{q_n(x)Q_n(x)}. \tag{4.1}$$

Lemma 4.1. *For any $1 < c < b$,*

$$\widetilde{E}(a,b) \subset \{x \in I\colon \; Q_{n+1}(x) \geqslant \max\{Q_n(x)^c, a^{c^{n+1}}\} \text{ i.o. } n\}.$$

Proof. Fix an $x \in \widetilde{E}(a,b)$ and $m \in \mathbb{N}$. Then there exists $k > m$ such that

$$Q_m(x) < a^{b^k c^{m-k}} \quad \text{and} \quad d_k(x) \geqslant a^{b^k}.$$

Set $f(n) = a^{b^k c^{n-k}}$. We have $Q_m(x) < f(m)$, $Q_k(x) \geqslant f(k)$. Let $m \leqslant n < k$ be the smallest integer such that $Q_n(x) < f(n)$ and $Q_{n+1}(x) \geqslant f(n+1)$. As a consequence, we have

$$Q_{n+1}(x) \geqslant \max\{Q_n(x)^c, a^{c^{n+1}}\}. \tag{4.2}$$

Fix $1 < c < b$, $a > 1$ and $t = 4(c+1)$. For any $Q \geqslant 2$, let

$$\mathscr{J}_Q = \Big\{ B\Big(\frac{P_n(x)}{q_{n-1}(x)Q_n(x)}, \; \frac{2^{n+1}}{Q_n(x)^{(1+c)}}\Big)\colon \; x \in (0,1], Q_n \geqslant a^{c^{n+1}/t}, \; Q_n = Q \Big\}.$$

Then

$$\mathscr{I}_Q = \Big\{ B\Big(\frac{P_n(x)}{q_{n-1}(x)Q_n(x)}, \; \frac{2}{Q_n(x)^t}\Big)\colon \; x \in (0,1], \; Q_n(x) = Q \Big\}.$$

□

Lemma 4.2. *For any* $Q_0 \geqslant 2$, *we have*

$$\widetilde{E}(a,b) \subset \bigcup_{Q=Q_0}^{\infty}\Big(\bigcup_{B\in\mathscr{J}_Q} B \cup \bigcup_{B\in\mathscr{I}_Q} B\Big). \tag{4.3}$$

P r o o f. By the definition of P_n, Q_n and q_n, we have $q_n \geqslant 2^{-n}Q_n$ for all $n \geqslant 1$. Then, by (4.1), it follows that for any $x \in (0,1]$

$$\Big|x - \frac{P_n(x)}{q_{n-1}(x)Q_n(x)}\Big| \leqslant \frac{1}{q_n(x)Q_n(x)(d_{n+1}(x)-1)} \leqslant \frac{2^{n+1}}{Q_n(x)Q_{n+1}(x)}. \tag{4.4}$$

Fix $x \in \widetilde{E}(a,b)$. By Lemma 4.1, there exists $n \in \mathbb{N}$ such that $Q_n(x) \geqslant 2^n > Q_0$ and

$$Q_{n+1}(x) \geqslant \max\{Q_n(x)^c, a^{c^{n+1}}\}.$$

(i) If $Q_n(x) \geqslant a^{c^{n+1}/t}$, then by (4.4)

$$\Big|x - \frac{P_n(x)}{q_{n-1}(x)Q_n(x)}\Big| \leqslant \frac{2^{n+1}}{Q_n(x)Q_{n+1}(x)} \leqslant \frac{2^{n+1}}{Q_n(x)^{1+c}}.$$

Thus, $x \in B(P_n(x)/q_{n-1}(x)Q_n(x), 2^{n+1}/Q_n(x)^{(1+c)})$ as $Q_n(x) \geqslant a^{c^{n+1}/t}$.

(ii) If $Q_n(x) < a^{c^{n+1}/t}$, then $Q_{n+1}(x) \geqslant Q_n(x)^t$, thus,

$$\Big|x - \frac{P_n(x)}{q_{n-1}(x)Q_n(x)}\Big| \leqslant \frac{2^{n+1}}{Q_n(x)Q_{n+1}(x)} \leqslant \frac{2}{Q_n(x)^t}.$$

Therefore, $x \in B(P_n(x)/q_{n-1}(x)Q_n(x), 2/Q_n(x)^t)$. □

Lemma 4.3 ([10])**.** *Let* k, m *be natural numbers. Denote by* $S(m,k)$ *the number of vectors* $(k_1,\dots,k_n)$ *of natural numbers such that* $1 \leqslant n \leqslant k$ *and* $\prod_{j=1}^{n} k_j \leqslant m$, *i.e.,*

$$S(m,k) = \sharp\Big\{(k_1,\dots,k_n)\in\mathbb{N}^n\colon\ 1\leqslant n\leqslant k,\ \prod_{j=1}^{n} k_j \leqslant m\Big\}.$$

Then

$$S(m,k) \leqslant m(2+\log m)^{k-1}.$$

P r o o f. It can be done by induction. In [10], a detailed proof is given. □

Lemma 4.2 [The number of intervals in $\mathscr{J}_Q$]**.** *For any $k \geqslant 1$, take*

$$S_k = \{B \in \mathscr{J}_{\mathscr{Q}}\colon\ a^{k-1} \leqslant Q < a^k\}.$$

Then we have

$$\sharp S_k \leqslant a^k (2 + \log a^k)^{\log_c kt - 2}. \tag{4.5}$$

P r o o f. Lemma 4.3 will be applied to prove this assertion. Notice that P_n, Q_n and q_{n-1} can be uniquely determined by $d_1, \ldots, d_n$, hence

$$\begin{aligned}\sharp S_k &\leqslant \sharp\Big\{(d_1, \ldots, d_n)\colon\ a^{\frac{1}{t}c^{n+1}} \leqslant \prod_{j=1}^{n} d_j = Q,\ \text{and } a^{k-1} \leqslant Q < a^k\Big\}\\ &\leqslant \sharp\Big\{(d_1, \ldots, d_n)\colon\ \prod_{j=1}^{n} d_j \leqslant a^k,\ n+1 \leqslant \log_c kt\Big\}\\ &\leqslant a^k \left(2 + \log a^k\right)^{\log_c kt - 2}.\end{aligned}$$

□

Lemma 4.5 [The number of intervals in $\mathscr{I}_Q$]**.** *For any $Q \geqslant 2$, $\sharp\{B\colon\ B \in \mathscr{I}_Q\} \leqslant Q^3$.*

P r o o f. Since any $P_n/q_{n-1}Q_n$ is uniquely determined by the sequence $(d_1, \ldots, d_n)$, so that for any fixed $Q \geqslant 1$ we have

$$\begin{aligned}\sharp\{q_{n-1}\colon\ Q_n = Q\} &\leqslant \sharp\{q_{n-1}\colon\ q_{n-1} \leqslant Q\} \leqslant Q,\\ \sharp\{P_n\colon\ Q_n = Q\} &\leqslant \sharp\{P_n\colon\ P_n \leqslant Q^2\} \leqslant Q^2,\\ \sharp\{B\colon\ B \in \mathscr{I}_Q\} &= \sharp\{(P_n(x), q_{n-1}(x))\colon\ x \in (0,1], Q_n = Q\}\\ &\leqslant \sharp\{q_{n-1}\colon\ Q_n = Q\} \times \sharp\{P_n\colon\ Q_n = Q\} \leqslant Q^3.\end{aligned}$$

□

Definition 4.1 ([4], p. 42)**.** Let X be a metric space. If $F \subset X$ and $s \in [0, +\infty)$, for any $\delta > 0$ define

$$\mathscr{H}^s(F) = \lim_{\delta \to 0} \inf \Big\{\sum_{i=1}^{\infty} |U_i|^s\colon\ U_i \text{ is the } \delta\text{-covering of } F\Big\}.$$

Then $\mathscr{H}^s(F)$ is called the s-dimensional Hausdorff measure of F, and the Hausdorff dimension of the set F is defined by

$$\dim_H F = \inf\{s > 0\colon\ \mathscr{H}^s(F) = 0\}.$$

Proposition 4.1. *For any $a > 1$, $b > 1$, we have*

$$\dim_H \widetilde{E}(a,b) \leqslant \frac{1}{1+b}.$$

P r o o f. Fix $\varepsilon > 0$ and $\frac{1}{2}(1+b) < c < b$. Take $s = (1+2\varepsilon)/(1+c)$ and choose k_1 large enough such that, for any $k \geqslant k_1$,

$$(a^k(2+\log a^k)^{\log_c kt-2}) \cdot (a^{1+b}2(kt)^{\log_c 2}) \leqslant a^{k(1+\varepsilon)}. \tag{4.6}$$

Note that Lemma 4.2 gives a covering system of $\widetilde{E}(a,b)$, namely,

$$\widetilde{E}(a,b) \subset \bigcup_{Q=a^{k_1}}^{\infty} \biggl(\bigcup_{B\in\mathscr{J}_Q} B \cup \bigcup_{B\in\mathscr{I}_Q} B \biggr) = \biggl(\bigcup_{k=k_1+1}^{\infty} \bigcup_{B\in S_k} B \biggr) \cup \biggl(\bigcup_{Q=a^{k_1}}^{\infty} \bigcup_{B\in\mathscr{I}_Q} B \biggr).$$

So, we have

$$\mathscr{H}^s(\widetilde{E}(a,b)) \leqslant \liminf_{k_1\to\infty} \sum_{k=k_1+1}^{\infty} \sum_{B\in S_k} |B|^s + \liminf_{k_1\to\infty} \sum_{Q=a^{k_1}}^{\infty} \sum_{B\in\mathscr{I}_Q} |B|^s =: I_1 + I_2.$$

Note that for any $B \in S_k$,

$$|B| = \frac{2^{n+2}}{Q_n^{1+c}} \leqslant \frac{2(kt)^{\log_c 2}}{a^{(k-1)(1+c)}} \leqslant \frac{a^{1+b}2(kt)^{\log_c 2}}{a^{k(1+c)}}.$$

Then, by Lemma 4.4 and (4.6), we have

$$\begin{aligned}
I_1 &\leqslant \liminf_{n\to\infty} \sum_{k=k_1+1}^{\infty} \left(\frac{a^{1+b}2(kt)^{\log_c 2}}{a^{k(1+c)}} \right)^s \sharp S_k \\
&\leqslant \liminf_{k_1\to\infty} \sum_{k=k_1+1}^{\infty} \left(\frac{a^{1+b}2(kt)^{\log_c 2}}{a^{k(1+c)}} \right)^s a^k(2+\log a^k)^{\log_c kt-2} \\
&\leqslant \liminf_{n\to\infty} \sum_{k=k_1+1}^{\infty} \frac{1}{a^{k(1+c)s}} a^{k(1+\varepsilon)} = \liminf_{n\to\infty} \sum_{k=k_1+1}^{\infty} a^{-k\varepsilon} \\
&< \infty.
\end{aligned}$$

For I_2, by Lemma 4.5, we have

$$\begin{aligned}
I_2 &= \liminf_{k_1\to\infty} \sum_{Q=a^{k_1}}^{\infty} \frac{4^s}{Q^{ts}} \sharp\mathscr{I}_Q \leqslant \liminf_{k_1\to\infty} \sum_{Q=a^{k_1}}^{\infty} \frac{4^s}{Q^{ts}} \cdot Q^3 \\
&= \liminf_{k_1\to\infty} \sum_{Q=a^{k_1}}^{\infty} \frac{4^s}{Q^{4+8\varepsilon}} \cdot Q^3 < \infty.
\end{aligned}$$

Thus we have $\dim_H \widetilde{E}(a,b) \leqslant s$. Letting $c \to b$ and $\varepsilon \to 0$, we obtain

$$\dim_H \widetilde{E}(a,b) \leqslant \frac{1}{1+b}.$$

$\square$

4.2. Lower bound. The lower bound can be obtained as a simple consequence of a general method. Example 4.6 from [2] states the following:

Lemma 4.6 ([4]). *If $[0,1] = E_0 \supset E_1 \supset \ldots$ are sets each of which is a finite union of disjoint closed intervals, and each interval of E_{n-1} contains at least m_n intervals of E_n which are separated by gaps of length at least ε_n, and if $m_n \geqslant 2$ and $\varepsilon_n \geqslant \varepsilon_{n+1} > 0$, then*

$$\dim_H \bigcap_{n=1}^{\infty} E_n \geqslant \liminf_{n\to\infty} \frac{\log(m_1 \ldots m_{n-1})}{-\log(m_n \varepsilon_n)}. \tag{4.7}$$

Proof. We will apply Lemma 4.6 to the set

$$F_1 = \{x \in (0,1]\colon\ a^{b^n} < d_n(x) \leqslant 3a^{b^n} \text{ for all } n \geqslant 1\},$$

which is a subset of $E(a,b)$.

For any $n \geqslant 1$, we write

$$J(d_1,\ldots,d_n) = \bigcup_{a^{b^{n+1}} < d_{n+1} \leqslant 3a^{b^{n+1}}} \operatorname{cl} I(d_1,\ldots,d_n,d_{n+1})$$

for an n-th order realizable interval with respect to F_1, if $a^{b^k} < d_k \leqslant 3a^{b^k}$ for each $1 \leqslant k \leqslant n$. Then set

$$E_n = \bigcup J(d_1,\ldots,d_n),$$

where the union is taken over all n-th order realizable intervals. Then it is easy to see that

$$F_1 = \bigcap_{n=1}^{\infty} E_n.$$

It is evident that, for all $n \geqslant 1$,

$$m_n \geqslant \frac{1}{2} a^{b^n}. \tag{4.8}$$

Now we will give a bound estimation on the gap ε_n of any two n-th order realizable intervals. Note that for each n-th order realizable interval $J(d_1, \ldots, d_n)$, one has

$$J(d_1, \ldots, d_n) \subset I(d_1, \ldots, d_n).$$

Moreover, from Proposition 2.2, for n large enough, the gap between the right endpoint of the interval $I(d_1, \ldots, d_n)$ and the right endpoint of the interval $J(d_1, \ldots, d_n)$ is

$$\begin{aligned}(4.9)\quad &\left(\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n - 1}\right) \\ &- \left(\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \ldots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)} \frac{1}{a^{b^{n+1}}}\right) \\ &\geqslant \frac{1}{2} \times \frac{1}{3^{2n}} \prod_{k=1}^{n} \frac{1}{a^{2b^k}} := \varepsilon_n.\end{aligned}$$

Hence, by Lemma 4.6, we have $\dim_H F_1 \geqslant 1/(1+b)$, and thus

$$\dim_H E(a,b) \geqslant \dim_H F_1 \geqslant \frac{1}{1+b}.$$

□

5. Proof of Theorem 3.2

To get Theorem 3.2, in the light of Theorem 3.1 we only need to show $\dim_H F \leqslant \frac{1}{2}$. Note that

$$F = \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \{x \in I\colon\ d_n(x) \geqslant M,\ \forall n \geqslant N\}.$$

For any $\varepsilon > 0$, take $M = M(\varepsilon)$ large enough such that

$$(5.1)\qquad \sum_{d=M}^{\infty} \left(\frac{1}{d(d-1)}\right)^{\frac{1}{2}+\varepsilon} < 1.$$

Set $F(M) = \{x \in I\colon\ d_n(x) \geqslant M, \forall n \geqslant 1\}$. Lemma 2.1 implies, for any $N \geqslant 1$,

$$\dim_H \{x \in I\colon\ d_n(x) \geqslant M, \forall n \geqslant N\} = \dim_H F(M).$$

So, $\dim_H F \leqslant \dim_H F(M)$. On the other hand,

$$\begin{aligned}\mathscr{H}^{\frac{1}{2}+\varepsilon}(F(M)) &\leqslant \liminf_{N\to\infty} \sum_{d_k\geqslant M, 1\leqslant k\leqslant N} |I(d_1,\ldots,d_N)|^{\frac{1}{2}+\varepsilon} \\ &= \liminf_{N\to\infty} \sum_{d_k\geqslant M, 1\leqslant k\leqslant N} \left(\prod_{k=1}^{N} \frac{1}{d_k(d_k-1)}\right)^{\frac{1}{2}+\varepsilon} \\ &= \liminf_{N\to\infty} \left(\sum_{d=M}^{\infty} \frac{1}{d(d-1)}\right)^{N(\frac{1}{2}+\varepsilon)} < 1.\end{aligned}$$

Thus,

$$\dim_H F \leqslant \dim_H F(M) \leqslant \frac{1}{2} + \varepsilon.$$

By letting $\varepsilon \to 0$, we obtain $\dim_H F \leqslant \frac{1}{2}$. □

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