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CONGRUENCE KERNELS OF DISTRIBUTIVE PJP-SEMILATTICES

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Abstract. A meet semilattice with a partial join operation satisfying certain axioms is a JP-semilattice. A PJP-semilattice is a pseudocomplemented JP-semilattice. In this paper we describe the smallest PJP-congruence containing a kernel ideal as a class. Also we describe the largest PJP-congruence containing a filter as a class. Then we give several characterizations of congruence kernels and cokernels for distributive PJP-semilattices.

Keywords: semilattice, distributivity, pseudocomplementation, congruence, kernel ideal, cokernel

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1. Introduction

Partial lattices have been studied by many authors. We refer the reader to [6], [7], [9], [10], [11] for partial lattices. Cornish and Noor [8], [12] have studied partial lattices which they preferred to call near lattices. A *near lattice* $\mathbf{N}$ is a meet semilattice such that for any $a, b \in \mathbf{N}$, $a \vee b$ exists whenever there is a common upper bound of a, b. We also refer the reader to the recent publications [3], [4], [5] for near lattices. Throughout the paper by semilattice we mean the meet semilattice. First we introduce the notion of the JP-semilattice.

A meet semilattice $\mathbf{S} = \langle \mathbf{S}; \wedge, \vee \rangle$ with a partial binary operation $\vee$ is said to be a *join partial semilattice* (or JP-*semilattice*) if for all $x, y, z \in S$,

(i) $x \vee x$ exists and $x \vee x = x$;
(ii) if $x \vee y$ exists, then $y \vee x$ exists and $x \vee y = y \vee x$;
(iii) if $x \vee y$, $y \vee z$, $(x \vee y) \vee z$ exist, then $x \vee (y \vee z)$ exists and $(x \vee y) \vee z = x \vee (y \vee z)$;
(iv) if $x \vee y$ exists, then $x = x \wedge (x \vee y)$;
(v) if $y \vee z$ exists, then $(x \wedge y) \vee (x \wedge z)$ exists for all $x \in S$.

Observe that not every semilattice needs to be a JP-semilattice, for example, the semilattice $\mathbf{P}$ given in Figure 1 is not a JP-semilattice. Here $b \vee c$ exists, but $(a \wedge b) \vee (a \wedge c)$ does not. Moreover, it is easy to see that every near lattice is a JP-semilattice but the converse is not necessarily true, for example, the semilattice $\mathbf{M}$ given in Figure 1 is a JP-semilattice but not a near lattice.

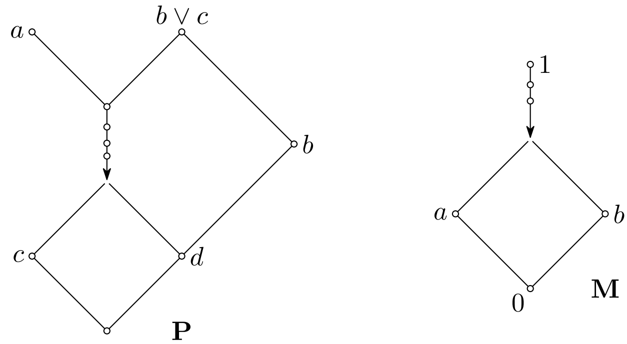

Fig. 1

A JP-semilattice $\mathbf{S}$ is said to be JP-*distributive* iff for all $x, y, z \in S$ such that $y \vee z$ exists one has $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (remember the right-hand side exists by condition (v) above). It is evident that every distributive semilattice, considered as a JP-semilattice, is distributive (see [13]). However, the converse is not true. Consider the JP-semilattice $\mathcal{M}_\infty$ given by Figure 2. It is JP-distributive but not distributive as a semilattice. The authors have studied JP-distributive JP-semilattices in [1].

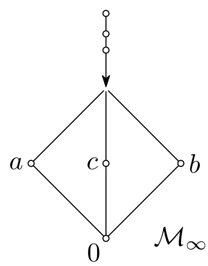

Fig. 2

Let $\mathbf{S}$ be a JP-semilattice with smallest element 0 and let $a \in S$. An element $d \in S$ is called the *pseudocomplement* of $a \in S$ if $a \wedge d = 0$ and for all $x \in S$, $a \wedge x = 0$ implies $x \leqslant d$. Clearly the pseudocomplement of an element is unique whenever it exists. The pseudocomplement of an element $a \in S$ is denoted by a^*. A JP-semilattice is said to be a *pseudocomplemented JP-semilattice* (or simply a PJP-*semilattice*) if every element has a pseudocomplement.

Let $\mathbf{S}$ be a pseudocomplemented JP-semilattice. The set

$$\mathrm{Sk}(S) = \{a^*\colon\ a \in S\}$$

is called the *skeleton* of S. The elements of $\mathrm{Sk}(S)$ are called *skeletal*. It is evident that $\sup\{a^*, b^*\}$ in $\mathrm{Sk}(S)$ always exists and we denote it by $a^* \veebar b^*$. That is, for any $a, b \in \mathrm{Sk}(S)$ we have $a \veebar b = \sup\{a, b\}$ in $\mathrm{Sk}(S)$. The following identities hold for PJP-semilattices as they hold for pseudocomplemented semilattices (see [9]). We often use the identities in this paper.

Lemma 1.1.

(a) $a \leqslant a^{**}$;
(b) $a \leqslant b$ *implies* $b^* \leqslant a^*$;
(c) $a^* = a^{***}$;
(d) $0^* = 1$, *the largest element of* S;
(e) $a \wedge b^* = a \wedge (a \wedge b)^*$;
(f) $a \in \mathrm{Sk}(S) \Leftrightarrow a = a^{**}$;
(g) $a, b \in \mathrm{Sk}(S) \Rightarrow a \wedge b = (a \wedge b)^{**}$;
(h) $a, b \in \mathrm{Sk}(S) \Rightarrow a \veebar b = (a^* \wedge b^*)^*$.

As in the case of distributive lattices, JP-distributivity in JP-semilattices does not imply the existence of pseudocomplements. For example, consider the distributive JP-semilattice $\mathcal{M}_\infty$ given in Figure 2. Clearly, $\mathcal{M}_\infty$ is not pseudocomplemented. In this paper we concentrate our attention on distributive PJP-semilattices.

Let $\mathbf{S}$ be a PJP-semilattice. A semilattice congruence θ on S is called a JP-*congruence* on S, if $x_1 \equiv y_1(\theta)$ and $x_2 \equiv y_2(\theta)$ implies $x_1 \vee x_2 \equiv y_1 \vee y_2(\theta)$ whenever $x_1 \vee x_2$ and $y_1 \vee y_2$ exist. A JP-congruence θ on S is called a PJP-*congruence* on S, if $x \equiv y(\theta) \Rightarrow x^* \equiv y^*(\theta)$. A non-empty subset I of a JP-semilattice $\mathbf{S}$ is called an *ideal* of S if the following conditions hold:

(i) if $i \in I$, $j \in S$ and $j \leqslant i$, then $j \in I$, and
(ii) if $i, j \in I$ and $i \vee j$ exists, then $i \vee j \in I$.

In Section 2 we give a useful characterization of PJP-congruences. We also give a description of the smallest PJP-congruence containing a certain ideal as a class.

Let θ be a PJP-congruence on S. Then $\ker(\theta) = \{x \in S\colon\ x \equiv 0(\theta)\}$ is called the *kernel* of θ. A subset J of S is said to be a *congruence kernel* if $J = \ker(\theta)$ for some PJP-congruence θ on S. Observe that in the PJP-semilattice $\mathbf{M}$ given in Figure 1, the ideal $I = \{0, a, b\}$ is not a kernel of any PJP-congruence on M. If $0 \equiv a(\theta)$ for any PJP-congruence θ on M, then $1 \equiv a^* = b(\theta)$, that is, $0 \equiv 1(\theta)$. Thus I is not a PJP-congruence kernel. An ideal I of a PJP-semilattice $\mathbf{S}$ is called a *kernel* *ideal* if $I = \ker(\theta)$ for some PJP-congruence θ on S. The set of all kernel ideals will be denoted by $\mathrm{KI}(S)$. Congruence kernels have been studied by Cornish [6] for pseudocomplemented distributive lattices and by Blyth [2] for pseudocomplemented semilattices. In this paper we characterize congruence kernels of distributive PJP-semilattices. In Section 3 we give characterizations of kernel ideals of distributive PJP-semilattices.

Let $\mathbf{S}$ be a PJP-semilattice. Let θ be a PJP-congruence on S. Then

$$\mathrm{Coker}(\theta) = \{x \in S\colon\ x \equiv 1(\theta)\}$$

is called the *cokernel* of θ. A subset J of S is said to be a *congruence cokernel* if $J = \mathrm{Coker}(\theta)$ for some PJP-congruence θ on S. A filter F of S is called a $*$-*filter* if

$$f^{**} \in F \Rightarrow f \in F.$$

In Section 4, we study cokernel filters. Here we characterize $*$-filters as a cokernel filters.

2. PJP-congruences

For the basic properties of pseudocomplementation we refer the reader to [9]. First we have the following useful characterization of PJP-congruences.

Theorem 2.1. *Let* $\mathbf{S}$ *be a* PJP-*semilattice. Then a* JP-*congruence* θ *on* S *is a* PJP-*congruence if and only if*

$$x \equiv 0(\theta) \Rightarrow x^* \equiv 1(\theta).$$

P r o o f. If θ is a PJP-congruence, then clearly the condition holds. Conversely, let θ be a JP-congruence such that the condition holds. Let $x \equiv y(\theta)$. Then $x^* \wedge y \equiv x^* \wedge x = 0(\theta)$ and so $(x^* \wedge y)^* \equiv 1(\theta)$. This implies

$$\begin{aligned} x^* &= x^* \wedge 1 \\ &\equiv x^* \wedge (x^* \wedge y)^*(\theta) \\ &= x^* \wedge y^* \quad \text{(by Lemma 1.1 (e))}. \end{aligned}$$

Similarly, we have $y^* \equiv x^* \wedge y^*(\theta)$. Hence $x^* \equiv y^*(\theta)$ and therefore θ is a PJP-congruence. $\square$

The following theorem gives us a description of the smallest PJP-congruence containing a certain ideal as a class.

Theorem 2.2. *Let* **S** *be a distributive* PJP*-semilattice and let I be an ideal of S such that $i, j \in I$ implies $(i^* \wedge j^*)^* \in I$. Define a binary relation $\Theta(I)$ on S by*

$$x \equiv y(\Theta(I)) \quad \text{if and only if} \quad x \wedge i^* = y \wedge i^* \text{ for some } i \in I.$$

Then $\Theta(I)$ is the smallest PJP-congruence containing I as a class.

Proof. Clearly, $\Theta(I)$ is both reflexive and symmetric. To prove that it is transitive, let $x \equiv y(\Theta(I))$ and $y \equiv z(\Theta(I))$. Then $x \wedge i^* = y \wedge i^*$ and $y \wedge j^* = z \wedge j^*$ for some $i, j \in I$. Then by the assumption $k = (i^* \wedge j^*)^* \in I$. We have

$$\begin{aligned} x \wedge k^* &= x \wedge (i^* \wedge j^*)^{**} = x \wedge (i^* \wedge j^*) \quad \text{(by Lemma 1.1 (b))} \\ &= (x \wedge i^*) \wedge j^* = (y \wedge i^*) \wedge j^* = (y \wedge j^*) \wedge i^* \\ &= (z \wedge j^*) \wedge i^* = z \wedge (i^* \wedge j^*) = z \wedge (i^* \wedge j^*)^{**} \\ &= z \wedge k^*. \end{aligned}$$

Hence $x \equiv z(\Theta(I))$. Thus $\Theta(I)$ is transitive.

Let $x \equiv y(\Theta(I))$ and $s \equiv t(\Theta(I))$. Then there are $i, j \in I$ with $k = (i^* \wedge j^*)^* \in I$ such that $x \wedge i^* = y \wedge i^*$ and $s \wedge j^* = t \wedge j^*$. Hence

$$\begin{aligned} (x \wedge s) \wedge k^* &= (x \wedge s) \wedge (i^* \wedge j^*)^{**} \\ &= (x \wedge s) \wedge (i^* \wedge j^*) \quad \text{(by Lemma 1.1 (b))} \\ &= (x \wedge i^*) \wedge (s \wedge j^*) = (y \wedge i^*) \wedge (t \wedge j^*) \\ &= (y \wedge t) \wedge (i^* \wedge j^*)^{**} \\ &= (y \wedge t) \wedge k^*. \end{aligned}$$

Also, if $x \vee s$ and $y \vee t$ exist, then

$$\begin{aligned} (x \vee s) \wedge k^* &= (x \wedge k^*) \vee (s \wedge k^*) \text{ as } S \text{ is a distributive JP-semilattice} \\ &= (x \wedge i^* \wedge j^*) \vee (s \wedge i^* \wedge j^*) \quad \text{(by Lemma 1.1 (b))} \\ &= (y \wedge i^* \wedge j^*) \vee (t \wedge i^* \wedge j^*) \\ &= (y \wedge k^*) \vee (t \wedge k^*) \\ &= (y \vee t) \wedge k^* \text{ as } S \text{ is a distributive JP-semilattice} \\ &= (y \vee t) \wedge k^*. \end{aligned}$$

Hence $\Theta(I)$ is a JP-congruence. To prove that $\Theta(I)$ is a PJP-congruence, let $x \equiv 0(\Theta(I))$. Then $x \wedge i^* = 0 \wedge i^* = 0$. This implies $i^* \leqslant x^*$. Hence $x^* \wedge i^* = i^* = 1 \wedge i^*$. This implies $x^* \equiv 1(\Theta(I))$. Hence by Theorem 2.1, $\Theta(I)$ is a PJP-congruence.

Finally, let θ be a PJP-congruence containing I as a class and let $x \equiv y(\Theta(I))$. Then $x \wedge i^* = y \wedge i^*$ for some $i \in I$. Since θ is a PJP-congruence containing I as a class, we have $i \equiv 0(\theta)$. This implies $i^* \equiv 1(\theta)$. Hence

$$x = x \wedge 1 \equiv x \wedge i^*(\theta) = y \wedge i^* \equiv y \wedge 1(\theta) = y.$$

Therefore $\Theta(I)$ is the smallest congruence containing I as a class. $\square$

3. Kernel ideals

Not every ideal of a JP-distributive PJP-semilattice is a kernel ideal. For a counterexample, consider the distributive PJP-semilattice $\mathbf{M}$ given in Figure 1. Let $I = \{0, a, b\}$. Then I is an ideal of M but not a kernel ideal, since $0 \equiv a(\theta)$ for some PJP-congruence θ on M implies $1 \equiv b(\theta)$.

We have the following characterization of kernel ideals.

Theorem 3.1. *An ideal I of a distributive* PJP*-semilattice* $\mathbf{S}$ *is a kernel ideal of* $\mathbf{S}$ *if and only if*

$$i, j \in I \Rightarrow (i^* \wedge j^*)^* \in I.$$

Proof. Let I be a kernel ideal of $\mathbf{S}$. Then $I = \ker\theta$ for some PJP-congruence θ. If $i, j \in I$, then $i \equiv 0(\theta)$ and $j \equiv 0(\theta)$. This implies immediately that $i^* \equiv 1(\theta)$ and $j^* \equiv 1(\theta)$. Hence $i^* \wedge j^* \equiv 1(\theta)$. This implies $(i^* \wedge j^*)^* \equiv 0(\theta)$. Thus $(i^* \wedge j^*)^* \in I$.

Conversely, let I be an ideal of S and suppose the condition holds. Then by Theorem 2.2, the binary relation $\Theta(I)$ on S defined by

$$x \equiv y(\Theta(I)) \quad \text{if and only if} \quad x \wedge i^* = y \wedge i^* \text{ for some } i \in I$$

is a PJP-congruence containing the ideal I as a class. So it is enough to show that I is a kernel ideal of $\Theta(I)$. For all $i \in I$, by taking $i = j$ in the condition we have $i^{**} \in I$. Hence

$$\begin{aligned} x \equiv 0(\Theta(I)) &\Leftrightarrow x \wedge i^* = 0 \text{ for some } i \in I \\ &\Leftrightarrow x \leqslant i^{**} \text{ for some } i \in I \\ &\Leftrightarrow x \in I. \end{aligned}$$

Thus I is a kernel ideal. $\square$

Theorem 3.2. *Let* **S** *be a distributive* PJP*-semilattice. An ideal I of S is a kernel ideal if and only if*

(i) $i \in I$ *implies* $i^{**} \in I$;

(ii) *for every* $i, j \in I$ *there is* $k \in I$ *such that* $i^* \wedge j^* = k^*$.

Proof. Let I be a kernel ideal. Then by taking $i = j$ in Theorem 3.1 we have $i \in I \Rightarrow i^{**} \in I$. Thus (i) holds. Let $i, j \in I$. Put $k = (i^* \wedge j^*)^*$, then by Theorem 3.1, $k \in I$. Also $k^* = i^* \wedge j^*$. Thus (ii) holds.

Conversely, let I be an ideal and $i, j \in I$. Then by (ii), there is $k \in I$ such that $k^* = i^* \wedge j^*$. Thus by (i), $k^{**} = (i^* \wedge j^*)^* \in I$. Hence by Theorem 3.1, I is a kernel ideal. $\square$

Theorem 3.3. *Let* **S** *be a distributive* PJP*-semilattice. A principal ideal $I = (x]$ of S is a kernel ideal if and only if $x \in \mathrm{Sk}(S)$.*

Proof. Suppose $I = (x]$ is a kernel ideal, then $x^{**} \in I$. This implies $x^{**} \leqslant x$. But $x \leqslant x^{**}$. Hence $x = x^{**} \in \mathrm{Sk}(S)$.

Conversely, let $I = (x]$ be a principal ideal and $x \in \mathrm{Sk}(S)$. Then by Lemma 1.1 (f), we have $x = x^{**}$. Let $i, j \in I$. Then $i, j \leqslant x$. This implies $x^* \leqslant i^* \wedge j^*$. Thus $(i^* \wedge j^*)^* \leqslant x^{**} = x$. This implies $(i^* \wedge j^*)^* \in I$. Hence by Theorem 3.1, I is a kernel ideal. $\square$

It is well known that the binary relation $\psi(I)$ on a semilattice **S** defined by

$$x \equiv y(\psi(I)) \text{ if and only if } x \wedge a \in I \Leftrightarrow y \wedge a \in I \text{ for any } a \in S$$

is the largest semilattice congruence containing the ideal I as a class.

Now we have the following result for distributive JP-semilattices.

Theorem 3.4. *Let* **S** *be a distributive* JP*-semilattice and let I be an ideal of S. Then $\psi(I)$ is the largest* JP*-congruence containing I as a class.*

Proof. It is enough to show that $\psi(I)$ has the substitution property for the partial operation $\vee$. Let $x \equiv y(\psi(I))$, $s \equiv t(\psi(I))$ and let $x \vee s$, $y \vee t$ exist. Since **S** is a distributive JP-semilattice, for any $a \in S$ we have that $(x \wedge a) \vee (s \wedge a)$, $(y \wedge a) \vee (t \wedge a)$ exist and $(x \vee s) \wedge a = (x \wedge a) \vee (s \wedge a)$, $(y \vee t) \wedge a = (y \wedge a) \vee (t \wedge a)$. Thus

$$\begin{aligned}(x \vee s) \wedge a \in I &\Leftrightarrow (x \wedge a) \vee (s \wedge a) \in I \\ &\Leftrightarrow x \wedge a \in I \text{ and } s \wedge a \in I \\ &\Leftrightarrow y \wedge a \in I \text{ and } t \wedge a \in I \\ &\Leftrightarrow (y \wedge a) \vee (t \wedge a) \in I \Leftrightarrow (y \vee t) \wedge a \in I.\end{aligned}$$

Thus $x \vee s \equiv y \vee t(\psi(I))$. Hence $\psi(I)$ is the largest JP-congruence. $\square$

The following result is a description of the largest PJP-congruence containing a kernel ideal as a class.

Theorem 3.5. *Let* $\mathbf{S}$ *be a distributive* PJP*-semilattice. If* I *is a kernel ideal of* S*, then* $\psi(I)$ *is the largest* PJP*-congruence containing* I *as a class.*

Proof. By Theorem 3.4, $\psi(I)$ is a largest JP-congruence. Let $x \equiv 0(\psi(I))$. Then $x \in I$. Now for any $a \in S$,

$$\begin{aligned} x^* \wedge a \in I &\Rightarrow (x^* \wedge (x^* \wedge a)^*)^* \in I, \text{ by Theorem 3.1} \\ &\Rightarrow (x^* \wedge a^*)^* \in I, \text{ by Lemma 1.1 (e)} \\ &\Rightarrow a \in I, \text{ since } a \leqslant a^{**} \leqslant (x^* \wedge a^*)^* \\ &\Rightarrow 1 \wedge a \in I. \end{aligned}$$

Also

$$1 \wedge a = a \in I \Rightarrow x^* \wedge a \in I.$$

Thus $x^* \equiv 1(\psi(I))$. Hence by Theorem 2.1, $\psi(I)$ is a PJP-congruence. □

$*$-ideal. An ideal I of a JP-semilattice is called a $*$-*ideal* if it satisfies condition (i) of Theorem 3.2, that is,

$$i \in I \text{ implies } i^{**} \in I.$$

Clearly, every kernel ideal of a distributive PJP-semilattice is a $*$-ideal. Consider the distributive PJP-semilattice $\mathbf{M}$ given in Figure 1. Here the ideal $I = \{0, a, b\}$ is a $*$-ideal but not a kernel ideal.

Theorem 3.6. *Let* $\mathbf{S}$ *be a distributive* PJP*-semilattice. Every principal* $*$*-ideal* I *of* $\mathbf{S}$ *can be written as* $(a^{**}]$ *for some* $a \in I$*. Moreover, for any* $a \in S$ *the principal ideal* $I = (a^{**}]$ *is a kernel ideal.*

Proof. Let I be a principal $*$-ideal of $\mathbf{S}$. Then $I = (a]$ for some $a \in S$. Since I is a $*$-ideal, for $a \in I$ we have $a^{**} \in I$. Thus $a^{**} \leqslant a$. But $a \leqslant a^{**}$. Hence $I = (a^{**}]$ for some $a \in S$.

Moreover, for any $a \in S$, since $a^{**} \in \mathrm{Sk}(S)$, so by Theorem 3.3, $I = (a^{**}]$ is a kernel ideal. □

Theorem 3.7. *A* $*$*-ideal* I *of a distributive* PJP*-semilattice is a kernel ideal if and only if* $i^{**} \veebar j^{**} \in I$ *for all* $i, j \in I$*.*

P r o o f. For any $i, j \in I$ we have

$$\begin{aligned}(i^* \wedge j^*)^* &= (i^{***} \wedge j^{***})^* \text{ by Lemma 1.1 (c)}\\ &= i^{**} \veebar j^{**} \text{by Lemma 1.1 (h).}\end{aligned}$$

By Theorem 3.1, I is a kernel ideal if and only if $i, j \in I$ implies $i^{**} \veebar j^{**} \in I$. $\square$

Glivenko congruence. Let $\mathbf{S}$ be a distributive PJP-semilattice. The binary relation G on S defined by

$$x \equiv y(G) \Leftrightarrow x^{**} = y^{**}$$

is a semilattice congruence called the *Glivenko congruence*. It is evident that G is compatible with $*$. We shall show that G is a PJP-congruence.

Let I be an ideal. Define

$$I^0 = \{x \in S \colon x \wedge i = 0 \text{ for all } i \in I\}.$$

Theorem 3.8. *I^0 is a kernel ideal.*

P r o o f. Let $x, y \in I^0$. Then $x \wedge i = y \wedge i = 0$ for all $i \in I$. Hence $i \leqslant x^*, y^*$ and consequently, $(x^* \wedge y^*)^* \leqslant i^*$. This implies $(x^* \wedge y^*)^* \wedge i \leqslant i^* \wedge i = 0$. Hence $(x^* \wedge y^*)^* \in I^0$. Thus by Theorem 3.1, I^0 is a kernel ideal. $\square$

Lemma 3.9. *If $x \equiv y(\psi(I))$, then $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$.*

P r o o f. Let $x \equiv y(\psi(I))$. Then $x \wedge x^* = 0 \equiv y \wedge x^*(\psi(I))$. Therefore $y \wedge x^* \in I$. Similarly, $x \wedge y^* \in I$. Hence $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$ as I is a kernel ideal. $\square$

Theorem 3.10. *Let I be a kernel ideal of a distributive* PJP*-semilattice* $\mathbf{S}$. *Then $\psi(I) \wedge \psi(I^0) = G$.*

P r o o f. Let $x \equiv y(\psi(I) \wedge \psi(I^0))$. Then by Lemma 3.9, we have $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$ and $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I^0$ whence $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* = 0$. This implies

$$x \wedge y^* \leqslant (x \wedge y^*)^{**} \leqslant [(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* = 0.$$

Thus $x \wedge y^* = 0$. Hence $y^* \leqslant x^*$. Similarly, $x^* \leqslant y^*$. This implies $x^* = y^*$ and consequently, $x^{**} = y^{**}$. Hence $x \equiv y(G)$.

Conversely, let $x \equiv y(G)$. Since $a \equiv a^{**}(G)$ for any $a \in S$, we have $x \wedge a \equiv x \wedge a^{**}(G)$, $y \wedge a \equiv y \wedge a^{**}(G)$ and $x \wedge a \equiv y \wedge a^{**}(G)$. Hence $(x \wedge a)^{**} = (x \wedge a^{**})^{**}$, $(y \wedge a)^{**} = (y \wedge a^{**})^{**}$ and $(x \wedge a)^{**} = (y \wedge a^{**})^{**}$. Now for any $a \in S$,

$$\begin{aligned} x \wedge a \in I &\Leftrightarrow (x \wedge a)^{**} \in I \text{ as } I \text{ is a kernel ideal of } S \\ &\Leftrightarrow (y \wedge a^{**})^{**} \in I \\ &\Leftrightarrow (y \wedge a)^{**} \in I \\ &\Leftrightarrow y \wedge a \in I. \end{aligned}$$

Also, for all $i \in I$,

$$\begin{aligned} x \wedge a \in I^0 &\Leftrightarrow (x \wedge a) \wedge i = 0 \\ &\Leftrightarrow x \wedge (a \wedge i) = 0 \\ &\Leftrightarrow x \leqslant (a \wedge i)^* \\ &\Leftrightarrow x^{**} \leqslant (a \wedge i)^* \\ &\Leftrightarrow y^{**} \leqslant (a \wedge i)^* \\ &\Leftrightarrow y \leqslant (a \wedge i)^* \\ &\Leftrightarrow y \wedge (a \wedge i) = 0 \\ &\Leftrightarrow y \wedge a \in I^0. \end{aligned}$$

Hence $x \equiv y(\psi(I) \wedge \psi(I^0))$. Therefore $G = \psi(I) \wedge \psi(I^0)$. □

Corollary 3.11. *G is a* PJP*-congruence.*

Proof. This is immediate from the fact that $\psi(I) \wedge \psi(I^0)$ is a PJP-congruence. □

4. Congruence cokernels

Let **S** be a JP-semilattice. A non-empty subset F of S is called a *filter* of S if

(i) $a \in F$ and $b \in S$ with $a \leqslant b$ implies $b \in F$, and
(ii) $a, b \in F$ implies $a \wedge b \in F$.

Now we have the following lemma.

Lemma 4.1. *Let* **S** *be a* JP*-semilattice. Then every cokernel of S is a filter.*

Proof. Let $F = \mathrm{Coker}(\theta)$ for some PJP-congruence θ. If $x, y \in F$, then $x \equiv 1(\theta)$ and $y \equiv 1(\theta)$. Hence $x \wedge y \equiv 1(\theta)$. Thus $x \wedge y \in F$. Now let $x \in F$ and $x \leqslant y$. Then $x = x \wedge y \equiv 1 \wedge y(\theta) = y$. Thus $y \equiv 1(\theta)$. Hence $y \in F$. Therefore F is a filter. □

Let $\mathbf{S}$ be a JP-semilattice and let F be a filter of $\mathbf{S}$. Define a binary relation $\Theta(F)$ on S by

$$x \equiv y(\Theta(F)) \quad \text{if and only if} \quad x \wedge f = y \wedge f \text{ for some } f \in F.$$

Theorem 4.2. *Let F be a filter of a distributive JP-semilattice $\mathbf{S}$. Then the relation $\Theta(F)$ on S is a JP-congruence containing F as a class. Moreover, if $\mathbf{S}$ has a largest element 1, then $\Theta(F)$ is the smallest JP-congruence containing F as a class.*

Proof. Clearly $\Theta(F)$ is an equivalence relation. Let $x \equiv y(\Theta(F))$ and $s \equiv t(\Theta(F))$. Then $x \wedge f_1 = y \wedge f_1$ and $s \wedge f_2 = t \wedge f_2$ for some $f_1, f_2 \in F$. This implies

$$(x \wedge s) \wedge (f_1 \wedge f_2) = (x \wedge f_1) \wedge (s \wedge f_2) = (y \wedge f_1) \wedge (t \wedge f_2) = (y \wedge t) \wedge (f_1 \wedge f_2).$$

Since $f_1 \wedge f_2 \in F$, we have $x \wedge s \equiv y \wedge t(\Theta(F))$.

Also, if $x \vee s$ and $y \vee t$ exist, then

$$\begin{aligned}(x \vee s) \wedge (f_1 \wedge f_2) &= (x \wedge f_1 \wedge f_2) \vee (s \wedge f_1 \wedge f_2) \\ &= (y \wedge f_1 \wedge f_2) \vee (t \wedge f_1 \wedge f_2) = (y \vee t) \wedge (f_1 \wedge f_2).\end{aligned}$$

Thus $\Theta(F)$ is a JP-congruence. Clearly, $\Theta(F)$ contains F as a class.

Moreover, assume that S has a largest element 1. Let θ be any congruence on S containing F as a class. Assume $x \equiv y(\Theta(F))$. Then $x \wedge f = y \wedge f$ for some $f \in F$. This implies $x = x \wedge 1 \equiv x \wedge f(\theta)$. Similarly, $y \equiv y \wedge f(\theta)$. Hence $x \equiv y(\theta)$. Thus $\Theta(F)$ is the smallest JP-congruence containing F as a class. □

The following result is the description of the smallest PJP-congruence containing a filter as a class.

Theorem 4.3. *Let $\mathbf{S}$ be a PJP-semilattice and let F be a filter of S. Then $\Theta(F)$ is the smallest PJP-congruence containing F as a class.*

Proof. By Theorem 4.2, $\Theta(F)$ is a JP-congruence containing F as a class. Let $x \equiv 0(\Theta(F))$. Then $x \wedge f = 0$ for some $f \in F$. This implies $f \leqslant x^*$. Thus $x^* \in F$. Hence $x^* \equiv 1(\Theta(F)$. Hence by Theorem 2.1, we have that $\Theta(F)$ is a PJP-congruence. □

Corollary 4.4. *Every filter of a* PJP*-semilattice is a cokernel.*

P r o o f. It is clear from the fact that for any filter F of $\mathbf{S}$ we have

$$x \in F \Leftrightarrow x \equiv 1(\Theta(F)).$$

$\square$

$*$-filters. First we prove the following useful result.

Lemma 4.5. *Let* $\mathbf{S}$ *be a distributive* PJP*-semilattice. If* $a \vee b$ *exists, then*

$$(a \vee b)^* = a^* \wedge b^*.$$

P r o o f. We have $(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*) = 0 \vee 0 = 0$. Let $(a \vee b) \wedge x = 0$. Then $(a \wedge x) \vee (b \wedge x) = 0$. Hence $a \wedge x = 0$ and $b \wedge x = 0$. This implies $x \leqslant a^*, b^*$. Hence $x \leqslant a^* \wedge b^*$. Therefore $(a \vee b)^* = a^* \wedge b^*$. $\square$

For every filter F of S define

$$F_* = \{x \in S \colon\ x^* \in F\}.$$

Lemma 4.6. *Let* $\mathbf{S}$ *be a distributive* PJP*-semilattice and* F *a filter of* $\mathbf{S}$*. Then* F_* *is a kernel ideal of* $\mathbf{S}$*.*

P r o o f. Let $x, y \in F_*$. Then $x^*, y^* \in F$. If $x \vee y$ exists, then by Lemma 4.5 we have $(x \vee y)^* = x^* \wedge y^* \in F$ as F is a filter. Hence $x \vee y \in F_*$. Let $x \in F_*$ and $y \leqslant x$. Then $y^* \geqslant x^* \in F$. This implies $y^* \in F$. Thus $y \in F_*$. Hence F_* is an ideal.

To prove that F_* is a kernel ideal, let $x, y \in F_*$. Then $x^*, y^* \in F$ so that $(x^* \wedge y^*)^{**} = x^* \wedge y^* \in F$ and consequently $(x^* \wedge y^*)^* \in F_*$. Hence by Theorem 3.1, F_* is a kernel ideal. $\square$

For every $I \in KI(S)$ define

$$I_* = \{x \in S \colon\ x^* \in I\}.$$

Lemma 4.7. *Let* $\mathbf{S}$ *be a distributive* PJP*-semilattice and* I *a kernel ideal of* $\mathbf{S}$*. Then* I_* *is a* $*$*-filter of* $\mathbf{S}$*.*

P r o o f. Let $x, y \in I_*$. Then $x^*, y^* \in I$. So by Theorem 3.1, we have $(x \wedge y)^* = (x \wedge y)^{***} = (x^{**} \wedge y^{**})^* \in I$. Hence $x \wedge y \in I_*$. Now let $x \in I_*$ and $y \geqslant x$. Then $y^* \leqslant x^* \in I$ so that $y^* \in I$ and consequently, $y \in I_*$. Hence I_* is a filter. Let $x^{**} \in I_*$. Then $x^* = x^{***} \in I$ and hence $x \in I_*$. Therefore I_* is a $*$-filter. $\square$

The following theorem is a characterization of $*$-filters.

Theorem 4.8. *A filter F of a* JP*-distributive* PJP*-semilattice is a $*$-filter if and only if $(F_*)_* = F$.*

P r o o f. Let $(F_*)_* = F$ and let $x^{**} \in F$. Since F is a filter, F_* is a kernel ideal. Hence $x^* \in F_*$ and so $x \in (F_*)_* = F$. Thus F is a $*$-filter.

Conversely, let F be a $*$-filter. Then

$$\begin{aligned} x \in (F_*)_* &\Leftrightarrow x^* \in F_* \\ &\Leftrightarrow x^{**} \in F \\ &\Leftrightarrow x \in F \quad (\Rightarrow \text{ as } F \text{ is a } *\text{-filter and} \Leftarrow \text{ as } F \text{ is a filter}). \end{aligned}$$

□

D-filter. A filter F of a PJP-semilattice **S** is called a D-*filter* if it contains the dense filter $D = \{x \in S\colon\ x^* = 0\}$.

Theorem 4.9. *Every $*$-filter is a* D*-filter but the converse is not true.*

P r o o f. Let F be a $*$-filter and let $d \in D$. Then $d^{**} = 1 \in F$ which implies that $d \in F$. Hence F contains D. Thus F is a D-filter.

To prove the converse is not true, consider the distributive PJP-semilattice **N** given in Figure 3. The filter $[c)$ is a D-filter but not a $*$-filter.

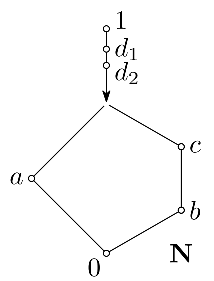

Fig. 3

□

Let **S** be a PJP-semilattice. A PJP-congruence θ on S is called a *boolean congruence* if the factor PJP-semilattice $\mathbf{S}/\theta$ is a Boolean lattice.

Theorem 4.10. *A* PJP*-congruence θ is a boolean congruence if and only if $x \equiv x^{**}(\theta)$ for all $x \in X$.*

P r o o f. This is immediate from the fact that $([x](\theta))^* = [x^*](\theta)$. □

Theorem 4.11. *Let* **S** *be a distributive* PJP*-semilattice. Then the following conditions are equivalent:*

(i) *every* D*-filter is a* $*$*-filter;*

(ii) $\Theta(D)$ *is a boolean congruence.*

P r o o f. (i) $\Rightarrow$ (ii). For each $x \in S$ we have that $F = [x^{**}) \vee D$ is a D-filter and hence F is a $*$-filter. Since $x^{**} \in F$, we have $x \in F$. Thus $x = x^{**} \wedge d$ for some $d \in D$. This implies $x \wedge d = x^{**} \wedge d$. Hence $x \equiv x^{**}\Theta(D)$. Therefore, by Theorem 4.10, $\Theta(D)$ is a boolean congruence.

(ii) $\Rightarrow$ (i). Let F be a D-filter. By (ii), $\Theta(D)$ is a boolean congruence. Hence by Theorem 4.10 $x \equiv x^{**}(\Theta(D))$. Thus $x \wedge d = x^{**} \wedge d$ for some $d \in D$. If $x^{**} \in F$, then $x^{**} \wedge d \in F$ as $D \subseteq F$. Hence $x \wedge d \in F$ and consequently, $x \in F$. Thus F is a $*$-filter. $\square$

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