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On Weakly W_3-Symmetric Manifolds

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Abstract

The object of the present paper is to study *weakly W_3-symmetric manifolds* and its decomposability with the existence of such notions. Among others it is shown that in a decomposable weakly W_3-symmetric manifold both the decompositions are weakly Ricci symmetric.

Key words: weakly W_3-symmetric manifold, W_3-curvature tensor, decomposable manifold, scalar curvature, totally umbilical hypersurfaces, totally geodesic, mean curvature

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1 Introduction

The study of Riemann symmetric manifolds began with the work of Cartan [3]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan [3] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [36], semisymmetric manifold by Sinyukov [32] and Szabó [33], pseudosymmetric manifold in the sense of Mikeš [12, 13] and Deszcz [8], pseudosymmetric manifold in the sense of Chaki [4], generalized pseudosymmetric manifold by Chaki [5], weakly symmetric manifold by Selberg [21] and weakly symmetric manifold by Támassy and Binh [34]. It may be noted that the notion of weakly symmetric Riemannian manifolds by Selberg [21] is different and are not equivalent to that of Támassy and Binh [34]. In this connection it is mentioned that Mikeš [11] studied projective-symmetric and projective-recurrent affinely connected spaces. Also in [14] Mikeš and Tolobaev studied symmetric and projectively symmetric affinely connected spaces and it is shown that [14] there exist projectively m-symmetric spaces, the differ from k-symmetric spaces and projectively k-symmetric spaces $(k < m)$.

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamássy and Binh [34]. A non-flat Riemannian manifold (M^n, g), $n > 2$, is called a weakly symmetric manifold if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+ F(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X) \qquad (1)\end{aligned}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, $\chi(M^n)$ being the Lie algebra of smooth vector fields on M, where A, B, F, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. The existence of a $(WS)_n$ is proved by Prvanović [18]. Then De and Bandyopadhyay [7] gave an example of a $(WS)_n$ by a metric of Roter [19] and proved that in a $(WS)_n$, $B = F$ and $D = E$ [7]. Hence the defining condition of a $(WS)_n$ reduces to the following form:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+ B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + D(V)R(Y,Z,U,X). \qquad (2)\end{aligned}$$

The example of a $(WS)_n$ given in [7] was of vanishing scalar curvature. However, there are various proper examples of a $(WS)_n$ given by Shaikh and Jana [27], which are of non-vanishing scalar curvatures. $(WS)_n$ is also studied by Altay [1], Binh [2], Hui, Matsuyama and Shaikh [10], Özen and Altay ([15, 16]), Shaikh et. al. ([22, 23, 24, 25, 26, 27, 28, 29, 31]).

Also in 1993 Tamássy and Binh [35] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold $(M^n, g), n > 2$, is called weakly Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(Y,X), \qquad (3)$$

where A, B and D are three non-zero 1-forms and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

In 1973, Pokhariyal [17] introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance. The W_3-curvature tensor of type (0,4) is defined by

$$\begin{aligned}W_3(Y,Z,U,V) &= R(Y,Z,U,V)\\ &+ \frac{1}{n-1}\big[S(Y,V)g(Z,U) - S(Y,U)g(Z,V)\big], \qquad (4)\end{aligned}$$

where R is the curvature tensor of type (0,4) and S is the Ricci tensor of type (0,2). The present paper deals with a Riemannian manifold (M^n, g), $n > 2$, (the condition $n > 2$ is assumed throughout the paper) whose W_3-curvature tensor is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_X W_3)(Y, Z, U, V) &= A(X)W_3(Y, Z, U, V) + B(Y)W_3(X, Z, U, V)\\ &+ F(Z)W_3(Y, X, U, V) + D(U)W_3(Y, Z, X, V) + E(V)W_3(Y, Z, U, X) \quad (5)\end{aligned}$$

for all vector fields X, Y, Z, U, $V \in \chi(M^n)$, where A, B, F, D and E are 1-forms (not simultaneously zero). Such a manifold will be called a *weakly W_3-symmetric manifold* and is denoted by $(WW_3S)_n$, where the first 'W' stands for the word *weakly* and 'W_3' represents the '*W_3-curvature tensor*'. Here A, B, F, D, E are said to be the associated 1-forms of the manifold.

The paper is organized as follows. Section 2 is concerned with preliminaries. It is shown that in a $(WW_3S)_n$, the associated 1-forms $B \neq F$ and $D = E$. Hence the defining condition (5) of a $(WW_3S)_n$ turns into the following form:

$$\begin{aligned}(\nabla_X W_3)(Y, Z, U, V) &= A(X)W_3(Y, Z, U, V) + B(Y)W_3(X, Z, U, V)\\ &+ F(Z)W_3(Y, X, U, V) + D(U)W_3(Y, Z, X, V) + D(V)W_3(Y, Z, U, X), \quad (6)\end{aligned}$$

where A, B, F and D are 1-forms (not simultaneously zero).

Section 3 is devoted to the study of Einstein $(WW_3S)_n$. Every $(WS)_n$ is a $(WW_3S)_n$. However, the converse is not true. In this section it is proved that an Einstein $(WW_3S)_n$ with vanishing scalar curvature is a $(WS)_n$. Also it is shown that an Einstein $(WW_3S)_n$ is a $(WS)_n$ if and only if the scalar curvature of the manifold vanishes. Section 4 deals with the decomposable $(WW_3S)_n$ and a full classification of such a manifold is given. It is proved that in a decomposable $(WW_3S)_n$, one of the decomposition is Ricci symmetric as well as locally symmetric but the other decomposition is a manifold of constant curvature. Shaikh and Jana [27] already proved that every $(WS)_n$ is not a $(WRS)_n$, in general. In this paper it is shown that if a Riemannian manifold (M^n, g) is a decomposable $(WW_3S)_n$ such that $M = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, then M_1 is $(WRS)_p$ and M_2 is $(WRS)_{n-p}$.

Recently Özen and Altay [15] studied the totally umbilical hypersurfaces of weakly and pseudosymmetric spaces. Again Özen and Altay [16] also studied the totally umbilical hypersurfaces of weakly concircular and pseudo concircular symmetric spaces. In this connection it may be mentioned that Shaikh, Roy and Hui [30] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces. Section 5 deals with the study of totally umbilical hypersurfaces of weakly W_3-symmetric manifolds. Finally, in the last section, the existence of $(WW_3S)_n$ and decomposable $(WW_3S)_n$ is ensured by an interesting example.

2 Preliminaries

In this section, some formulas are derived, which will be useful to the study of decomposable $(WW_3S)_n$. Let $\{e_i \colon i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type (0,2) and the scalar curvature r of the manifold are given by the following:

$$S(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i)$$

and

$$r = \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} g(Qe_i, e_i),$$

where Q is the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y. Now from (4), we have the following:

$$\sum_{i=1}^{n} W_3(e_i, Z, U, e_i) = \frac{1}{n-1}\big[(n-2)S(Z,U) + rg(Z,U)\big], \tag{7}$$

$$\sum_{i=1}^{n} W_3(Y, e_i, e_i, V) = 2\ S(Y,V), \tag{8}$$

$$\sum_{i=1}^{n} W_3(Y, Z, e_i, e_i) = 0 = \sum_{i=1}^{n} W_3(e_i, e_i, U, V). \tag{9}$$

Also from (4) it follows that

$$\left.\begin{array}{rl} (i) & W_3(Y,Z,U,V) \neq -W_3(Z,Y,U,V), \\ (ii) & W_3(Y,Z,U,V) = -W_3(Y,Z,V,U), \\ (iii) & W_3(Y,Z,U,V) \neq W_3(U,V,Y,Z), \\ (iv) & W_3(Y,Z,U,V) + W_3(Z,U,Y,V) + W_3(U,Y,Z,V) \neq 0. \end{array}\right\} \tag{10}$$

In view of (4) we obtain by virtue of Bianchi identity that

$$\begin{aligned} &(\nabla_X W_3)(Y,Z,U,V) + (\nabla_Y W_3)(Z,X,U,V) + (\nabla_Z W_3)(X,Y,U,V) \\ &= \frac{1}{n-1}\Big[(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Y,U)g(Z,V) + (\nabla_Y S)(Z,V)g(X,U) \\ &- (\nabla_Y S)(Z,U)g(X,V) + (\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(X,U)g(Y,V)\Big]. \end{aligned} \tag{11}$$

Proposition 2.1 *The defining condition of a* $(WW_3S)_n$ *can always be expressed in the form* (6)*.*

Proof Interchanging U and V in (5), we get

$$\begin{aligned} (\nabla_X W_3)(Y,Z,V,U) &= A(X)W_3(Y,Z,V,U) + B(Y)W_3(X,Z,V,U) \\ &+ F(Z)W_3(Y,X,V,U) + D(V)W_3(Y,Z,X,U) + E(U)W_3(Y,Z,V,X). \end{aligned} \tag{12}$$

Adding (5) and (12), we obtain by virtue of (10)(ii) that

$$\lambda(U)W_3(Y,Z,X,V) + \lambda(V)W_3(Y,Z,X,V) = 0, \tag{13}$$

where $\lambda(X) = D(X) - E(X)$ for all X.

If we choose a particular vector field ρ such that $\lambda(\rho) \neq 0$, then putting $U = V = \rho$ in (13), we get $W_3(Y, Z, X, \rho) = 0$. Again setting $V = \rho$ in (13), we obtain $W_3(Y, Z, X, U) = 0$ for all vector fields Y, Z, X and U, which contradicts to our assumption that the manifold is not W_3-flat. Hence we must have $\lambda(X) = 0$ for all X and consequently $D(X) = E(X)$ for all X. But in view of (10)(i), it follows that the relation $B = F$ does not hold in a $(WW_3S)_n$. Hence the defining condition of a $(WW_3S)_n$ can be written as (6). This proves the proposition. □

Proposition 2.2 *If in a $(WW_3S)_n$ the Ricci tensor vanishes then it is a $(WS)_n$.*

Proof Let us consider a $(WW_3S)_n$ such that the Ricci tensor vanishes, i.e., $S(X, Y) = 0$ for all X, Y. Then from (4), it follows that $W_3(Y, Z, U, V) = R(Y, Z, U, V)$. Consequently the relation (6) yields the relation (2). This proves the proposition. □

Proposition 2.3 *In a $(WW_3S)_n$ of non-zero constant scalar curvature $\frac{(2n-3)r}{3n-4}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector σ defined by $g(X, \sigma) = L(X) = B(X) + F(X) - A(X) \neq 0$ for all X.*

Proof By virtue of (6), (11) yields

$$\begin{aligned}
&A(X)W_3(Y, Z, U, V) + \{B(X) + F(X)\}W_3(Z, Y, U, V)\\
&\quad + A(Y)W_3(Z, X, U, V) + \{B(Y) + F(Y)\}W_3(X, Z, U, V)\\
&\quad + A(Z)W_3(X, Y, U, V) + \{B(Z) + F(Z)\}W_3(Y, X, U, V)\\
&\quad + D(U)\big[W_3(Y, Z, X, V) + W_3(Z, X, Y, V) + W_3(X, Y, Z, V)\big]\\
&\quad - D(V)\big[W_3(Y, Z, X, U) + W_3(Z, X, Y, U) + W_3(X, Y, Z, U)\big]\\
&= \frac{1}{n-1}\Big[(\nabla_X S)(Y, V)g(Z, U) - (\nabla_X S)(Y, U)g(Z, V) + (\nabla_Y S)(Z, V)g(X, U)\\
&- (\nabla_Y S)(Z, U)g(X, V) + (\nabla_Z S)(X, V)g(Y, U) - (\nabla_Z S)(X, U)g(Y, V)\Big]. \quad (14)
\end{aligned}$$

Setting $Y = V = e_i$ in (14) and taking summation over i, $1 \leq i \leq n$ and using (7)–(9), we get

$$\begin{aligned}
&\frac{1}{n-1}A(X)\{(n-2)S(Z, U) + rg(Z, U)\} - 2\{B(X) + F(X)\}S(Z, U)\\
&\qquad + W_3(Z, X, U, \rho_1) + W_3(X, Z, U, \rho_2) + W_3(X, Z, U, \rho_3)\\
&\quad - 2A(Z)S(X, U) + \frac{1}{n-1}\{B(Z) + F(Z)\}\{(n-2)S(X, U) + rg(X, U)\}\\
&\qquad + D(U)\left[\frac{1}{n-1}\{(n-2)S(Z, X) + rg(Z, X)\} - 2S(Z, X)\right]\\
&\qquad - \big[W_3(\rho_4, Z, X, U) + W_3(Z, X, \rho_4, U) + W_3(X, \rho_4, Z, U)\big]
\end{aligned}$$

$$= \frac{1}{n-1}\big[dr(X)g(Z,U) - 2(\nabla_X S)(Z,U) \\ + \frac{1}{2}dr(Z)g(X,U) - (n-1)(\nabla_Z S)(X,U)\big], \tag{15}$$

where $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$, $F(X) = g(X,\rho_3)$ and $D(X) = g(X,\rho_4)$ for all X. Again contracting (15) over Z and U, we obtain

$$(3n-4)L(QX) - (2n-3)r\ L(X) = \frac{1}{2}(n-2)dr(X), \tag{16}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ for all X, Y and $L(X) = g(X,\sigma) = B(X) + F(X) - A(X) \neq 0$ and r is the scalar curvature of the manifold.

If the scalar curvature r of the manifold is non-zero constant then

$$dr(X) = 0 \quad \text{for all } X. \tag{17}$$

By virtue of (17), it follows from (16) that

$$L(QX) = \frac{2n-3}{3n-4} r L(X),$$

which implies that

$$S(X,\sigma) = \frac{2n-3}{3n-4} r g(X,\sigma).$$

This proves the proposition. □

3 Einstein $(WW_3S)_n$

Let us consider a $(WW_3S)_n$, which is an Einstein manifold. Then we have

$$S(X,Y) = \frac{r}{n} g(X,Y) \tag{18}$$

from which it follows that

$$dr(X) = 0 \quad \text{and} \quad (\nabla_Z S)(X,Y) = 0 \text{ for all } X, Y, Z. \tag{19}$$

If in an Einstein $(WW_3S)_n$, $r = 0$ then from (18) it follows that $S(X,Y) = 0$ for all X, Y and hence by virtue of Proposition 2.2, we can state the following:

Theorem 3.1 *An Einstein $(WW_3S)_n$ with vanishing scalar curvature is a $(WS)_n$.*

By virtue of (18) and (19), we have from (4) that

$$W_3(Y,Z,U,V) = R(Y,Z,U,V) \\ + \frac{r}{n(n-1)}\big[g(Y,V)g(Z,U) - g(Y,U)g(Z,V)\big] \tag{20}$$

and

$$(\nabla_X W_3)(Y, Z, U, V) = (\nabla_X R)(Y, Z, U, V). \tag{21}$$

In view of (20) and (21), (6) yields

$$\begin{aligned}
(\nabla_X R)(Y, Z, U, V) &= A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\
&+ F(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + D(V)R(Y, Z, U, X) \\
&+ \frac{r}{n(n-1)}\big[A(X)\{g(Y, V)g(Z, U) - g(Y, U)g(Z, V)\} \\
&+ B(Y)\{g(X, V)g(Z, U) - g(X, U)g(Z, V)\} \\
&+ F(Z)\{g(Y, V)g(Z, X) - g(Y, U)g(X, V)\} \\
&+ D(U)\{g(Y, V)g(Z, X) - g(Y, X)g(Z, V)\} \\
&+ D(V)\{g(Y, X)g(Z, U) - g(Y, U)g(Z, X)\}\big].
\end{aligned} \tag{22}$$

Now if the Einstein $(WW_3S)_n$ is a $(WS)_n$, then using (2) in (22), we get

$$\begin{aligned}
&[B(Z) - F(Z)]R(Y, X, U, V) \\
&= \frac{r}{n(n-1)}\big[A(X)\{g(Y, V)g(Z, U) - g(Y, U)g(Z, V)\} \\
&+ B(Y)\{g(X, V)g(Z, U) - g(X, U)g(Z, V)\} \\
&+ F(Z)\{g(Y, V)g(Z, X) - g(Y, U)g(X, V)\} \\
&+ D(U)\{g(Y, V)g(Z, X) - g(Y, X)g(Z, V)\} \\
&+ D(V)\{g(Y, X)g(Z, U) - g(Y, U)g(Z, X)\}\big].
\end{aligned} \tag{23}$$

Setting $X = U = e_i$ in (23) and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}
&[B(Z) - F(Z)]S(Y, V) \\
&= \frac{r}{n(n-1)}\big[A(Z)g(Y, V) - A(Y)g(Z, V) - (n-1)B(Y)g(Z, V) \\
&+ (n-1)F(Z)g(Y, V) + D(Z)g(Y, V) - D(Y)g(Z, V)\big].
\end{aligned} \tag{24}$$

Using (18) in (24), we obtain

$$\begin{aligned}
&\frac{r}{n}[B(Z) - F(Z)]g(Y, V) \\
&= \frac{r}{n(n-1)}\big[A(Z)g(Y, V) - A(Y)g(Z, V) - (n-1)B(Y)g(Z, V) \\
&+ (n-1)F(Z)g(Y, V) + D(Z)g(Y, V) - D(Y)g(Z, V)\big].
\end{aligned} \tag{25}$$

Contracting (25) over Y and Z, we get

$$r\{B(Z) - F(Z)\} = 0. \tag{26}$$

Since in a $(WW_3S)_n$, $B \neq F$. Then from (26), we must obtain $r = 0$. Thus we can state the following:

Theorem 3.2 *If an Einstein $(WW_3S)_n$ is a $(WS)_n$ then the scalar curvature of the manifold vanishes.*

Combining Theorem 3.1 and Theorem 3.2, we can state the following:

Theorem 3.3 *An Einstein $(WW_3S)_n$ is a $(WS)_n$ if and only if the scalar curvature of the manifold vanishes.*

4 Decomposable $(WW_3S)_n$

A Riemannian manifold (M^n, g) is said to be decomposable or product manifold ([20], [37]) if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{27}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \ldots, x^p$ denoted by $\bar{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \ldots, x^n$ denoted by $\overset{*}{x}$; $a, b, c, \ldots$ run from 1 to p and $\alpha, \beta, \gamma, \ldots$ run from $p+1$ to n. The two parts of (27) are the metrics of M_1^p $(p \geq 2)$ and M_2^{n-p} $(n-p \geq 2)$ which are called the decompositions of the decomposable manifold $M^n = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$.

Let (M^n, g) be a Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}, \overset{*}{V} \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$, the following relations hold [37]:

$$R(\overset{*}{X}, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, \overset{*}{Y}, \bar{Z}, \overset{*}{U}) = R(\bar{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),$$
$$(\nabla_{\overset{*}{X}} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}) = (\nabla_{\overset{*}{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}),$$
$$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}); R(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}) = \overset{*}{R}(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),$$
$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); S(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{S}(\overset{*}{X}, \overset{*}{Y}),$$
$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}); (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}),$$

and

$$r = \bar{r} + \overset{*}{r},$$

where r, $\bar{r}$, and $\overset{*}{r}$ are scalar curvatures of M, M_1, M_2 respectively.

Let us consider a Riemannian manifold (M^n, g), which is a decomposable $(WW_3S)_n$. Then $M^n = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$.

Now from (4), we find

$$\begin{aligned} &W_3(\overset{*}{Y},\bar{Z},\bar{U},\bar{V})=0 \\ &\quad = W_3(\bar{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = W_3(\bar{Y},\overset{*}{Z},\bar{U},\bar{V}) = W_3(\bar{Y},\bar{Z},\overset{*}{U},\bar{V}), \end{aligned} \tag{28}$$

$$W_3(\bar{Y},\overset{*}{Z},\overset{*}{U},\bar{V}) = \frac{1}{n-1}S(\bar{Y},\bar{V})g(\overset{*}{Z},\overset{*}{U}), \tag{29}$$

$$W_3(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) = \frac{1}{n-1}S(\overset{*}{Y},\overset{*}{V})g(\bar{Z},\bar{U}), \tag{30}$$

$$W_3(\overset{*}{Y},\bar{Z},\overset{*}{U},\bar{V}) = -\frac{1}{n-1}S(\overset{*}{Y},\overset{*}{U})g(\bar{Z},\bar{V}), \tag{31}$$

$$W_3(\bar{Y},\overset{*}{Z},\bar{U},\overset{*}{V}) = -\frac{1}{n-1}S(\bar{Y},\bar{U})g(\overset{*}{Z},\overset{*}{V}), \tag{32}$$

$$(\nabla_{\overset{*}{X}}W_3)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0 = (\nabla_{\bar{X}}W_3)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}). \tag{33}$$

Again from (6), we find

$$\begin{aligned} &(\nabla_{\bar{X}}W_3)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = \\ &A(\bar{X})W_3(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(\bar{Y})W_3(\bar{X},\bar{Z},\bar{U},\bar{V}) + F(\bar{Z})W_3(\bar{Y},\bar{X},\bar{U},\bar{V}) \\ &\quad + D(\bar{U})W_3(\bar{Y},\bar{Z},\bar{X},\bar{V}) + D(\bar{V})W_3(\bar{Y},\bar{Z},\bar{U},\bar{X}), \end{aligned} \tag{34}$$

$$A(\overset{*}{X})W_3(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0, \tag{35}$$

$$B(\overset{*}{Y})W_3(\bar{X},\bar{Z},\bar{U},\bar{V}) = 0, \tag{36}$$

$$F(\overset{*}{Y})W_3(\bar{Y},\bar{X},\bar{U},\bar{V}) = 0, \tag{37}$$

$$D(\overset{*}{U})W_3(\bar{Y},\bar{Z},\bar{X},\bar{V}) = 0. \tag{38}$$

$$B(\overset{*}{Y})W_3(\bar{X},\overset{*}{Z},\overset{*}{U},\bar{*}V) + F(\overset{*}{Z})W_3(\overset{*}{Y},\bar{X},\overset{*}{U},\bar{V}) = 0, \tag{39}$$

$$B(\bar{Y})W_3(\overset{*}{X},\bar{Z},\bar{U},\overset{*}{V}) + F(\bar{Z})W_3(\bar{Y},\overset{*}{X},\bar{U},\overset{*}{V}) = 0, \tag{40}$$

$$\begin{aligned} (\nabla_{\bar{X}}W_3)(\bar{Y},\overset{*}{Z},\overset{*}{U},\bar{V}) \;=\; & A(\bar{X})W_3(\bar{Y},\overset{*}{Z},\overset{*}{U},\bar{V}) + B(\bar{Y})W_3(\bar{X},\overset{*}{Z},\overset{*}{U},\bar{V}) \\ & + D(\bar{V})W_3(\bar{Y},\overset{*}{Z},\overset{*}{U},\bar{X}), \end{aligned} \tag{41}$$

$$\begin{aligned} (\nabla_{\overset{*}{X}}W_3)(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) \;=\; & A(\overset{*}{X})W_3(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) + B(\overset{*}{Y})W_3(\overset{*}{X},\bar{Z},\bar{U},\overset{*}{V}) \\ & + D(\overset{*}{V})W_3(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{X}). \end{aligned} \tag{42}$$

Also from (6), we obtain

$$\begin{aligned} (\nabla_{\overset{*}{X}}W_3)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = {} & A(\overset{*}{X})W_3(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) + B(\overset{*}{Y})W_3(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) \\ & + F(\overset{*}{Z})W_3(\overset{*}{Y},\overset{*}{X},\overset{*}{U},\overset{*}{V}) + D(\overset{*}{U})W_3(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) \\ & + D(\overset{*}{V})W_3(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{X}), \end{aligned} \tag{43}$$

$$A(\bar{X})W_3(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{44}$$

$$B(\bar{Y})W_3(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{45}$$

$$F(\bar{Z})W_3(\overset{*}{Y},\overset{*}{X},\overset{*}{U},\overset{*}{V}) = 0, \tag{46}$$

$$D(\bar{U})W_3(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) = 0. \tag{47}$$

From (35)–(38) we conclude that either
(I) $A = B = F = D = 0$ on M_2, or,
(II) M_1 is W_3-flat.
Firstly, we consider the case (I). Then from (42), it follows that

$$(\nabla_{\overset{*}{X}} W_3)(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) = 0,$$

which implies by virtue of (30) that

$$(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) = 0 \tag{48}$$

and hence the decomposition M_2 is Ricci symmetric.

Also from (43), we have

$$(\nabla_{\overset{*}{X}} W_3)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0$$

and hence

$$(\nabla_{\overset{*}{X}} R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) + \frac{1}{n-1}\Big[(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V})g(\overset{*}{Z},\overset{*}{U}) - (\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V})g(\overset{*}{Z},\overset{*}{V})\Big] = 0,$$

which yields by virtue of (48) that $(\nabla_{\overset{*}{X}} R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0$, i.e., the decomposition M_2 is locally symmetric.

Secondly, we assume that M_1 is W_3-flat. Then we have

$$R(\bar{Y},\bar{Z},\bar{U},\bar{V}) = -\frac{1}{n-1}\big[S(\bar{Y},\bar{V})g(\bar{Z},\bar{U}) - S(\bar{Y},\bar{U})g(\bar{Z},\bar{V})\big]. \tag{49}$$

Contracting (49) over $\bar{Y}$ and $\bar{V}$, we obtain

$$S(\bar{Z},\bar{U}) = -\frac{\bar{r}}{n-2}g(\bar{Z},\bar{U}). \tag{50}$$

In view of (50), (49) yields

$$R(\bar{Y},\bar{Z},\bar{U},\bar{V}) = \frac{\bar{r}}{(n-1)(n-2)}\big[g(\bar{Y},\bar{V})g(\bar{Z},\bar{U}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})\big]$$

that is, the decomposition M_1 is a manifold of constant curvature.

Thus we can state the following:

Theorem 4.1 *Let (M^n, g) be a Riemannian manifold such that*

$$M = M_1^p \times M_2^{n-p}, \quad 2 \leq p \leq n-2.$$

If M^n is a $(WW_3S)_n$ then either (I) or (II) holds.

(I) $A = 0$, $B = 0$, $F = 0$, $D = 0$ on M_2 (resp. M_1), and hence M_2 (resp. M_1) is Ricci symmetric as well as locally symmetric.

(II) M_1 (resp. M_2) is W_3-flat and hence M_1 (resp. M_2) is a manifold of constant curvature.

Using (29) in (41), we get

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{V}) = A(\bar{X})S(\bar{Y}, \bar{V}) + B(\bar{Y})S(\bar{X}, \bar{V}) + D(\bar{V})S(\bar{Y}, \bar{X}). \tag{51}$$

Similarly by virtue of (30) we have from (42) that

$$(\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{V}) = A(\overset{*}{X})S(\overset{*}{Y}, \overset{*}{V}) + B(\overset{*}{Y})S(\overset{*}{X}, \overset{*}{V}) + D(\overset{*}{V})S(\overset{*}{Y}, \overset{*}{X}). \tag{52}$$

From (51) and (52), we can state the following:

Theorem 4.2 *Let (M^n, g) be a Riemannian manifold such that*

$$M = M_1^p \times M_2^{n-p}, \quad 2 \leq p \leq n-2.$$

If M is a $(WW_3S)_n$ then the decomposition M_1 is $(WRS)_p$ and the decomposition M_2 is $(WRS)_{n-p}$.

Using (30) and (32) in (40), we obtain

$$B(\bar{Y})S(\overset{*}{X}, \overset{*}{V})g(\bar{Z}, \bar{U}) - F(\bar{Z})S(\bar{Y}, \bar{U})g(\overset{*}{X}, \overset{*}{V}) = 0. \tag{53}$$

Contracting (53) over $\overset{*}{X}$ and $\overset{*}{V}$, we get

$$\overset{*}{r}\, B(\bar{Y})g(\bar{Z}, \bar{U}) - (n-p)F(\bar{Z})S(\bar{Y}, \bar{U}) = 0. \tag{54}$$

Again contracting (54) over $\bar{Z}$ and $\bar{U}$, we get

$$F(Q\bar{Y}) = r_1 F(\bar{Y}) \text{ where } r_1 = \frac{p}{n-p}\, \overset{*}{r}\,. \tag{55}$$

Similarly it follows from (39) that

$$F(Q\,\overset{*}{Y}) = r_2 F(\overset{*}{Y}) \text{ where } r_2 = \frac{n-p}{p} \bar{r}. \tag{56}$$

Hence we can state the following:

Theorem 4.3 *Let (M^n, g) be a Riemannian manifold such that*

$$M = M_1^p \times M_2^{n-p}, \quad 2 \leq p \leq n-2.$$

If M is a $(WW_3S)_n$ then the relations (55) *and* (56) *hold.*

5 Totally umbilical hypersurfaces of $(WW_3S)_n$

Let $(\bar{V}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (V, g) be a hypersurface of $(\bar{V}, \bar{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values $1, 2, \dots, n$ and Latin indices take values $1, 2, \dots, (n+1)$. Let N^α be the components of a local unit normal to (V, g). Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \tag{57}$$

$$\bar{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \quad \bar{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \tag{58}$$

$$y_i^\alpha y_j^\beta g^{ij} = \bar{g}^{\alpha\beta} - N^\alpha N^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \tag{59}$$

The hypersurface (V, g) is called a totally umbilical hypersurface ([6],[9]) of $(\bar{V}, \bar{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \qquad y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{60}$$

where the scalar function H is called the mean curvature of (V, g) given by $H = \frac{1}{n} \sum g^{ij} \Omega_{ij}$. If, in particular, $H = 0$, i.e.,

$$\Omega_{ij} = 0, \tag{61}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.

The equation of Weingarten for (V, g) can be written as $N_{,j}^\alpha = -\frac{H}{n} y_j^\alpha$. The structure equations of Gauss and Codazzi ([6],[9]) for (V, g) and $(\bar{V}, \bar{g})$ are respectively given by

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \tag{62}$$

$$\bar{R}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = H_{,i}\ g_{jk} - H_{,j}\ g_{ik}, \tag{63}$$

where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively, and

$$B_{ijkl}^{\alpha\beta\gamma\delta} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta, \quad B_i^\alpha = y_i^\alpha, \quad G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}.$$

Also we have ([6], [9])

$$\bar{S}_{\alpha\delta} B_i^\alpha B_j^\delta = S_{ij} - (n-1) H^2 g_{ij}, \tag{64}$$

$$\bar{S}_{\alpha\delta} N^\alpha B_i^\delta = (n-1) H_{,i}, \tag{65}$$

where S_{ij} and $\bar{S_{\alpha\delta}}$ are the Ricci tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively.

In terms of local coordinates the relations (4) and (6) can be written as

$$(W_3)_{hijk} = R_{hijk} + \frac{1}{n-1} \Big[S_{hk} g_{ij} - S_{hj} g_{ik} \Big], \tag{66}$$

$$\begin{aligned}(W_3)_{hijk,l} \;&=\; A_l(W_3)_{hijk} + B_h(W_3)_{lijk} + F_i(W_3)_{hljk} \\ &+\; D_j(W_3)_{hilk} + D_k(W_3)_{hijl}.\end{aligned} \tag{67}$$

By virtue of (62) and (64), we have from (66) that

$$(W_3)_{ijkl} = (\bar{W}_3)_{\alpha\beta\gamma\delta}B^{\alpha\beta\gamma\delta}_{ijkl} + 2H^2G_{ijkl}. \tag{68}$$

Let $(\bar{V}, \bar{g})$ be a weakly W_3-symmetric manifold. Then we get

$$\begin{aligned}(\bar{W}_3)_{\beta\gamma\sigma\alpha,\delta} \;&=\; A_\delta(\bar{W}_3)_{\beta\gamma\sigma\alpha} + B_\beta(\bar{W}_3)_{\delta\gamma\sigma\alpha} + F_\gamma(\bar{W}_3)_{\beta\delta\sigma\alpha} \\ &+\; D_\sigma(\bar{W}_3)_{\beta\gamma\delta\alpha} + D_\alpha(\bar{W}_3)_{\beta\gamma\sigma\delta},\end{aligned} \tag{69}$$

where A, B, F and D are 1-forms (not simultaneously zero).

Multiplying both sides of (69) by $B^{\alpha\beta\gamma\delta\sigma}_{hijkl}$ and then using (67) and (68), we get either $H = 0$ or

$$\begin{aligned}2H_{,k}G_{ijlh} \;&=\; H\big[A_kG_{ijlh} + B_iG_{kjlh} + F_jG_{iklh} \\ &+\; D_lG_{ijkh} + D_hG_{ijlk}\big].\end{aligned} \tag{70}$$

Transvecting (70) by $g^{ih}g^{jl}$, we obtain

$$2nH_{,k} = \big[nA_k + B_k + F_k + 2D_k\big]H \quad \text{for all } k. \tag{71}$$

This leads to the following:

Theorem 5.1 *If the totally umbilical hypersurface of a $(WW_3S)_n$ is a $(WW_3S)_n$ then either the manifold is a totally geodesic hypersurface or the associated 1-forms A, B, F and D are related by the relation* (71).

We now consider that the space (V, g) is totally geodesic hypersurface, i.e.,

$$H = 0. \tag{72}$$

In view of (72), (68) yields

$$(\bar{W}_3)_{\alpha\beta\gamma\delta}B^{\alpha\beta\gamma\delta}_{ijkl} = (W_3)_{ijkl}. \tag{73}$$

Using (73) in (69), we have the relation (67). Thus we can state the following:

Theorem 5.2 *The totally geodesic hypersurface of a $(WW_3S)_n$ is $(WW_3S)_n$.*

6 Example of $(WW_3S)_n$ and decomposable $(WW_3S)_n$

Example 6.1 Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \colon 0 < x^4 < 1\}$ be a manifold endowed with the metric

$$\begin{aligned}ds^2 = g_{ij}dx^idx^j \;&=\; (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^4)^2\right] + (dx^2)^2 + (dx^3)^2, \\ &i, j = 1, 2, 3, 4.\end{aligned} \tag{74}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, W_3-curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{14} = \Gamma^4_{44} = \frac{2}{3x^4} = -\Gamma^4_{11},$$
$$R_{1441} = -\frac{2}{3(x^4)^{\frac{2}{3}}}, S_{11} = S_{44} = -\frac{2}{3(x^4)^2},$$
$$r = -\frac{4}{3(x^4)^{\frac{10}{3}}} \neq 0, (W_3)_{1414} = \frac{4}{9(x^4)^{\frac{2}{3}}} = -(W_3)_{4114},$$
$$(W_3)_{1212} = \frac{2}{9(x^4)^2} = (W_3)_{1313} = -(W_3)_{4224} = -(W_3)_{4334},$$

$$(W_3)_{1212,4} = (W_3)_{1313,4} = -(W_3)_{4224,4} = -(W_3)_{4334,4} = -\frac{20}{27(x^4)^3}, \tag{75}$$

$$(W_3)_{1414,4} = -\frac{40}{27(x^4)^{\frac{5}{3}}} = -(W_3)_{4114,4}, \tag{76}$$

and the components that can be obtained from these by the symmetry properties, where ',' denotes the covariant differentiation with respect to the metric tensor. Therefore, the manifold M^4 with the considered metric is a Riemannian manifold, which is neither W_3-flat nor W_3 symmetric and is of non-vanishing scalar curvature.

We shall now show that this (M^4, g) is a $(WW_3S)_4$, that is, it satisfies (67).

In terms of local coordinate system we consider the components of the 1-forms A, B, F and D as follows:

$$\begin{aligned} A(\partial_i) = A_i &= -\frac{10}{3x^4} \quad \text{for } i = 4 \\ &= 0 \quad \text{otherwise}, \\ B_i = F_i = D_i &= 0 \quad \text{for } i = 1, 2, 3, 4, \end{aligned} \tag{77}$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

In our M^4 with the considered 1-forms, (67) reduces to the following equations:

$$\begin{aligned} (W_3)_{121l,i} &= A_i(W_3)_{121l} + B_1(W_3)_{i21l} + F_2(W_3)_{1i1l} \\ &+ D_1(W_3)_{12il} + D_l(W_3)_{121i}, \end{aligned} \tag{78}$$

$$\begin{aligned} (W_3)_{12t2,i} &= A_i(W_3)_{12t2} + B_1(W_3)_{i2t2} + F_2(W_3)_{1it2} \\ &+ D_t(W_3)_{12i2} + D_2(W_3)_{12ti}, \end{aligned} \tag{79}$$

$$\begin{aligned} (W_3)_{1u12,i} &= A_i(W_3)_{1u12} + B_1(W_3)_{iu12} + F_u(W_3)_{1i12} \\ &+ D_1(W_3)_{1ui2} + D_2(W_3)_{1u1i}, \end{aligned} \tag{80}$$

$$\begin{aligned}(W_3)_{t212,i} &= A_i(W_3)_{t212} + B_t(W_3)_{i212} + F_2(W_3)_{ti12} \\ &+ D_1(W_3)_{t2i2} + D_2(W_3)_{t21i},\end{aligned} \tag{81}$$

$$\begin{aligned}(W_3)_{131u,i} &= A_i(W_3)_{131u} + B_1(W_3)_{i31u} + F_3(W_3)_{1i1u} \\ &+ D_1(W_3)_{13iu} + D_u(W_3)_{131i},\end{aligned} \tag{82}$$

$$\begin{aligned}(W_3)_{13t3,i} &= A_i(W_3)_{13t3} + B_1(W_3)_{i3t3} + F_3(W_3)_{1it3} \\ &+ D_t(W_3)_{13i3} + D_3(W_3)_{13ti},\end{aligned} \tag{83}$$

$$\begin{aligned}(W_3)_{1p13,i} &= A_i(W_3)_{1p13} + B_1(W_3)_{ip13} + F_p(W_3)_{1i13} \\ &+ D_1(W_3)_{1pi3} + D_3(W_3)_{1p1i},\end{aligned} \tag{84}$$

$$\begin{aligned}(W_3)_{t313,i} &= A_i(W_3)_{t313} + B_t(W_3)_{i313} + F_3(W_3)_{ti13} \\ &+ D_1(W_3)_{t3i3} + D_3(W_3)_{t31i},\end{aligned} \tag{85}$$

$$\begin{aligned}(W_3)_{14lp,i} &= A_i(W_3)_{14lp} + B_1(W_3)_{i4lp} + F_4(W_3)_{1ilp} \\ &+ D_1(W_3)_{14ip} + D_p(W_3)_{141i},\end{aligned} \tag{86}$$

$$\begin{aligned}(W_3)_{14t4,i} &= A_i(W_3)_{14t4} + B_1(W_3)_{i4t4} + F_4(W_3)_{1it4} \\ &+ D_t(W_3)_{14i4} + D_4(W_3)_{14ti},\end{aligned} \tag{87}$$

$$\begin{aligned}(W_3)_{t414,i} &= A_i(W_3)_{t414} + B_t(W_3)_{i414} + F_4(W_3)_{ti14} \\ &+ D_4(W_3)_{t4i4} + D_4(W_3)_{t41i},\end{aligned} \tag{88}$$

$$\begin{aligned}(W_3)_{411l,i} &= A_i(W_3)_{411l} + B_4(W_3)_{i11l} + F_1(W_3)_{4i1l} \\ &+ D_1(W_3)_{41il} + D_l(W_3)_{411i},\end{aligned} \tag{89}$$

$$\begin{aligned}(W_3)_{41t4,i} &= A_i(W_3)_{41t4} + B_4(W_3)_{i1t4} + F_1(W_3)_{4it4} \\ &+ D_t(W_3)_{41i4} + D_4(W_3)_{41ti},\end{aligned} \tag{90}$$

$$\begin{aligned}(W_3)_{4t14,i} &= A_i(W_3)_{4t14} + B_4(W_3)_{it14} + F_t(W_3)_{4i14} \\ r &+ D_1(W_3)_{4ti4} + D_4(W_3)_{4t1i},\end{aligned} \tag{91}$$

$$\begin{aligned}(W_3)_{1114,i} &= A_i(W_3)_{1114} + B_1(W_3)_{i114} + F_1(W_3)_{1i14} \\ &+ D_1(W_3)_{11i4} + D_4(W_3)_{111i},\end{aligned} \tag{92}$$

$$\begin{aligned}(W_3)_{q114,i} &= A_i(W_3)_{q114} + B_q(W_3)_{i114} + F_1(W_3)_{qi14}\\ &+ D_1(W_3)_{q1i4} + D_4(W_3)_{q11i},\end{aligned} \tag{93}$$

$$\begin{aligned}(W_3)_{422l,i} &= A_i(W_3)_{422l} + B_4(W_3)_{i22l} + F_2(W_3)_{4i2l}\\ &+ D_2(W_3)_{42il} + D_l(W_3)_{422i},\end{aligned} \tag{94}$$

$$\begin{aligned}(W_3)_{v224,i} &= A_i(W_3)_{v224} + B_v(W_3)_{i224} + F_2(W_3)_{vi224}\\ &+ D_2(W_3)_{v2i4} + D_4(W_3)_{v22i},\end{aligned} \tag{95}$$

$$\begin{aligned}(W_3)_{42s4,i} &= A_i(W_3)_{42s4} + B_4(W_3)_{i2s4} + F_2(W_3)_{4is4}\\ &+ D_s(W_3)_{42i4} + D_4(W_3)_{42si},\end{aligned} \tag{96}$$

$$\begin{aligned}(W_3)_{4s24,i} &= A_i(W_3)_{4s24} + B_4(W_3)_{is24} + F_s(W_3)_{4i24}\\ &+ D_2(W_3)_{4si4} + D_4(W_3)_{4s2i},\end{aligned} \tag{97}$$

$$\begin{aligned}(W_3)_{433l,i} &= A_i(W_3)_{433l} + B_4(W_3)_{i33l} + F_3(W_3)_{4i3l}\\ &+ D_3(W_3)_{43il} + D_l(W_3)_{433i},\end{aligned} \tag{98}$$

$$\begin{aligned}(W_3)_{v334,i} &= A_i(W_3)_{v334} + B_v(W_3)_{i334} + F_3(W_3)_{vi34}\\ &+ D_3(W_3)_{v3i4} + D_4(W_3)_{v33i},\end{aligned} \tag{99}$$

$$\begin{aligned}(W_3)_{4434,i} &= A_i(W_3)_{4434} + B_4(W_3)_{i434} + F_4(W_3)_{4i34}\\ &+ D_3(W_3)_{44i4} + D_4(W_3)_{443i},\end{aligned} \tag{100}$$

$$\begin{aligned}(W_3)_{4434,i} &= A_i(W_3)_{4434} + B_4(W_3)_{i434} + F_4(W_3)_{4i34}\\ &+ D_3(W_3)_{44i4} + D_4(W_3)_{443i},\end{aligned} \tag{101}$$

where $i = 1, 2, 3, 4$; $l = 1, 2, 3, 4$; $t = 2, 3, 4$; $u = 1, 3, 4$; $p = 1, 4$; $v = 1, 2, 3$; $q = 2, 3$; $s = 3, 4$, since for the cases other than (78)–(101), the components of each term of (67) either vanishes identically or the relation (67) holds trivially using the skew-symmetry property of W_3.

Now using (75) and (77), it follows for $i = 4$ that, right hand side of (78) (for $l = 2$) $= A_4\ (W_3)_{1212} = -\frac{20}{27(x^4)^3} = (W_3)_{1212,4} =$ left hand side of (78) (for $l = 2$).

For $i = 1, 2, 3$, the relation (77) implies that both sides of equation (78) are equal. By the similar arguement, it can be easily seen that the equation (79)–(101) holds. Thus the manifold under consideration is weakly W_3-symmetric manifold.

Hence we can state the following:

Theorem 6.2 *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (74). Then (M^4, g) is a weakly W_3-symmetric manifold with non-vanishing scalar curvature, which is neither W_3-flat nor W_3-symmetric.*

Example 6.3 Let $M = \{(x^1, x^2, x^3, \dots, x^n) \in \mathbb{R}^n : 0 < x^4 < 1\}$ be a manifold endowed with the metric

$$ds^2 = \left[(x^4)^{\frac{4}{3}} - 1\right]\left[(dx^1)^2 + (dx^4)^2\right] + \delta_{ab}dx^a dx^b, \tag{102}$$

where δ_{ab} is the kronecker delta and a, b run from 1 to n. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{14} = \Gamma^4_{44} = \frac{2}{3x^4} = -\Gamma^4_{11}, R_{1441} = -\frac{2}{3(x^4)^{\frac{2}{3}}},$$
$$S_{11} = S_{44} = -\frac{2}{3(x^4)^2}, r = -\frac{4}{3(x^4)^{\frac{10}{3}}} \neq 0,$$
$$(W_3)_{1414} = \frac{2(n-2)}{3(n-1)(x^4)^{\frac{2}{3}}} = -(W_3)_{4114},$$
$$(W_3)_{1212} = (W_3)_{1313} = \frac{2}{3(n-1)(x^4)^2} = -(W_3)_{4224} = -(W_3)_{4334},$$
$$(W_3)_{1k1k} = (W_3)_{4k4k} = \frac{2}{3(n-1)(x^4)^2},$$

$$\begin{aligned}(W_3)_{1212,4} = (W_3)_{1313,4} &= -(W_3)_{4224,4} = -(W_3)_{4334,4} \\ &= -\frac{20}{9(n-1)(x^4)^3},\end{aligned} \tag{103}$$

$$(W_3)_{1414,4} = -\frac{20(n-2)}{9(n-1)(x^4)^{\frac{5}{3}}} = -(W_3)_{4114,4}, \tag{104}$$

$$(W_3)_{1k1k,4} = (W_3)_{4k4k,4} = -\frac{20}{9(n-1)(x^4)^3} \quad \text{for } 5 \leq k \leq n. \tag{105}$$

If we consider the components of the 1-forms A, B, F and D as

$$\begin{aligned}A(\partial_i) = A_i &= -\frac{10}{3x^4} \quad \text{for } i = 4 \\ &= 0 \quad \text{otherwise },\\ B_i = F_i = D_i &= 0 \quad \text{for } i = 1, 2, \dots, n,\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, then it can be easily shown that M^n is a $(WW_3S)_n$, which is not W_3-symmetric. Hence we can state the following:

Theorem 6.4 *Let (M^n, g), $n \geq 4$, be a Riemannian manifold endowed with the metric given in* (102). *Then (M^n, g) is a weakly W_3-symmetric manifold with non-vanishing scalar curvature, which is not W_3-symmetric.*

Let (M_1^4, g_1) be a Riemannian manifold in Example 6.1 and $(\mathbb{R}^{n-4}, g_0)$ be an $(n-4)$-dimensional Euclidean space with standard metric g_0. Then (M^n, g) in Example 6.2 is a product manifold of (M_1^4, g_1) and $(\mathbb{R}^{n-4}, g_0)$. Thus we can state the following:

Theorem 6.5 *Let (M^n, g), $n \geq 4$, be a Riemannian manifold endowed with the metric given in* (102). *Then $(M^n, g)(n \geq 4)$ is a decomposable weakly W_3-symmetric manifold $(M_1^4, g_1) \times (\mathbb{R}^{n-4}, g_0)$ with non-vanishing scalar curvature, which is not W_3-symmetric.*

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